Large Charge in $\mathcal{N}=2$ SCFTs

- work in progress w/
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- Goal: generalize beautiful rank-1 story of
  - Hellerman, Maeda 1710.07336 [HM]
  - H, M, Orlando, Reffert, Watanabe 1804.01535, ...

  to rank $r>1$ $\mathcal{N}=2$ SCFTs.

- Outline
  (1) Review [HM]
  (2) Copy [HM] @ $r>1$? ~ yes
  (3) Copy [HMORW] @ $r>1$?
  or
  Can we determine the 4-derivative WZ-like & F-terms in the effective theory?

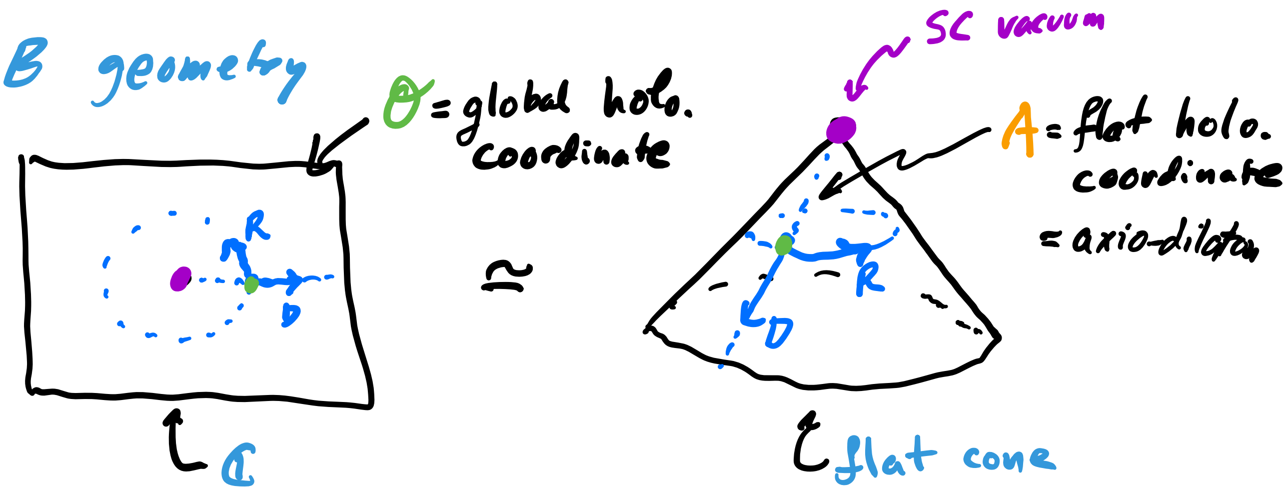
  ~ some ideas ...

(1) Review [HM]: rank-1 Coulomb branches

- $\mathcal{O}(x)$ generates CB chiral ring. $D(\mathcal{O}) = R(\mathcal{O}) \doteq \Delta$.
- $B_n \doteq \log\left(\langle \mathcal{O}^n(x)\, \bar{\mathcal{O}}^n(0)\rangle_{\mathbb{R}^4} \cdot |x|^{2n\Delta}\right)$

(*) $$\overset{n\to\infty}{\simeq} \underbrace{\log(n!) + b_{-1}\cdot n}_{\text{2-deriv terms @ saddle pt. soln}} + \underbrace{\alpha\cdot\log(n)}_{\text{WZ term}} + \underbrace{\mathcal{O}(n^0)}_{\geq 4\text{-deriv terms, fluctuations}}$$

- CB geometry



Broken Dil + $U(1)_R$ $\Rightarrow$ hol. $\mathbb{C}^*$ homothety

$$\mathcal{L}_{2\partial} \sim |\mathcal{O}|^{-2+\frac{2}{\Delta}}\, \partial\mathcal{O}\cdot\partial\bar{\mathcal{O}} \sim (\mathrm{Im}\,\tau)\, \partial A\cdot\partial\bar{A} \qquad \underline{\text{free}}$$

$$\mathcal{O} \propto (A)^{\Delta} \qquad \text{coeff. determines } b_{-1} \text{ in } (*)$$

