Large Charge on a Conformal Manifold

Adar Sharon

Focus Week on Quantum Mechanical Systems
at Large Quantum Number

Based on 2008.01106 with M. Watanabe

Large Charge

For all operators of charge $Q \gg 1$, what is the smallest dimension $\Delta(Q)$?

CFTs fall into different phases:

1. Superfluid phase: the theory is dominated by a Goldstone mode for the spontaneously broken symmetry. $\Delta(Q) \sim Q^{\frac{d}{d-1}}$
   Example: 3d $\mathcal{N} = 2$ theory with $W = \Phi^3$ ("SUSY Ising model").

Large Charge

For all operators of charge $Q \gg 1$, what is the smallest dimension $\Delta(Q)$?

CFTs fall into different phases:

1. Superfluid phase: the theory is dominated by a Goldstone mode for the spontaneously broken symmetry. $\Delta(Q) \sim Q^{\frac{d}{d-1}}$
   Example: 3d $\mathcal{N} = 2$ theory with $W = \Phi^3$ ("SUSY Ising model").

2. Free scalar phase: dominated by axio-dilaton. $\Delta(Q) \sim Q$
   Example: 3d $\mathcal{N} = 2$ theory with $W = XYZ$ ("XYZ model")

Large Charge

For all operators of charge $Q \gg 1$, what is the smallest dimension $\Delta(Q)$?

CFTs fall into different phases:

1. Superfluid phase: the theory is dominated by a Goldstone mode for the spontaneously broken symmetry. $\Delta(Q) \sim Q^{\frac{d}{d-1}}$
   Example: 3d $\mathcal{N} = 2$ theory with $W = \Phi^3$ (“SUSY Ising model”).

2. Free scalar phase: dominated by axio-dilaton. $\Delta(Q) \sim Q$
   Example: 3d $\mathcal{N} = 2$ theory with $W = XYZ$ (“XYZ model”)

3. Free fermion phase. $\Delta(Q) \sim Q^{\frac{d}{d-1}}$

Large Charge

Superfluid phase: 3d $\mathcal{N} = 2$ with $W = \Phi^3$.

At large charge, dimensional analysis gives one possible term:

$$K = |\Phi|^{3/2}$$

Fermions decouple, and we are left with an effective action for $\phi = |\phi|e^{i\chi}$. Integrating out $|\phi|$ we find

$$\mathcal{L} = |\partial\chi|^3$$

From which we find

$$\Delta(Q) \sim Q^{3/2}$$

Large Charge

Free scalar phase: 3d $\mathcal{N} = 2$ theory with $W = XYZ$.

Theory has a moduli space ($X \neq 0, Y = Z = 0$ + permutations). X^n is BPS, and so

$$\Delta_{X^n} = \frac{2}{3}n$$

Due to BPS bound, this must be smallest dimension. So

$$\Delta(Q) \sim Q$$

Universality Classes

Focus on two phases:

1. Superfluid phase: $\Delta \sim Q^{\frac{d}{d-1}}$
2. Free scalar phase: $\Delta \sim Q$

Are these really fundamentally different?

Look for a parameter τ which interpolates between these phases:

$\Delta \sim Q$ $\quad\longrightarrow\quad$ $\Delta \sim Q^{\frac{d}{d-1}}$ $\quad Q\tau$

Free scalar phase $\quad\quad$ Superfluid phase

Universality Classes

$\Delta \sim Q$ $\qquad$ $\Delta \sim Q^{\frac{d}{d-1}}$

$\longrightarrow Q\tau$

Free scalar phase $\qquad$ Superfluid phase

Examples:

1. The $O(2)$ model with $\tau = 4 - d = \epsilon$ using the ϵ expansion [Badel,Cuomo,Monin,Rattazzi; Watanabe; Arias-Tamargo,Rodriguez-Gomez,Russo].

