Thermal one-point functions and single-valued polylogarithms: a spurious remark or something deeper?

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Introduction and Motivation

CFTs in nontrivial geometries: generalities

There is a deep connection between the RG and the coupling of CFTs to nontrivial background geometries.

Conformal transformations of d-dimensional flat space (i.e. with metric $\eta_{\mu\nu}$) are coordinate reparametrizations $x^\mu \mapsto x'^\mu(x)$ that preserve the norm of vectors up to a scale factor.

$$x^\mu \mapsto x'^\mu : ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \mapsto ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu \equiv \Omega^2(x) ds^2$$

$$\Rightarrow \eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = \Omega^2(x) \eta_{\rho\sigma}$$

- Hence, conformal transformations are equivalent to a local rescaling of the flat metric $\eta_{\mu\nu} \mapsto \Omega^2(x)\eta_{\mu\nu}$ i.e.
  – *either* we work with the coordinates x'^μ and metric $\eta_{\mu\nu}$,
  – *or* we use the original coordinates x^μ and the metric $\Omega^2(x)\eta_{\mu\nu}$.
- In $d = 2$ $x'^\mu(x)$ are general analytic functions of x^μ.
- In $d > 2$ $x'^\mu(x)$ are at most quadratic in x^μ.

CFTs in nontrivial geometries: generalities

CFTs are those QFTs that have a number (sometime finite in $d = 2$, certainly infinite in $d > 2$) of <u>*quasiprimary*</u> *local operators $\mathcal{O}(x)$ that under conformal transformations behave as*

$$x^\mu \mapsto x'^\mu : \mathcal{O}(x)\Big|_\eta \mapsto \mathcal{O}'(x')\Big|_\eta = \Omega^\Delta(x)\mathcal{O}(x')\Big|_\eta \equiv \Omega^\Delta(x)\mathcal{O}(x)\Big|_{\Omega^2\eta}$$

where Δ is the scaling dimension. (I considered scalars for simplicity.)

- The conformal Ward identities express the invariance of correlation functions under conformal transformations

$$x^\mu \mapsto x'^\mu : \langle\mathcal{O}(x)...\rangle\Big|_\eta \mapsto \langle\mathcal{O}'(x')...\rangle\Big|_\eta = \Omega^\Delta(x)\langle\mathcal{O}(x')...\rangle\Big|_\eta$$
$$\equiv \Omega^\Delta(x)\langle\mathcal{O}(x)...\rangle\Big|_{\Omega^2\eta} \equiv \langle\mathcal{O}(x)...\rangle\Big|_\eta$$

- The simplest example: the 2pt function of a scalar field $\phi(x)$ with dimension Δ. Under the scale transformation $x^\mu \to x'^\mu = \lambda x^\mu$ for which $\Omega = \lambda$ we have

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{x_{12}^{2\Delta}} \to \langle\phi'(x_1')\phi'(x_2')\rangle = \lambda^{2\Delta}\frac{1}{(x_{12}')^{2\Delta}} \equiv \langle\phi(x_1)\phi(x_2)\rangle$$

CFTs in nontrivial geometries: generalities

CFT correlators in the nontrivial - but conformally flat - geometry $\Omega^2\eta_{\mu\nu}$ are fully determined by the flat space correlators.

- Consider the 2-pt function of scalars in flat space. Going to spherical coordinates $\vec{x} \to (r, \Omega_{d-1})$ with $ds^2 = dr^2 + r^2 d\Omega^2_{d-1}$ one has

$$\langle \mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\rangle = \frac{1}{\vec{x}_{12}^{2\Delta}} \to$$

$$\to \langle \mathcal{O}(r_1,\theta)\mathcal{O}(r_2,0)\rangle = \frac{1}{(\vec{r}_1 - \vec{r}_2)^{2\Delta}} = \frac{1}{(r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta)^\Delta}$$

- If there was a conformal transformation of flat space with metric $\eta_{\mu\nu}$ to the metric $\Omega^2(x)\eta_{\mu\nu}$, the Ward identity would give

$$[\Omega(r_1,\theta)\Omega(r_2,0)]^{-\Delta}\langle \mathcal{O}(r_1,\theta)\mathcal{O}(r_2,0)\rangle\Big|_\eta = \langle \mathcal{O}(r_1,\theta)\mathcal{O}(r_2,0)\rangle\Big|_{\Omega^2\eta}$$

CFTs in nontrivial geometries: example in $d = 2$

- In $d = 2$ all analytic transformations are conformal transformations since for $ds^2 = dx^2 + dy^2 = dzd\bar{z}\,,\;\; z = x + iy\,,\;\; \bar{z} = x - iy$ and under the general analytic transformations $z \mapsto z' = f(z)$, $\bar{z} \mapsto \bar{z}' = \bar{f}(\bar{z})$ we have

$$ds^2 \mapsto ds'^2 = \partial_z f(z) \partial_{\bar{z}} \bar{f}(\bar{z}) dz d\bar{z} \equiv \Omega^2(z, \bar{z}) ds^2$$

- Choosing $f(z) = L \ln(z/L)$ and $\bar{f}(\bar{z}) = L \ln(\bar{z}/L)$ we find

$$ds^2 \mapsto ds'^2 = \frac{L^2}{z\bar{z}} dz d\bar{z} = \frac{L^2}{x^2 + y^2}[dx^2 + dy^2] \equiv \frac{L^2}{r^2}[dr^2 + r^2 d\theta^2]$$

- Now, we observe the the metric shown in the last equality is actually the reparametrization of the metric on $\mathbb{R} \times S^1$ i.e. under

$$r = L e^{\frac{\rho}{L}} \;\;\Rightarrow\;\; ds^2 \equiv \frac{L^2}{r^2}[dr^2 + r^2 d\theta^2] = d\rho^2 + L^2 d\theta^2$$

CFTs in nontrivial geometries: example in $d = 2$

- So we have

$$\langle \mathcal{O}(r_1,\theta)\mathcal{O}(r_2,0)\rangle\Big|_{\Omega^2\eta} \equiv \langle \mathcal{O}(\rho_1,\theta)\mathcal{O}(\rho_2,0)\rangle\Big|_{\mathbb{R}\times S^1} = \frac{1}{L^{2\Delta}}\frac{1}{\left(2\cosh\frac{\rho_1-\rho_2}{L} - 2\cos\theta\right)^{\Delta}}$$

- The metric reparametrization above generalises for all $d > 2$ as

$$r = Le^{\frac{\rho}{L}} \;\Rightarrow\; ds^2 \equiv \frac{L^2}{r^2}[dr^2 + r^2 d\Omega_{d-1}^2] = d\rho^2 + L^2 d\Omega_{d-1}^2$$

However, there is no *conformal* transformation of flat d-dimensional space with $\Omega^2(r,\Omega_{d-1}) = L^2/r^2$, so *CFT correlation functions on* $\mathbb{R}\times S^{d-1}$ *cannot be determined by those on* $\mathbb{R}^d$.