- Add sources $\quad \mathcal{L}_S \sim -n\Delta\left[\log(A)\cdot\delta(x-x_1) + \log(\bar{A})\cdot\delta(x-x_2)\right]$

so $\quad B_n = \log\left[\langle e^{-S_{EFT}-S_S}\rangle / \langle e^{-S_{EFT}}\rangle\right]$

$$\simeq -\left(S_{EFT}+S_s\right)\Big|_{\text{saddle pt.}} + \text{fluctuations}$$

Find saddle pt. $\begin{cases} A(x) \sim \sqrt{n}\cdot \hat{f} \\ \bar{A}(x) \sim \sqrt{n}\cdot \hat{g} \end{cases}$ $O(n^0)$

• Then by $\mathbb{C}^\times$

$$\mathcal{L}_{EFT} \supset A^{4-2m-p}\,\partial^{2m} A^p$$

$$\sim O(n^{2-m}) \to \begin{cases} m=0 \to \text{const.} ✓ \\ m=1 \to 2\partial ✓ \\ m\geq 2 \to O(n^0) \end{cases}$$

& contrib frm $S_s \sim \log(n!) + b_{-1}\, n + O(n^0)$

Wick contract. ⮥ ⮡ normalization

But:
$$\int_{\mathbb{R}^4}^{L} d^4x \left\{ \left(\frac{\partial A}{A}\right)^4 + \left(\frac{\partial A}{A}\right)^2\left(\frac{\partial \bar{A}}{\bar{A}}\right)^2 + \left(\frac{\partial \bar{A}}{\bar{A}}\right)^4 \right\}\Bigg|_{s.p.} \sim \log L \cdot O(n^0)$$

IR divergence indicates contribution from dil + $U(1)_R$ WZ term

In fact, $\mathcal{N}=2$ SUSY $\Rightarrow$ unique combination & matches precisely 4∂ WZ term

[de Wit, Grisaru, Roček 9601115
Dine, Seiberg 9705057
Bobev, Elvang, Olson 1312.2925
HMORW 1804.01535]

Regulate IR by Weyl: $\mathbb{R}^4 \to S^4$ & get term

$$\sim \Delta a\,(\log A)\cdot \chi(S^4) \sim \log n$$

$$\Rightarrow \quad \alpha = 2\Delta a \doteq 2\left(a_{CFT} - a_V\right)$$

⮡ free vector multiplet central charge

(2) Repeat at rank $r>1$

- $\{\theta_i,\ i=1\cdots r\}$ freely generates CB chiral ring

  (already an assumption!)

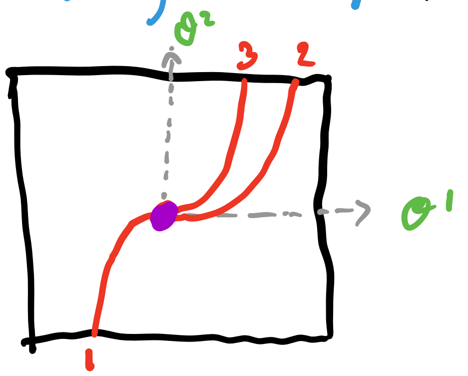
$$D(\theta_i) = R(\theta_i) = \Delta_i$$

- look at $\quad B_n \doteq \log\left(\langle \theta^n(x)\, \bar{\theta}^n(0)\rangle\, |x|^{2n\cdot\Delta}\right), \quad |n|\to\infty$

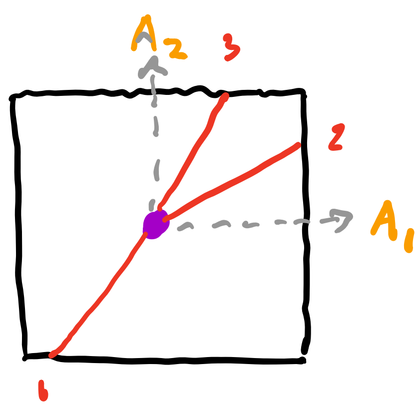
$$n \doteq (n_1 \cdots n_r) \qquad \Delta \doteq (\Delta_1 \cdots \Delta_r) \qquad n\cdot\Delta \doteq \sum_i n_i \Delta_i \qquad |n| \doteq \sum_i n_i$$