2. The $O(N)$ model with $\tau = 1/N$ [Alvarez-Guame,Orlando,Reffert].

Is there a unitary, continuous example relating these two phases?

Conformal Manifold

Consider the 3d $\mathcal{N} = 2$ theory with three chiral superfields X, Y, Z and superpotential

$$W = g(XYZ + \tau(X^3 + Y^3 + Z^3))$$

The theory has a $U(1)_R$-symmetry:

$$X \to e^{2i\alpha} X, \quad Y \to e^{2i\alpha} Y, \quad Z \to e^{2i\alpha} Z$$

Can study the theory at large $U(1)_R$ charge.

Conformal Manifold

Consider the 3d $\mathcal{N} = 2$ theory with three chiral superfields X,Y,Z and superpotential

$$W = g(XYZ + \tau(X^3 + Y^3 + Z^3))$$

Important points:

- $\tau = 0$: XYZ model (free scalar phase)
- $\tau = \infty$: (3 copies of) the 3d $\mathcal{N} = 2$ Ising model (superfluid phase)

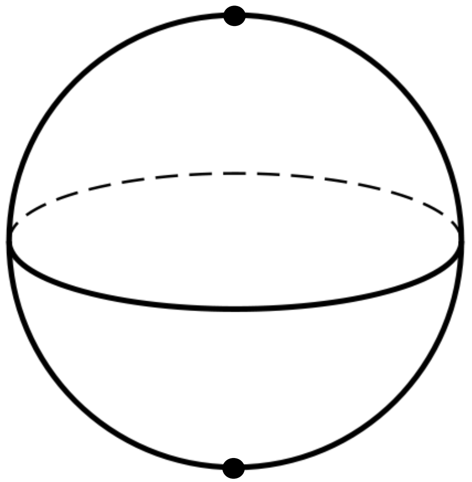
τ is **exactly marginal** [Leigh-Strassler], and so interpolates between the phases.

Conformal Manifold

$$W = g(XYZ + \tau(X^3 + Y^3 + Z^3))$$

Conformal manifold is compact and topologically a sphere.

$\tau = \infty$



$\tau = 0$

Conformal Manifold

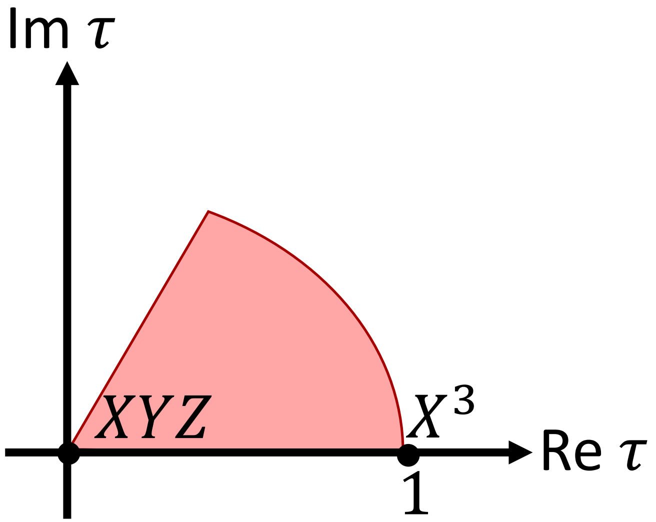
$$W = g(XYZ + \tau(X^3 + Y^3 + Z^3))$$

There exists an S_4 duality relating different values of τ [Baggio,Bobev,Chester,Lauria,Pufu].
Generators:

$$\tau \to \omega\tau, \quad \tau \to \bar{\tau}, \quad \tau \to \frac{\tau+2\omega^2}{\omega\tau-1}$$

Where $\omega = e^{\frac{2\pi i}{3}}$.

Can thus focus on a “fundamental domain” in τ-plane.



Conformal Manifold

$$W = g(XYZ + \tau(X^3 + Y^3 + Z^3))$$

Unfortunately, effective action at large charge is extremely complicated (many possible terms).