CFTs in nontrivial geometries: Weyl invariance

- One possibility is to consider *Weyl invariant QFTs*. These are QFTs that have operators $\mathcal{O}(x)$ with a definite behaviour under general *local Weyl rescalings* of the flat metric

$$\eta_{\mu\nu} \mapsto \Omega^2(x)\eta_{\mu\nu} \ \Rightarrow \ \mathcal{O}(x) \mapsto \mathcal{O}'(x) = \Omega^{\Delta}(x)\mathcal{O}(x)$$

- The above behaviour is *independent of the spin of the operator* $\mathcal{O}(x)$. This leads to the corresponding Ward identities expressing *Weyl invariance of correlation functions* as

$$\Omega^{\Delta}(x)\langle\mathcal{O}(x)...\rangle\Big|_{\Omega^2\eta} \equiv \langle\mathcal{O}(x)...\rangle\Big|_{\eta}$$

- So, for *Weyl invariant theories*

$$[\Omega(r_1,\theta)\Omega(r_2,0)]^{-\Delta}\langle\mathcal{O}(r_1,\theta)\mathcal{O}(r_2,0)\rangle\Big|_{\eta} = \langle\mathcal{O}(r_1,\theta)\mathcal{O}(r_2,0)\rangle\Big|_{\Omega^2\eta} =$$

$$\equiv \langle\mathcal{O}(\rho_1,\theta)\mathcal{O}(\rho_2,0)\rangle\Big|_{\mathbb{R}\times S^{d-1}} = \frac{1}{L^{2\Delta}}\frac{1}{\left(2\cosh\frac{\rho_1-\rho_2}{L} - 2\cos\theta\right)^{\Delta}}$$

CFTs in nontrivial geometries: Weyl invariance

- Weyl invariance connects background effective actions with the RG. Consider the renormalised effective action $W_r[g_{\mu\nu}, J]$ of a theory coupled to $g_{\mu\nu}(x)$ and a scalar source $J(x)$ with Weyl-weight (i.e. scaling dimension) $d - \Delta$. For constant Weyl rescalings we have

$$\delta g^{\mu\nu}(x) = 2g^{\mu\nu}(x)\,,\;\; \delta J(x) = (d-\Delta)J(x)$$

  hence we have the following local form if the RG flow equation

$$\mu\frac{d}{d\mu}W_r = \left(\mu\frac{\partial}{\partial\mu} + \int d^d x\sqrt{g}\left(2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} + (d-\Delta)J(x)\frac{\delta}{\delta J(x)}\right)\right)W_r = 0$$

  μ is the RG mass scale, absent in CFTs except in the presence of anomalies.

- Conformal invariance in flat space implies Weyl invariance in a general background (??) [e.g. M. Luty et. al. (2017)].

CFTs in nontrivial geometries: recap

In two-dimensional CFTs correlators in the *thermal* geometry $\mathbb{R} \times S^1_L$ are fully determined by those on $\mathbb{R}^2$.

In $d > 2$ it appears that only for *Weyl invariant* theories correlators in $\mathbb{R} \times S^{d-1}_L$ are fully determined by those on $\mathbb{R}^d$.

However, the thermal geometries $S^1_L \times \mathbb{R}^{d-1}$, although conformally flat, are not related *neither* by a conformal transformation *nor* by a Weyl rescaling to $\mathbb{R}^d$ for $d > 2$.

We generally need additional data to describe CFTs in thermal geometries for $d > 2$. What are these additional data, and how are they related to the RG properties of QFTs?

The conformal OPE in nontrivial geometries

The conformal OPE in nontrivial geometries: generalities

- The conformal OPE is the statement that quasiprimary operators form a complete basis for operator products in a CFT i.e. for scalars

$$\phi(x_1)\phi(x_2) = \frac{1}{x_{12}^{2\Delta}}\mathbb{1} + \sum_{\mathcal{O}_s} \frac{1}{x_{12}^{2(\Delta - \frac{\Delta_s}{2} + \frac{s}{2})}}[x_{12} \cdot \mathcal{O}_s(x_2)]$$

  where $[x_{12} \cdot \mathcal{O}_s(x_2)]$ denotes the spin-s, dimension-Δ_s contribution with all its descendants.

- Using the OPE for correlators in nontrivial geometries we could evaluate them <u>if we knew</u> the 1-pt functions $\langle \mathcal{O}_s(x) \rangle$ for the relevant quasiprimary operators.
- Nevertheless, for 1-pt functions we generically have

$$\langle \mathcal{O}(x) \rangle_{\Omega^2 \eta_{\mu\nu}} = [\Omega(x)]^{-\Delta} \langle \mathcal{O}(x) \rangle_{\eta_{\mu\nu}} = 0 \ \text{ for } \ \mathcal{O}(x) \neq \mathbb{1}$$

  and we need to be careful.

The conformal OPE in nontrivial geometries: $d = 2$ example

- Since since *all* 1-pt functions vanish on $\mathbb{R}^d$, namely $\langle \mathcal{O}(x) \rangle \Big|_\eta = 0$ and in $d = 2$ the plane is conformally related to the *thermal* geometry it would appear that *all* 1-pt functions of quasiprimary operators vanish.
- Nevertheless, in $d = 2$ there exist operators transforming anomalously i.e. the energy momentum tensor

$$T(z) \to T'(z') = [f'(z)]^2 T(z') + \frac{c}{12}\{f(z), z\}\mathbb{1}\,, \quad \{f, z\} = \frac{f''' f' - \frac{3}{2} f''^2}{f'^2}$$

- Using the above one obtains

$$\langle T(z) \rangle_{\mathbb{R}\times S^1_\beta} = -\frac{c}{24}\frac{1}{L^2}$$

- We conclude that thermal correlation functions in $d = 2$ do receive contributions from nontrivial 1pt functions of *non-quasiprimary* operators i.e. from conformal anomalies. Setting $u = \rho \cos\phi$ and $L\theta = \rho \sin\phi$ the thermal 2-pt function becomes [J. Cardy (1986)]

$$\langle \phi(\rho, \phi)\phi(0, 0) \rangle = \frac{1}{\rho^{2\Delta_\phi}}\left[1 - \frac{\Delta_\phi}{12}\frac{\rho^2}{L^2}\cos 2\phi + \cdots\right]$$

The conformal OPE in nontrivial geometries: $S^1_\beta \times \mathbb{R}^{d-1}$

- In the $S^1_\beta \times \mathbb{R}^{d-1}$ geometry the 1-pt functions of scalar quasiprimaries can only depend on a single parameter as

$$\langle \mathcal{O}(x) \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} = \langle \mathcal{O}(0) \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}$$

- For $SO(d)$ irreducible tensors we have

$$\langle T_{\mu\nu\ldots}(0) \rangle_{S^1_\beta \times \mathbb{R}^{d-1}} = \frac{b_T}{\beta^{\Delta_T}} (\hat{e}_\mu \hat{e}_\nu \ldots - \text{traces})$$

where $x^\mu = (\tau, \mathrm{x})$ are coordinates on $S^1_\beta \times \mathbb{R}^{d-1}$ with period $\tau \sim \tau + \beta$, $r = |\mathrm{x}|$ and $\theta \in [0, \pi]$ is a polar angle when $\mathbb{R}^{d-1}$ is written in spherical coordinates. $\hat{e}_\mu$ are unit vectors in the τ-direction.