$$\theta^n \doteq \prod_i \theta_i^{n_i}$$

- CB geometry $(r=2)$



$\simeq$



$(\not\simeq \mathbb{C}^2)$

$\simeq \mathbb{C}^2$
cplx structure

sp. Kahler geom
not nec. flat
rays = $\mathbb{C}^*$-orbits

● = CFT vacuum
—$_i$ = $U(1) \times CFT_i$ vacua $\Big\}$ "singularities"
$\uparrow$ (rank-1 CFT)

$(A_1, \cdots, A_r)$ = "special coord.s" $\qquad \Delta(A_i) = 1$

$A_i = A_i(\theta_j)$ • locally holomorphic but non-analytic at singularities

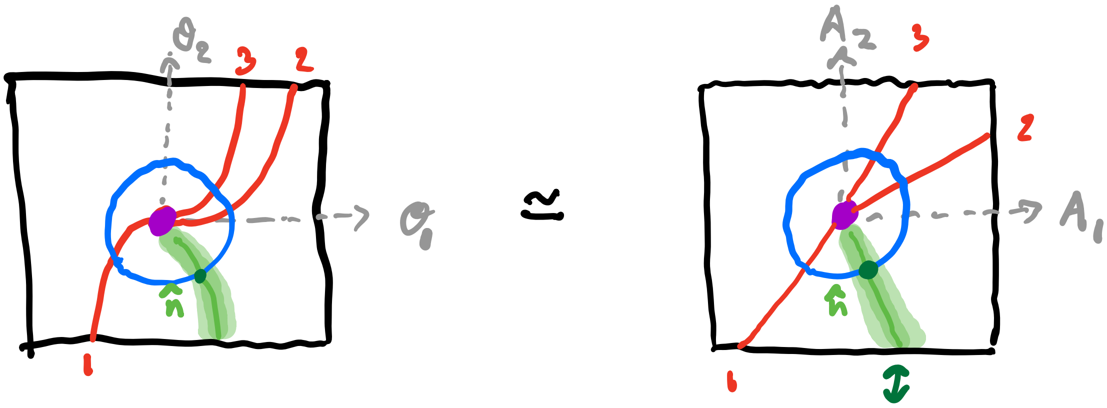
• can be very complicated

$$\mathcal{L}_{2\partial} \sim [\underbrace{\mathrm{Im}\,\tau^{ij}(A)}]\,\partial A_i \cdot \partial \bar{A}_j$$

• Curved CB metric + complicated $\{A_i\} \leftrightarrow \{\mathcal{O}_i\}$ relation means adding sources & solving for saddle point configuration is technically complicated.

– But, instead of looking at $\mathcal{O}^n$ 2-pt fnc, choose instead

$$P^{|n|}_{\hat{n}} = P^{|n|}_{\hat{n}}(\mathcal{O}_i) \begin{cases} \text{fixed } |n| \to \infty \\ \text{"localized" in transverse} \\ \hat{n} \in \mathbb{P}^{r-1} \text{ to } \mathbb{C}^\times \end{cases}$$



≈ flat 2d cone

≈ free EFT.

– Or, make simplifying assumption

CB is "isotrivial" $\Leftrightarrow$ flat AE $\Leftrightarrow$ $\tau^{ij}(A) = \text{constant}$.

• $\exists$ large class at all ranks

includes some gauge theories, $\mathcal{N}=3$ CFTs, ...