Examples of allowed terms in scale-invariant effective action:

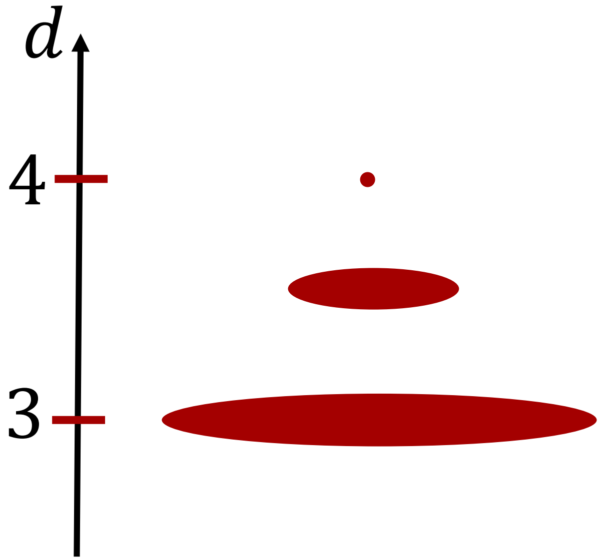
- $|X|^{\alpha_1}|Y|^{\alpha_2}|Z|^{\alpha_3}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 3/2$
- $|\alpha_1 X + \alpha_2 Y + \alpha_3 Z|^{3/2}$

Symmetries are not enough to make this problem tractable.

Conformal Manifold

$$W = g(XYZ + \tau(X^3 + Y^3 + Z^3))$$

Instead, study the conformal manifold in the $\mathrm{d} = 4 - \epsilon$ expansion.



Dimension at Large Charge

Compute the operator dimension at large charge, $\Delta(Q, \tau, \epsilon)$.

Following [Badel,Cuomo,Monin,Rattazzi], we compute

$$\Delta(Q, \tau, \epsilon) = \frac{1}{\epsilon}\Delta_0(\lambda_1, \lambda_2) + \Delta_1(\lambda_1, \lambda_2) + \cdots$$

where $\lambda_1 = \epsilon Q$, $\lambda_2 = \epsilon Q|\tau|^2$.

Consists of finding a semiclassical saddle at fixed background charge.

Dimension at Large Charge

$$\Delta(Q,\tau,\epsilon) = \frac{1}{\epsilon}\Delta_0(\lambda_1,\lambda_2) + \Delta_1(\lambda_1,\lambda_2) + \cdots$$

- Δ_0 (classical contribution): solve EOMs to find vevs

$$X = \cdots, Y = \cdots, Z = \cdots$$

and plug the solution back into the energy density on the cylinder.

Dimension at Large Charge

$$\Delta(Q,\tau,\epsilon) = \frac{1}{\epsilon}\Delta_0(\lambda_1,\lambda_2) + \Delta_1(\lambda_1,\lambda_2) + \cdots$$

- Δ_0 (classical contribution): solve EOMs to find vevs

$$X = \cdots, Y = \cdots, Z = \cdots$$

and plug the solution back into the energy density on the cylinder.

- Δ_1: look at fluctuations around saddle point. Consists of summing Casimir energies:

$$\Delta_1 = \frac{1}{2}\sum_{fields}\sum_{l\in modes} n_l\omega_l\,,$$

where ω_l is the dispersion relation at level l and n_l is the level l.

Large Charge Expansion

Start with classical contribution Δ_0.