- Then, the thermal two-point function takes the generic form

$$\langle \phi(x) \phi(0) \rangle_\beta \equiv g(r, \cos\theta) = \sum_{\mathcal{O}_s} a_{\mathcal{O}_s} \left(\frac{r}{\beta} \right)^{\Delta_{\mathcal{O}_s}} \frac{C_s^\nu(\cos\theta)}{r^{2\Delta_\phi}}$$

The conformal OPE in nontrivial geometries: $S^1_\beta \times \mathbb{R}^{d-1}$

- $C_s^\nu(\cos\theta)$ are Gegenbauer polynomials with $\nu = d/2 - 1$.
- The coefficients $a_{\mathcal{O}_s}$ are given by

$$a_{\mathcal{O}_s} = \frac{s!}{2^s(\nu)_s}\frac{g_{\phi\phi\mathcal{O}_s} b_{\mathcal{O}_s}}{C_{\mathcal{O}_s}}$$

  with $C_{\mathcal{O}_s}$ and $g_{\phi\phi\mathcal{O}_s}$ the corresponding 2-pt and 3-pt function coefficients, and $(a)_n$ the Pochhammer symbol.
- The unit operator $\mathbb{1}$ is the unique operator with dimension zero, and here

$$a_{\mathbb{1}} = \frac{2^{2\Delta_\phi - d}\Gamma(\Delta_\phi)}{\pi^{\frac{d}{2}}\Gamma(\frac{d}{2} - \Delta_\phi)}$$

  so that the momentum-space two-point function is unit-normalized.

The conformal OPE in nontrivial geometries: $d = 2$ example

- The thermal 2-pt function gives (there is an extra factor of 2 in the normalization of the $d = 2$ Gegenbauers)

$$a_T = \frac{g_T b_T}{C_T} = -\frac{\Delta_\phi}{12}$$

- Using the Ward identity we then find

$$g_T = \frac{d\Delta_\phi}{d-1} \Rightarrow b_T = -\frac{C_T}{24}$$

confirming that the e.m. tensor coefficient C_T coincides with the conformal anomaly c in $d = 2$.

- There is yet another coefficient $\tilde{c}$ related to thermal 1-pt functions

$$\langle T_{\tau\tau} \rangle_{\mathbb{R}\times S^{d-1}_\beta} = -(d-1)[f_\beta - f_\infty] = \frac{b_T}{\beta^d} = -2(d-1)\frac{\zeta(d)}{\beta^d}\tilde{c}$$

where f_β is the free energy density. For $d = 2$ we see that $\tilde{c} \sim c$.

The conformal OPE in nontrivial geometries: recap

The conformal OPE can be used to study correlators in nontrivial geometries.

For general $d > 2$ we need additional data, in the form of 1-pt functions, to determine the correlators.

Nevertheless, in $d = 2$ it appears as if the thermal correlator is fully determined by the plane result. This is a consequence of the fact that the only nonzero 1-pt functions are those of anomalously transforming operators, and they depend on the central charge. The latter cancels in the OPE.

The OPE inversion formula

The OPE inversion formula: the Euclidean formula

- Further information regarding the thermal 2-pt function can be obtained using an *OPE inversion formula*. **[L. Iliesiu et. al. 1802.10266 (JHEP)]**
- Complexifying Δ one defines the spectral function $a(\Delta, s)$ via

$$g(r, \cos\theta) = \sum_s \oint_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{d\Delta}{2\pi i} a(\Delta, s) \frac{C_s^\nu(\cos\theta)}{r^{2\Delta_\phi-\Delta}}$$

  whose poles at $\Delta = \Delta_{\mathcal{O}_s}$ with residues $-a_{\mathcal{O}_s}$ yield the physical spectrum.
- Assuming that the physical poles lie on the right of the imaginary axis one can close the contour clockwise for $r < 1$ (we set $\beta = 1$ from now on) if $a(\Delta, s)$ does not grow exponentially at infinity.

The OPE inversion formula: the Euclidean formula

- We can then use the orthogonality of Gegenbauer polynomials to project on a spin-s state and then integrate with a suitable power in the region of convergence $r \in [0, 1]$ to obtain $a(\Delta, s)$ as

$$a(\Delta, s) = \frac{1}{N_{s,\nu}} \int_0^1 \frac{dr}{r^{\Delta - 2\Delta_\phi + 1}} \int_{-1}^1 dx \, (1 - x^2)^{\nu - \frac{1}{2}} \, C_s^\nu(x) g(r, x)$$

where

$$x = \cos\theta \,, \quad N_{s,\nu} = \frac{2^{1-2\nu} \pi \Gamma(s + 2\nu)}{(s + \nu)\Gamma(s + 1)\Gamma^2(\nu)}$$

- This is termed **Euclidean inversion formula.**

The OPE inversion formula: the Lorenzian formula

- Writing $x = \cos\theta = (w + 1/w)/2$ with $w = e^{i\theta}$ one can transform the Euclidean formula into a contour integral over the unit circle in the complex-w plane.
- To exploit further the analytic structure of the 2-pt function $g(r, \cos\theta)$ one would like to allow w to explore the full complex plane. This can be done by a suitable complexification of the Euclidean variables r, θ, defining $z = rw$ and $\bar{z} = r/w$ which are now independent real variables.
- As a function of w, i.e. in the w-plane, $g(r, w)$ has the cuts $(-\infty, -1/r)$, $(-r, 0)$, $(0, r)$ and $(1/r, \infty)$. One also has to assume that it does not grow faster than w^{s_0} (resp. $1/w^{s_0}$) for large (resp. small) w for some constant $s_0 > 0$.

The OPE inversion formula: the Lorenzian formula

- Moreover, one needs to use the analytic extension of the Gegenbauer polynomials to the whole complex plane as **[M. Costa et. al. 1209.4355 (JHEP)]**

$$C_s^{\nu}(w) = \frac{\Gamma(s+2\nu)}{\Gamma(\nu)\Gamma(s+\nu+1)}(F_s(1/w)e^{i\nu\pi} + F_s(w)e^{-i\nu\pi})$$

where

$$F_s(w) = w^{s+2\nu}{}_2F_1(s+2\nu,\nu;s+\nu+1;w^2)$$

The OPE inversion formula: the Lorenzian formula

- Then, the integral giving $a(\Delta, s)$ will receive contributions from the discontinuities across the **cuts** of $g(r, w)$ as well as from the **arcs** at infinity. The final result is

$$a(\Delta, s) = a_{\text{Disc}}(\Delta, s) + \theta(s_0 - s)\, a_{\text{arcs}}(\Delta, s)$$

where

$$a_{\text{Disc}}(\Delta, s) = K_s \int_0^1 \frac{d\bar{z}}{\bar{z}} \int_1^\infty \frac{dz}{z} \Big[(z\bar{z})^{\Delta_\phi - \frac{\Delta}{2} - \nu} \times (z - \bar{z})^{2\nu} F_s\left(\sqrt{\frac{\bar{z}}{z}}\right) \text{Disc}[g(z, \bar{z})] \Big]$$

with

$$K_s = (1 + (-1)^s) \frac{\Gamma(s+1)\Gamma(\nu)}{4\pi\Gamma(s+\nu)}$$

- The discontinuity relevant for the evaluation of the above integral is the one across the cut $(1/r, \infty)$, as all others are related to it.