(cf. Cecotti, Del Zotto, Martone, Moscrop 2108.10884)

• A_i are free, but still nontrivially related to $\mathcal{O}_i$

[E.g. $SU(3)$ $\mathcal{N}=4$: $\begin{cases} \mathcal{O}_1 = A_1^2 + A_1 A_2 + A_2^2 \\ \mathcal{O}_2 = A_1 A_2 (A_1 + A_2) \end{cases}$

$$\Leftrightarrow \quad A_i^3 - A_i \mathcal{O}_1 + \mathcal{O}_2 = 0 \;]$$

- So, consider $B_n = \log\left(\langle A^n(x)\, \bar{A}^n(0)\rangle |x|^{2|n|}\right)$

$$A^n \doteq \prod_i A_i^{n_i} \quad \text{on isotrivial CB.}$$

$$n = |n|\,\hat{n} \qquad \hat{n} = \left(\frac{n_1}{|n|}, \dots, \frac{n_r}{|n|}\right)$$

Then $S_{2\partial} + S_{source}$ saddle point analogous to rank-1

$$A_j(x) \sim \sqrt{n_j}\, e^{i\varphi_j} \frac{|x_1 - x_2|}{(x-x_1)^2} \Rightarrow \langle A_j \rangle \sim \sqrt{n_j}\, e^{i\varphi_j} \equiv m_j \sqrt{|n|}$$

so saddle point solution supported along

$$A_i \propto m_i$$

direction on CB.

- $S_{EFT} + S_{source}\big|_{s.p.} \rightsquigarrow$

$$B_n \sim \underbrace{\sum_i \log(n_i!)}_{\text{Wick}} + \underbrace{b_{-1}(\hat{n})\,|n|}_{\text{normaliz.}} + \underbrace{\mathcal{O}(|n|^0)}_{\geq 4\partial,\ \text{fluct.}} + \underbrace{WZ}_{?}$$

- But now $\exists$ many IR-divergent 4∂ terms

$$\mathcal{L}_{4\partial} \supset \frac{(\partial A_i \cdot \partial A_j)(\partial A_k \cdot \partial A_l)}{A_i A_j A_k A_l} + \dots$$

Presumably all contribute to WZ term in m-direction on CB where axio-dilaton is

$$\phi \sim m_i \, [\mathrm{Im}\, \tau^{ij}] \, A_j$$

This gives $B_n \sim +2\alpha \left(\log |n|\right) + O(|n|^0 \log \hat{n}_i)$ contribution.

- Conclusion (?) If only interested in $O(\log|n|)$ contrib. to B_n, it will be given by anomaly $\alpha = 2\Delta a = 2(a_{CFT} - 2a_V)$

- Problems: $m_j \propto \sqrt{n_j}\, e^{i\beta_j}$

  what determines relative $U(1)_R$ phases?

  - In isotrivial case β_j are completely arbitrary at classical, 2d level ...
  - Did for unphysical correlator

    $$\langle A^n \bar{A}^n \rangle$$ multi-valued on CB

    instead of physical correlator

    $$\langle \mathcal{O}^n \bar{\mathcal{O}}^n \rangle$$

(3) 4-derivative terms on CB

- Vector multiplet $\mathcal{A} = (A, \lambda^i_\alpha, F_{\mu\nu})$
- Off-shell superfield in $\mathcal{N}=2$ suspace $(x^\mu, \theta^i_\alpha, \bar{\theta}^i_{\dot\alpha})$

$\mathcal{A}$ = chiral sufield $\Rightarrow \bar{D}^i_{\dot\alpha} \mathcal{A} = 0$

$\Rightarrow \mathcal{A}(x,\theta) \sim A(x) + \theta\lambda(x) + \theta\sigma^{\mu\nu}\theta\, F^+_{\mu\nu}(x) + \cdots$

$F_{\mu\nu}$ satisfies Bianchi identity, & θ^3, θ^4 not indep't

$\Rightarrow \mathcal{A}(x,\theta)$ is constrained: $D^{(i} \cdot D^{j)} \mathcal{A} = \bar{D}^{(i} \cdot \bar{D}^{j)} \bar{\mathcal{A}}$.

Scale & $U(1)_R$ - inv't 4∂ D-terms

$$\mathcal{L}_{4\partial\text{-}D} \sim \int d^8\theta \, \log[f(\mathcal{A}_i)] \cdot \log[\bar{g}(\bar{\mathcal{A}}_i)]$$

with $f(\mathcal{A}_i), g(\mathcal{A}_i)$ homogeneous in dimension.