Find four solutions to EOMs at fixed charge:

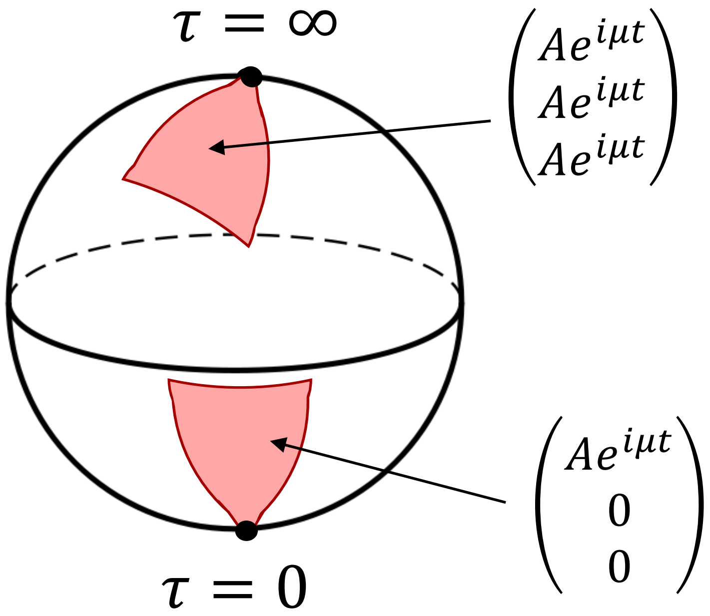
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} Ae^{i\mu t} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Ae^{i\mu t} \\ Ae^{i\mu t} \\ Ae^{i\mu t} \end{pmatrix}, \begin{pmatrix} Ae^{i\mu t} \\ Ae^{i\mu t} \\ Ae^{i\mu t + \frac{2\pi i}{3}} \end{pmatrix}, \begin{pmatrix} Ae^{i\mu t} \\ Ae^{i\mu t} \\ Ae^{i\mu t - \frac{2\pi i}{3}} \end{pmatrix}$$

For constants μ, A.

Which is the global minimum?

Interchange of Operator Dominance

It turns out that the global minimum changes as we vary τ.



Focus on fundamental domain.

Dimension at Large Charge

Compute $\Delta = \frac{1}{\epsilon}\Delta_0 + \Delta_1$. Long and boring computation….

Three interesting limits to consider:

1. For $Q \gg \frac{1}{\epsilon|\tau|^2}$, expect a superfluid phase
2. For $Q \ll \frac{1}{\epsilon|\tau|^2}$ and $Q \ll \frac{1}{\epsilon}$, expect free scalar phase
3. For $Q \ll \frac{1}{\epsilon|\tau|^2}$ and $Q \gg \frac{1}{\epsilon}$, expect BPS behavior

Dimension at Large Charge

Interesting limits:

1. For $Q \gg \frac{1}{\epsilon|\tau|^2}$, i.e. close to the X^3 fixed point, we find

$$\Delta \sim \frac{1}{\epsilon|\tau|^2}(\epsilon|\tau|^2 Q)^{\frac{4}{3}}\left(1+\frac{\epsilon}{9}\log(\epsilon|\tau|^2 Q)\right)$$

In this limit we expected

$$\Delta \sim Q^{\frac{d}{d-1}} = Q^{\frac{4-\epsilon}{3-\epsilon}} = Q^{\frac{4}{3}}(1+\frac{\epsilon}{9}\log Q+\cdots)$$

And so this exactly fits at leading order in ϵ.

Dimension at Large Charge

Interesting limits:

2. For $Q \ll \frac{1}{\epsilon|\tau|^2}$ and $Q \ll \frac{1}{\epsilon}$, i.e. close to the XYZ fixed point, we find

$$\Delta = \frac{d-1}{3}Q + \frac{\epsilon Q}{6(2+|\tau|^2)}\left(|\tau|^2 Q - 4 + 2\sqrt{4-5|\tau|^2+|\tau|^4} + |\tau|^2\right)$$

Note:

- Saturate the BPS bound at $\tau = 0$ (as expected for XYZ model), $\Delta_{BPS}(\mathrm{Q}) = \frac{d-1}{3}Q$.
- Also saturate bound at $\epsilon \to 0$.
- $Q > 3$ it is always above the BPS bound.