The OPE inversion formula: explicit example

Inversion formulae (i.e. spectral decompositions) are most useful when there is an independent evaluation of the correlation functions.

- For bosons (scalars) the simplest ansatz is to consider the momentum-space thermal 2-pt function

$$G^{(d)}(\omega_n, \mathrm{p}) = \frac{1}{\omega_n^2 + \mathrm{p}^2 + m_{\mathrm{th}}^2}\,,\, \omega_n = 2\pi n\,, n = 0, \pm 1, \pm 2, \ldots,$$

- This is motivated by known work on thermal field theory which shows that fields develop generically a thermal mass m_{th} at finite temperature.
- We are actually asking whether the simple ansatz above can define a thermal CFT. We make no reference to a Lagrangian, although it is known that the 2-pt function can be obtained, for example, in the large-N limit of the $O(N)$ model [T. P. et. al. hep-th/9803149 (PLB)].

The OPE inversion formula: explicit example

- In arbitrary-d the above 2-pt function can be Fourier-transformed to

$$G^{(d)}(\tau, \mathrm{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n=-\infty}^{\infty} \left(\frac{m_{\mathrm{th}}}{|X_n|} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(m_{\mathrm{th}}|X_n|)\,,\ X_n = (\tau - n, \mathrm{x})$$

where $K_\alpha(x)$ is the modified Bessel function of the second kind.

- Defining $z = \tau + i|\mathrm{x}|$ we find $|X_n| = \sqrt{(n-z)(n-\bar{z})}$.
- We focus on **odd** $d = 2k+1$, $k = 1, 2, \ldots$, and in that case we may write

$$G^{(2k+1)}(\tau, \mathrm{x}) = \frac{1}{2^{k+1}\pi^k} \sum_{n=-\inf ty}^{\infty} \frac{m_{\mathrm{th}}^{k-1}}{|X_n|^k} e^{-m_{\mathrm{th}}|X_n|} \sum_{p=0}^{k-1} \frac{L_{k,p}}{(m_{\mathrm{th}}|X_n|)^p}$$

with

$$L_{k,p} = \frac{(k-1+p)!}{2^p p!(k-1-p)!}$$

The OPE inversion formula: explicit example

- The latter coefficients also appear in the Bessel polynomials

$$y_n(x) = \sum_{p=0}^{n} L_{n+1,p}\, x^p = \sqrt{\frac{2}{\pi x}}\, e^{1/x} K_{n+\frac{1}{2}}(1/x)$$

- The relevant discontinuity $\mathrm{Disc}(G^{(d)})$ follows simply from understanding the discontinuity of the function

$$f^{(k)}(x) = \frac{a^{k-1}}{(\sqrt{x})^k}\, e^{-a\sqrt{x}} \sum_{p=0}^{k-1} \frac{L_{k,p}}{(a\sqrt{x})^p}$$

across the cut due to the square-root branch point at $x = 0$.

- Assuming that the cut goes from $x = 0$ to $x = \infty$ we find that

$$\mathrm{Disc}(f^{(k)}(x)) = \frac{2}{x^{k-1}}\Big(\frac{1}{\sqrt{-x}} U_k(x) \cos(a\sqrt{-x}) + V_k(x) \sin(a\sqrt{-x})\Big)$$

$$U_k(x) = \tfrac{1}{2}\big(\theta_{k-1}(\sqrt{x}) + \theta_{k-1}(-\sqrt{x})\big)$$

$$V_k(x) = \frac{1}{2\sqrt{x}}\big(\theta_{k-1}(\sqrt{x}) - \theta_{k-1}(-\sqrt{x})\big)$$

with $\theta_n(x) = x^n y_n(1/x)$ the so-called reverse Bessel polynomials.

The OPE inversion formula: explicit example

- We can now calculate the spectral function $a(\Delta, s)$. For the discontinuity part we find

$$a^{(k)}_{\mathrm{Disc},0}(\Delta, s) = (1+(-1)^s)\frac{1}{2^{2s+k}s!}\frac{\Gamma(k-\frac{1}{2})}{\Gamma(k+s-\frac{1}{2})} \times \sum_{n=0}^{k-1+s}\frac{2^{n+1}}{n!}\frac{(2(k-1+s)-n)!}{(k-1+s-n)!}m_{\mathrm{th}}^n\,\mathrm{Li}_{2k-1+s-n}(e^{-m_{\mathrm{th}}})$$

  where $\mathrm{Li}_\alpha(z) = \sum_{n=1}^{\infty} z^n/n^\alpha$ is the polylogarithm.
- The result above follows just from the leading term in a $\bar{z}$-expansion of the inversion formula. It gives the contributions of higher-spin conserved currents with $\Delta = d-2+s$.
- Subleading terms in the $\bar{z}$-expansion, denoted as $a^{(k)}_{\mathrm{Disc},1}, a^{(k)}_{\mathrm{Disc},2}, \ldots$, would give the contributions of higher-twist operators.

Gap equations from the inversion formula: bosonic case

- The arc part $a^{(d)}_{\text{arcs}}(\Delta, s)$ is nonzero only for $s = 0$. We find

$$a^{(d)}_{\text{arcs}}(\Delta, 0) = \frac{1}{2^{\Delta - \frac{d-5}{2}}\sqrt{\pi}} m_{\text{th}}^{\Delta} \Gamma\Big(-\frac{\Delta}{2}\Big)\Gamma\Big(-\frac{\Delta - d + 2}{2}\Big)$$

- Notice that for $m_{\text{th}} = 0$ only the $\Delta = 0$ term survives giving the contribution of the identity operator. This, along with the corresponding $m_{\text{th}} = 0$ contributions from $a^{(k)}_{\text{Disc}}(\Delta, s)$, yield the **spectrum of generalized free CFTs**.
- When $m_{\text{th}} \neq 0$ and for $\Delta > 0$ the above yields contributions of an infinite tower of scalar operators with $\Delta = 2m$, $m = 1, 2, \ldots$, as well as contributions with $\Delta = d - 2 + 2l$, $l = 0, 1, 2, \ldots$.
- The former correspond to operators of the form σ^m, $m = 1, 2, \ldots$, where σ is the **shadow of** ϕ^2.

The conformal OPE in nontrivial geometries: recap

We have seen that thermal one-point functions represent additional data needed to describe CFTs in nontrivial (i.e. thermal) geometries.