- Has singularities @ $f(A_i) = 0$ or $g(A_i) = 0$
- Should coincide with CB singularities

- CB singularities S_i are loci of vanishing central charge:

$$Z_{Q_i} = p \cdot A + q \cdot A_D \qquad A^i_D = \frac{\partial \mathcal{F}}{\partial A_i}$$

$Q_i = (p,q) \in \Lambda_i \equiv$ sublattice of the Electric & Magnetic

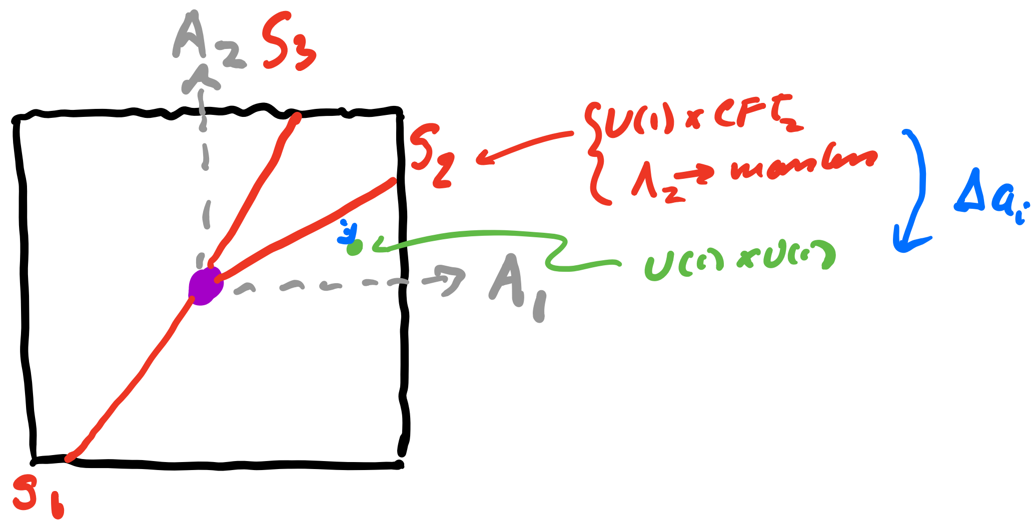
Charge lattice corresponding to states which become massless at S_i.

- Suggests:

$$\mathcal{L}_{4\partial\text{-}P} = \int d^8\theta \sum_i \Delta a_i \sum_{Q,P\in\Lambda_i} r_{Q,P} \log(Z_Q) \log(\overline{Z}_P)$$

- $\Delta a_i \doteq a_{CFT_i} - a_V$

Idea:



E.g., Approximate "transverse" $Dil_2 \times U(1)_{R_2}$ from CFT_2 @ S_2 singularity.

- What determines $r_{Q,P}$ coefficients?

- Guess: $r_{Q,P} = \delta_{Q,P} \cdot$ (multiplicity of massless BPS states)
  ↳ i.e., 1-particle

Note: BPS spectrum may be different in different "wedges" of CB.

- [Dine, Gray 9909020]: evidence works for $SU(N)$ $\mathcal{N}=4$ SYM

– w/ Cuomo, Martone, Tizzano for factorized isotrivial theories, some evidence ...

- Possible further checks:

  – 1-loop p.t. in gauge CFTs

  – localization results, e.g.

    Beccaria, Galvagno, Hasan 2001.06645

    Beccaria, Billò, Frau, Lerda, Pini 2105.15113

  – correct "projection" onto effective axio-dilaton

$$\phi \sim m_i [\operatorname{Im} \tau^{ij}] A_j$$

for suitable choices of sources.

o Scale & $U(1)_R$-inv't 4∂ F-terms

$$\mathcal{L}_{4\partial\text{-}F} = \int d^4\theta \; \partial^\mu A_i \partial_\mu A_j \underbrace{\mathcal{G}^{ij}(A_k)}_{} + \text{c.c.}$$

- $\mathcal{G}^{ij}$ homogeneous of degree 0
- Contributes @ $\mathcal{O}(|n|^0)$ in large-$|n|$ expansion
- Holomorphic, so determinable following Seiberg-Witten sol'n for prepotential

$$\mathcal{L}_{2\partial\text{-}F} = \int d^4\theta \, \mathcal{F}(A_k) + \text{c.c.}$$

?