Dimension at Large Charge

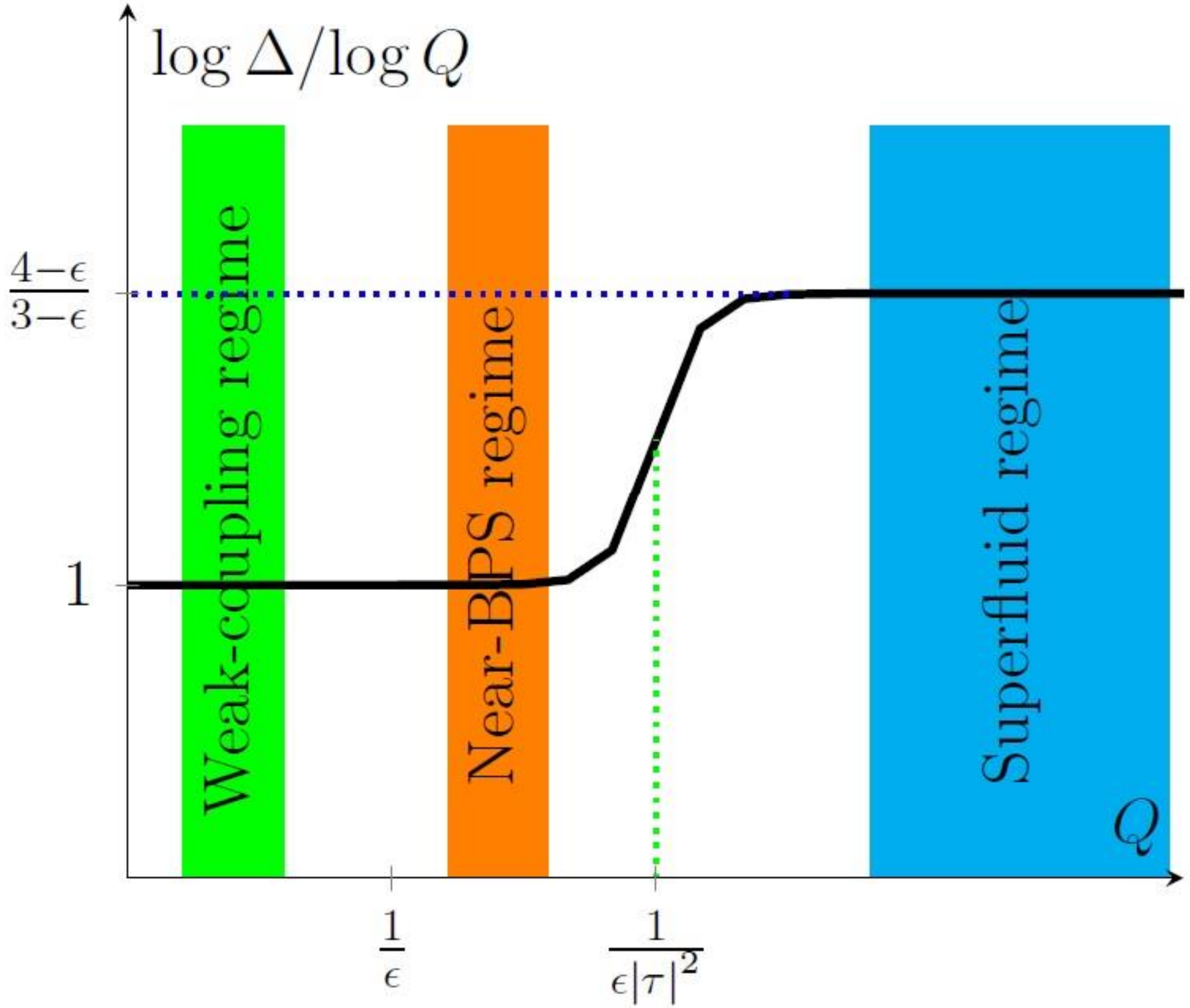
Interesting limits:

3. For $Q \ll \frac{1}{\epsilon|\tau|^2}$ and $Q \gg \frac{1}{\epsilon}$, still expect near-BPS behavior

$$\Delta = \frac{d-1}{3}Q + \frac{|\tau|^2 Q}{2}\epsilon Q\left(1 + 3\epsilon \log \epsilon Q\right)$$

Indeed see BPS behavior at $\tau \to 0$.