If one knew the thermal 2pt function one could read the thermal one-point functions of all operators in the spectrum using an conformal inversion formula of the OPE. Alternatively, knowing the thermal one-point functions one might reconstruct the thermal 2pt functions.

An important remaining question is: what are the physical parameters that determine the thermal one-point functions and how are they related to the RG propeprties of the CFT??

Thermal one-point functions from the effective action (thermal free energy)

Thermal one-point functions from the free energy

A few words about systems at imaginary chemical potential

- The canonical partition function of a system at finite temperature $T = 1/\beta$ with a global $U(1)$ charge operator $\hat{Q}$ having integer eigenvalues is

$$Z_c(\beta, Q) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta Q}\, \mathrm{Tr}\left[e^{-\beta \hat{H} - i\theta \hat{Q}}\right] = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta Q}\, Z_{gc}(\beta, \mu = -i\frac{\theta}{\beta})\,,$$

  where $Z_{gc}(\beta, \mu)$ is the grand canonical partition function with imaginary chemical potential μ.
- The latter function exhibits certain periodicity properties wrt θ. For example in $SU(N)$ gauge theories with the fermion number operator $\hat{Q}$ is a multiple of N in the confining phase and $Z_{gc}(\beta, \mu)$ will be periodic with a θ-period $2\pi/N$. However, in a high temperature deconfining phase one should find a 2π θ-period i.e. the breaking of $\mathbb{Z}_N$ symmetry signals a deconfining transition **[A. Roberge & N Weiss (1986)]**

Thermal one-point functions from the free energy

- Consider the Euclidean action action for a massive **complex** scalar field $\phi(x)$ in odd d-dimensions in the presence of an imaginary chemical potential (or equivalently in the presence of the temporal component of real gauge potential)

$$\mathcal{S}_E(\beta; m, \mu) = \int_0^\beta d\tau \int d^{d-1}\vec{x}\ |(\partial_\tau - i\mu)\phi|^2 + |\vec{\partial}\phi|^2 + m^2|\phi|^2 \,.$$

- To evaluate the grand canonical partition function and the corresponding free energy (grand canonical potential) of the theory one may cure the short distance singularity and other calibration issues by subtracting the zero temperature mass and chemical potential results to obtain

$$Z_{gc}(\beta; m, \mu) \equiv \frac{1}{Z(0; 0, 0)} \int (\mathcal{D}\bar{\phi})(\mathcal{D}\phi) e^{-S_E} = e^{-\beta F_{gc}(\beta; m, \mu)} \,.$$

Thermal one-point functions from the free energy

- By a simple scaling argument the grand canonical free energy is usually written as

$$F_{gc}(\beta; m, \mu) = \frac{V_{d-1}}{\beta^d}\mathcal{C}_d(\beta m, \beta\mu)\,,$$

  with V_{d-1} the spatial volume.
- The dimensionless function $\mathcal{C}_d$ has been extensively studied over the years e.g. [A. Castro-Neto et. al. (1992), T. Appelquist et. al. (1999), A. Leclair (2005)] as a measure of the degrees of freedom along the RG, although it does not appear to satisfy the requirements of a c-theorem [S. Giombi et. al. (2011)].
- Here I will point out some properties of $\mathcal{C}_d$ as a function of the parameters m, μ and notably also d. These properties are nicely presented using the set of complex variables

$$z = e^{-\beta m - i\beta\mu}\,,\ \ \bar{z} = e^{-\beta m + i\beta\mu} \Rightarrow \mathcal{C}_d(\beta m, \beta\mu) \equiv \mathcal{C}_d(z, \bar{z})\,.$$

Thermal one-point functions from the free energy

- The calculation of F_{gc} can be done along the lines described in the appendix of [T.P. et. al. (2018)] and yields

$$\mathcal{C}_d(z,\bar{z}) = -K_d \ln^d |z| - \frac{S_{d-1}}{(2\pi)^{d-1}} \left[i_d(z,\bar{z}) + \bar{i}_d(z,\bar{z}) \right] ,$$

where

$$K_d = \frac{\pi S_d}{d(2\pi)^d} \frac{1}{\sin(\pi d/2)} , \; S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} .$$

- For $d \geq 3$ the integral $i_d(z,\bar{z})$ is given by

$$i_d(z,\bar{z}) = \int_0^z \frac{dw}{w} \left(\ln w - \frac{1}{2} \ln \frac{z}{\bar{z}} \right) \left[\left(\ln w - \frac{1}{2} \ln \frac{z}{\bar{z}} \right)^2 - \ln^2 |z| \right]^{\frac{d-3}{2}} \ln(1-w)$$

and $\bar{i}_d(z,\bar{z})$ is obtained from above by exchanging $z \leftrightarrow \bar{z}$. For $d=1$ we have $i_1(z) = -\ln(1-z)$ and $\bar{i}_1(\bar{z}) = -\ln(1-\bar{z})$.

Thermal one-point functions from the free energy

- For odd d the calculation of $\mathcal{C}_d$ can be reduced to a finite series of iterating integrals and then to a finite double series using results such as

$$\int_0^z \frac{dw}{w}\left(\ln w - \frac{1}{2}\ln\frac{z}{\bar{z}}\right)^k \ln(1-w) = \sum_{\ell=0}^{k} \frac{(-1)^{\ell+1}k!}{\ell!}\ln^{\ell}|z| Li_{k+2-\ell}(z)\,,$$

where $Li_n(z)$ are the usual polylogarithms.

- The result can be written as

$$i_d(z,\bar{z}) + \bar{i}_d(z,\bar{z}) = -\frac{\Gamma\left(\frac{d+1}{2}\right)}{d-1} I_{d+2}(z,\bar{z})\,,$$

with

$$I_d(z,\bar{z}) = \sum_{n=0}^{\frac{d-3}{2}} \frac{(-1)^n(d-3-n)!}{\left(\frac{d-3}{2}-n\right)!}\frac{2^n\ln^n|z|}{n!}\left[Li_{d-2-n}(z) + Li_{d-2-n}(\bar{z})\right].$$

Thermal one-point functions from the free energy

- In the simple free field theory studied here m^2 and μ parametrize the deformations of the free Hamiltonian by the operators $\mathcal{O} = |\phi|^2$ and $\mathcal{Q} = i\bar{\phi}\overleftrightarrow{\partial_\tau}\phi$, the latter being the charge density defined from the $U(1)$ current $J_\mu = i\bar{\phi}\overleftrightarrow{\partial_\mu}\phi$.
- Evidently, the normalized thermal averages (integrated thermal one-point functions) of the above deformations are obtained as moments of the free energy. Assuming they are uniform i.e. $\int\langle O(x)\rangle = \beta V_{d-1}\langle O\rangle$ one obtains

$$\langle\mathcal{O}\rangle_d = \frac{1}{\beta^{d-2}}\hat{\mathrm{D}}\mathcal{C}_d(z,\bar{z})\,,\;\; \hat{\mathrm{D}} = \frac{1}{2\ln|z|}(z\partial_z + \bar{z}\partial_{\bar{z}})$$

$$\langle\mathcal{Q}\rangle_d = \frac{1}{\beta^{d-1}}\hat{\mathrm{L}}\mathcal{C}_d(z,\bar{z})\,,\;\; \hat{\mathrm{L}} = (z\partial_z - \bar{z}\partial_{\bar{z}})$$

- Namely, $\langle\mathcal{O}\rangle_d$ and $\langle\mathcal{Q}\rangle_d$ are the responses of $\mathcal{C}_d$ along the radial and angular directions in the two-dimensional space of massive and $U(1)$ deformations of the free CFTs.