Dimension at Large Charge



Interchange of Operator Dominance

Which operator has lowest dimension?

- At $\tau = 0$ (XYZ model): X^Q (or Y^Q or Z^Q)
- At $\tau = \infty$ (X^3 model): $(XYZ)^Q$

So as a function of τ we find a transition.

Can we go to $Q \sim O(1)$?

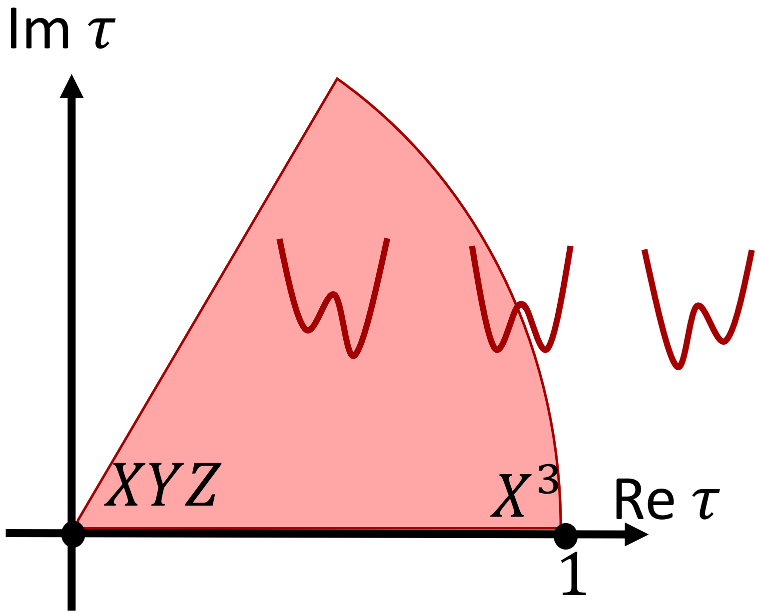
In $O(2)$ model and X^3 model, results obtained at large Q were extrapolated to $Q \sim O(1)$ and were in good agreement with numerical results. What about in our model?

Two issues with $Q \sim O(1)$:

1. At $Q \lesssim 2$ and small ϵ, the dimension is smaller than the BPS bound:
$$\Delta(\mathrm{Q}) - \Delta_{BPS}(\mathrm{Q}) \sim -\epsilon$$
2. Take $Q = 3$. There is a BPS operator with $Q = 3$. But we find
$$\Delta(\mathrm{Q} = 3) > \Delta_{BPS}(\mathrm{Q} = 3)$$

Can we go to $Q \sim O(1)$?

Possible subtlety: found four saddle points, which exchange dominance at the edges of the fundamental region:



When they are close to degenerate – expect nonperturbative corrections $\epsilon e^{-Q^{\#}}$.

Summary

1. Discussed large charge on a conformal manifold in the ϵ-expansion.

$$W = g(XYZ + \tau(X^3 + Y^3 + Z^3))$$

2. $\Delta(Q)$ transitions from free scalar phase to superfluid phase continuously as a function of τ.

3. Interesting phenomena at large charge:
   - Lowest-dimension operator depends on τ.
   - Multiple saddle points which exchange dominance as a function of τ.

Future Directions

1. Effective action directly in 3d?
2. Continuous transition from free fermion phase to another phase?
3. More examples; N=4 SYM? 3d CS-matter theories at large N?
4. More examples with instanton corrections? Explicit computations of these?

Thank You!