Thermal one-point functions from the free energy

- To further unveil the physical content of the above it is instructive to consider the case $d = 1$. This corresponds to a system of two noninteracting harmonic oscillators with frequency $\omega \equiv m$. The twisted partition function of the oscillators is given by

$$\mathcal{Z}_1 = Tr_{\mathcal{H}_{1,2}} e^{-\beta \hat{H} + i\beta\mu\hat{\mathcal{Q}}} .$$

- The Hamiltonian and the twist operator are respectively

$$\hat{H} = \sum_{i=1}^{2} \frac{\hat{p}_i^2}{2} + \frac{m^2 x_i^2}{2} , \;\; \hat{\mathcal{Q}}_i = \hat{a}_i^\dagger \hat{a}_i , \;\; \hat{\mathcal{Q}} = \hat{\mathcal{Q}}_1 - \hat{\mathcal{Q}}_2 ,$$

The trace in (1) is taken over the tensor product Hilbert space $\mathcal{H}_{1,2} \approx \{|n_1\rangle \otimes |n_2\rangle\}$, $n_1, n_2 = 0, 1, 2, ...$

Thermal one-point functions from the free energy

- Before I proceed with the trivial calculation, let me write the partition function in the slightly unconventional way as follows

$$\mathcal{Z}_1 = Tr_{\mathcal{H}_{1,2}} e^{-\beta(\hat{H}_0 + m^2 \hat{\mathcal{O}}) + i\beta\mu\hat{N}}\,, \;\; \hat{\mathcal{O}} = \frac{1}{2}(\hat{x}_i^2 + \hat{x}_2^2)$$

  and $\hat{H}_0 = (\hat{p}_1^2 + \hat{p}_2^2)/2$ is the *free* Hamiltonian or equivalently in this case the kinetic energy.

- Calculating the partition function one easily obtains

$$\mathcal{C}_1(z, \bar{z}) = -\ln|z| + \ln(1-z) + \ln(1-\bar{z})\,,$$

  This matches the result for the free energy above for $d = 1$.

Thermal one-point functions from the free energy

- The thermal averages of the operators $\hat{\mathcal{O}}$ and $\hat{\mathcal{Q}}$ are given by

$$\langle \hat{\mathcal{O}} \rangle_1 = -\frac{\beta}{2 \ln |z|} \left[1 + |z|^2 \left(\frac{1}{z(1-\bar{z})} + \frac{1}{\bar{z}(1-z)} \right) \right]$$
$$\langle \hat{\mathcal{Q}} \rangle_1 = |z|^2 \left(\frac{1}{z(1-\bar{z})} - \frac{1}{\bar{z}(1-z)} \right)$$

- Also recall that in our simple model there is a virial theorem at work relating the thermal averages of the operator $\hat{\mathcal{O}}$ and the Hamiltonan $\hat{H}$ as

$$2\omega^2 \langle \hat{\mathcal{O}} \rangle_1 = \langle \hat{H} \rangle \,,$$

  and therefore it is not surprising that the operator $\mathcal{O}$ is related to the radial Hamiltonian in general d-dimensions.

Thermal one-point functions from the free energy

- The generic formulae above yield an intimate connection among theories in different dimensions. The crucial point is the following result which can be proven by a direct calculation

$$\hat{\mathrm{D}}\mathcal{C}_d(z,\bar{z}) = -\frac{1}{4\pi}\mathcal{C}_{d-2}(z,\bar{z})\,,$$

  for $d = 1, 3, 5, ..$ with the boundary condition $\mathcal{C}_{-1}(z,\bar{z}) = -4\pi\langle\mathcal{O}\rangle_1$.
- Then from the above one obtains

$$\mathcal{C}_d(z,\bar{z}) = -4\pi\beta^d\langle\mathcal{O}\rangle_{d+2}$$

  which shows that the free energy of the d-dimensional theory is proportional to the one-point function of the operator $\mathcal{O}$ in $d+2$-dimensions. One also finds

$$\langle\mathcal{O}\rangle_d = -4\pi\beta^2\hat{\mathrm{D}}\langle\mathcal{O}\rangle_{d+2}\,,\quad \langle\mathcal{Q}\rangle_d = -4\pi\beta^2\hat{\mathrm{D}}\langle\mathcal{Q}\rangle_{d+2}\,,$$

Thermal one-point functions from the OPE: test

- In the case of a real scalar field studied in [T.P & A. Stergiou (2018)], ϕ^2 with dimension $\Delta_{\phi^2} = d-2$ was the only relevant operator in the OPE $\phi\times\phi$ which was *not a shadow operator*, and there were no currents with odd spin. In the case of the complex scalar field studied here there is another relevant operator which is not a shadow operator; this is the $s=1$ current J_μ with dimension $\Delta_J = d-1$. The contribution of the above two operators in the thermal two-point function is

$$g(r,\cos\theta) = \frac{a_{\mathbb{I}}^{(d)}}{r^{d-2}} + [\text{shadows}] + \frac{g^{(d)}_{\phi\phi\phi^2}}{C^{(d)}_{\phi^2}} b^{(d)}_{\phi^2} + r\cos\theta \frac{g^{(d)}_{\phi\phi J}}{C^{(d)}_J} b^{(d)}_J + \cdots$$

- The OPE coefficients depend on the thermal mass and the imaginary chemical potential, namely $b_{\mathcal{O}_s} = b_{\mathcal{O}_s}(z,\bar{z})$.

Thermal one-point functions from the OPE: test

- The operators ϕ^2 and J_0 correspond to the operators $\mathcal{O}$ and $\mathcal{Q}$ whose thermal one-point functions are given by the general formulae given previously.
- If one wanted the have a conformal OPE, then the arbitrary scale parameter introduced in the theory by b_{ϕ^2} should not be there. Requiring its vanishing is the condition (gap equation) suggested in **[T.P & A. Stergiou (2018)]** that determines the critical values of the thermal mass and chemical potential. For $\mu = 0$ this condition gives a class of algebraic equations for high order polylogarithms and it explains the critical behaviour of bosonic systems in generic even dimensions. (**see extra material).**

Thermal one-point functions from the OPE: test

- The thermal two-point function of the free theory is

$$g(r, \cos\theta; m, \mu) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n=-\infty}^{\infty} e^{in\mu} \left(\frac{m}{|X_{n,\zeta}|} \right)^{\nu} K_{\nu}\left(m|X_{n,\zeta}| \right) ,$$

  with $X_{n,\zeta} = \sqrt{(n-\zeta)(n-\bar{\zeta})}$, and $\zeta = re^{i\theta}$.
- The $n = 0$ term gives the usual zero temperature result for the massive two-point function, while the Bessel functions have polynomial expansions for d odd.
- In $d > 2$ the zero temperature contribution has a divergent $m \to 0$ expansion where two cut-off independent terms stand out: the term giving the massless two-point function and the r-independent term proportional to m^{d-2}. In the OPE expansion these correspond to the contribution of the unit and $|\phi|^2$ operators respectively.
- Notice that the contributions from shadow scalar operators come from the process of the cut-off subtraction and renormalization of the theory - which I assume that it can somehow be done and it does not affect the results.

Thermal one-point functions from the OPE: test

- After some algebra I find

$$\frac{g_{\phi\phi\phi^2}}{C_{\phi^2}} b_{\phi^2} = a_{\mathbb{I}} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-2)} \left[\frac{\Gamma\left(1-\frac{d}{2}\right)}{2\sqrt{\pi}} m^{d-2} + I_d(z,\bar{z}) \right] ,$$

$$\frac{g_{\phi\phi J}}{C_J} b_J = -\frac{1}{2} a_{\mathbb{I}} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-2)} \mathcal{I}_d(z,\bar{z})$$

where $a_{\mathbb{I}} = \sqrt{\frac{\pi}{2}} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{2^{\frac{d-3}{2}}} \frac{\Gamma(d-2)}{\Gamma\left(\frac{d-1}{2}\right)}$.

- The functions $I_d(z,\bar{z})$ was given above and $\mathcal{I}_d(z,\bar{z})$ is

$$\mathcal{I}_d(z,\bar{z}) = \sum_{n=0}^{\frac{d-1}{2}} \frac{(-1)^n (d-1-n)!}{\left(\frac{d-1}{2}-n\right)!} \frac{2^n \ln^n |z|}{n!} \left[Li_{d-1-n}(z) - Li_{d-1-n}(\bar{z}) \right] .$$

Thermal one-point functions as single-valued polylogarithms.

Thermal one-point functions as single-valued polylogarithms

- The results for the free energy and the thermal one-point functions are particular cases of the single-valued polylogarithms $P_w(z,\bar{z})$ constructed by Brown [F. C. Brown (2004), O. Schnetz (2013)]. Explicitly one observes that for $d \geq 1$

$$I_d(z,\bar{z}) = (-1)^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right) P_{w_d}(z,\bar{z}) \,, \quad w_d \equiv 0^{\{n_d\}} 10^{\{n_d\}} \,,$$

  with $n_d = (d-3)/2$.
- The index w_d denotes a "word" in the "two-letter alphabet" $\{0,1\}$ and one normalises such that $P_\emptyset = 1$ and $P_{0^{\{n\}}} = 2^n \ln^n |z|/n!$. Notice that $P_1(z,\bar{z}) = \ln(1-z) + \ln(1-\bar{z})$. It can be shown that the single-valued polylogarithm $P_{w_d}(z,\bar{z})$ satisfies for $d \geq 1$

$$z\partial_z P_{0w_d0}(z,\bar{z}) = P_{0w_d}(z,\bar{z}) \,, \quad \bar{z}\partial_{\bar{z}} P_{0w_d0}(z,\bar{z}) = P_{w_d0}(z,\bar{z}) \,,$$

  which are the form of Knizhnik-Zamolodhikov equations.

Thermal one-point functions as single-valued polylogarithms

- Actually, one can show that the above yields the second order differential equations studied e.g. in [J. Drummond (2012)]. The KZ-like equations then imply

$$\hat{\mathrm{L}} P_{0w_d0}(z,\bar{z}) = P_{0w_d}(z,\bar{z}) - P_{w_d0}(z,\bar{z})$$
$$\hat{\mathrm{D}} P_{0w_d0}(z,\bar{z}) = \frac{1}{n_d+1} P_{w_d}(z,\bar{z}) = \frac{2}{d+1} P_{0w_{d-2}0}(z,\bar{z}),$$

which are effectively a different form of the relationships given for the free energy and the thermal one-point functions.

Outlook

Outlook

- It is well-known that single-valued polylogarithms are closely related to Feynman graphs in gauge and string theories. The observation made in this work seems to connect two apparently different physical quantities; thermal one-point functions in odd-dimensional theories and multiloop graphs in $\mathcal{N} = 4$ SYM in $d = 4$.
- Contrasting these two manifestations of single-valued polylogarithms one notices that in the first case the variables z and $\bar{z}$ parametrise relevant deformations of free CFTs while in the second case parametrize spacetime points.
- Moreover, in the first case the order of the polylogarithms is related to spacetime dimension while in the latter case to the number of loops in the graphs.

Outlook

- It would be nice to understand further the implications, if any, of our observation. As a hint, one may consider interpreting the charge one-point functions as higher-dimensional generalizations of the twist one-point function operator in the system of two harmonic oscillators. For example in $d = 3$ we find [T.P. et. al. (2018)]

$$4\pi\beta^2\langle\mathcal{Q}\rangle_3 = -4iD(d)\,,\;\; D(z) = \Im Li_2(z) + \ln|z|\mathrm{Arg}(1-z)\,,$$

with $D(z)$ the famous Bloch-Wigner function (single-valued dilogarithm).

Outlook

- $D(z)$ appears in the simplest tree graph of $\mathcal{N}=4$ SYM

$$\frac{1}{\pi^2}\int\frac{d^4x}{(x-x_1)^2(x-x_2)^2(x-x_3)^2(x-x_4)^2}=\frac{1}{x_{14}^2x_{23}^2}\frac{4i}{z-\bar{z}}D(z)$$

  where the change of variables is

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,,v=\frac{x_{12}^2x_{34}^2}{x_{14}^2x_{23}^2}=(1-z)(1-\bar{z})\,,\quad\frac{v}{u}=z\bar{z}$$

- What is perhaps less known is that the integral above can be obtained as a limit that corresponds to a particular kind of twisting. Namely, using a result given [T.P. (1994)] one obtains after some algebra

$$\lim_{\epsilon\to 0}\Gamma(\epsilon)\left[G^{(4)}_{2-\epsilon}(v,Y)-v^{\epsilon}G^{(4)}_{2+\epsilon}(v,Y)\right]=\frac{4i}{z-\bar{z}}D(z)\,,$$

  where $G^{(d)}_{\Delta}(v,u)$ is the standard d-dimensional conformal block of a scalar operator with dimension Δ.

Outlook

- The important point, first stressed in [T.P. (1994)] and explicitly demonstrated in [F. Dolan & H. Osborn (2000)] is that the $\epsilon \to 0$ divergence cancels out in leaving a finite result. Hence, the first nontrivial ladder graph gives a twisting of shadow scalar conformal blocks. Perhaps this interpretation generalises to higher loops.
- One may also suspect the relevance of our results to recent works on large-charge expansions where imaginary chemical potentials play an important role [L. Alvarez-Gaume et. al. (2019]. In another direction, it would be interesting to study the manifestation of the shuffle algebra of the polylogarithms in the context of thermal field theories.

Extra material

Gap equations and vacuum structure of CFTs

- .. continuing the discussion of the gap equation: Notice that ϕ^2 operator enters the spectrum *both* from the discontinuity part *as well as* from the arc part of the spectral function.
- If we **demand the absence of this operator** from the spectrum the two above contributions must cancel each other. This gives rise to a condition that determines m_{th}, namely

$$\sum_{n=0}^{k-1} \frac{2^{n+1}}{n!} \frac{(2(k-1)-n)!}{(k-1-n)!} m_{\text{th}}^n \, \mathrm{Li}_{2k-1-n}(e^{-m_{\text{th}}}) = -\frac{1}{2\sqrt{\pi}} m_{\text{th}}^{2k-1} \Gamma(-k+\tfrac{1}{2})$$

- This is the so-called **gap equation** and it is here presented for any $d = 2k+1, k = 1, 2, \ldots$.

Gap equations and vacuum structure of CFTs

- The subleading terms in the give higher poles at $\Delta = d - 2 + 2l, l = 1, 2, \ldots,$. These correspond to scalar operators of the form $\phi \partial^{2l} \phi$.
- Such operators also arise from subleading terms in the $\bar{z}$ expansion of of the discontinuity parts of the spectral function., namely from $a^{(k)}_{\text{Disc},1}, a^{(k)}_{\text{Disc},2}, \ldots$. These operators **should also disappear from the spectrum** when the gap equation is satisfied.
- Although we have verified this in a couple of cases, we do not have a general proof as yet.

Gap equations from the inversion formula: bosonic case

- The arc contribution of the identity operator provides a quick consistency check of our computations. Since the identity operator has $\Delta = 0$ we see that the pole associated with it appears due to $\Gamma(-\frac{\Delta}{2})$.
- For the residue find

$$\underset{\Delta=0}{\mathrm{Res}}(a^{(d)}_{\mathrm{arcs}}(\Delta, 0)) = -\frac{2^{\frac{d-3}{2}}}{\sqrt{\pi}}\Gamma(\tfrac{d}{2} - 1)$$

- This exactly reproduces the correct normalization of the identity operator (in our conventions).

Gap equations and vacuum structure of CFTs

- The Bloch-Wigner function $D(z) \equiv D_2(z)$ gives the volume of ideal tetrahedra in Euclidean hyperbolic space $\mathcal{H}_3$ whose four vertices lie in $\partial\mathcal{H}_3$ at the points $0, 1, \infty$, and z (z is a dimensionless cross ratio here). These tetrahedra are the building blocks for general hyperbolic manifolds - the volume of the latter arises as the sum of ideal tetrahedra after a suitable triangulation [e.g. Zagier].
- It is known [Witten (98), Gukov (03)] that complex $SL(2, \mathbb{C})$ Chern-Simons theory with purely imaginary level corresponds, at least semi classically, to Euclidean three-dimensional gravity with negative cosmological constant. We have shown in [T.P. et. al. (18)] that $\langle \mathcal{Q} \rangle_3$ arises as a purely imaginary Chern-Simmons level in the 3d fermionic model coupled to background monopole operator. This way we may be able to understand the presence of hyperbolic volumes in the gap equations and eventually in the free energy of our models.

Gap equations and vacuum structure of CFTs

- It is also possible to study finite-temperature fermionic 2-pt functions using the inversion formula. The simplest case to consider is the singlet projection of the two-point functions of Dirac fermions $\psi_i(x)$, $\bar{\psi}_i(x)$ in odd dimensions,

$$\langle\psi_i(x)\bar{\psi}_i(0)\rangle_\beta \equiv \tilde{g}(r,\cos\theta) = \sum_{\tilde{\mathcal{O}}_s\neq\mathbb{1}} \tilde{a}_{\tilde{\mathcal{O}}_s}\left(\frac{r}{\beta}\right)^{\Delta_{\tilde{\mathcal{O}}_s}} \frac{C_s^\nu(\cos\theta)}{r^{2\Delta_\psi}}$$

  with $\Delta_\psi = \Delta_\phi + 1/2$ and $i,j = 1,2,\ldots,2^{\frac{d-1}{2}}$ spinor indices.
- This vanishes at zero temperature which is is a manifestation of the fact that the unit operator is absent in the finite-temperature OPE.
- The corresponding unit-normalized momentum-space 2-pt function is

$$\tilde{G}^{(d)}(\omega_n,\mathrm{p}) = \frac{\tilde{m}_{\mathrm{th}}}{\omega_n^2+\mathrm{p}^2+\tilde{m}_{\mathrm{th}}^2}$$

  where the fermionic Matsubara frequencies are $\omega_n = 2\pi(n+1/2)$, $n = 0,\pm1,\pm2,\ldots$.

Gap equations and vacuum structure of CFTs

- The fermionic propagator vanishes for $\tilde{m}_{\text{th}} = 0$ so we will only consider $\tilde{m}_{\text{th}} \neq 0$ in the fermionic case from now on. The calculations follow closely the bosonic case e.g. it is known that fermionic Matsubara sums reduce to a linear combination of bosonic ones.
- We then notice that by virtue of the relationship $\Delta_\psi = \Delta_\phi + 1/2$, the fermionic formulas can all be obtained from the bosonic ones by the simple shift $\Delta \to \Delta - 1$.
- The arc contributions in the fermionic case are thus given by

$$\tilde{a}^{(d)}_{\text{arcs}}(\Delta, 0) = -\frac{1}{2^{\Delta - \frac{d-3}{2}}\sqrt{\pi}}\,\tilde{m}_{\text{th}}^{\Delta-1}\,\Gamma\Big(-\frac{\Delta-1}{2}\Big)\,\Gamma\Big(-\frac{\Delta-d+1}{2}\Big)$$

- This gives operators of dimension $\Delta = 2m+1$ and $\Delta = d-1+2m$, $m = 0, 1, 2, \ldots$.

Gap equations and vacuum structure of CFTs

- The former are contributions that do not arise from the discontinuity part, having the form $\tilde{\sigma}^m$ with $\tilde{\sigma}$ the shadow field of $\bar{\psi}\psi$. Note that, as expected, there is no contribution from the unit operator.
- The latter provide contributions from operators of the form $\bar{\psi}\partial^{2m}\psi$ that coincide with those coming from the discontinuity.
- The fermionic gap equation is the condition for the cancellation of the latter operators from the spectrum and it reads

$$\sum_{n=0}^{k-1} \frac{2^{n+1}}{n!} \frac{(2(k-1)-n)!}{(k-1-n)!} \tilde{m}_{\text{th}}^{n+1} \,\text{Li}_{2k-1-n}(-e^{-\tilde{m}_{\text{th}}}) = -\frac{1}{2\sqrt{\pi}} \tilde{m}_{\text{th}}^{2k} \Gamma(-k+\tfrac{1}{2})$$

Gap equations and vacuum structure of CFTs

The Lorentzian inversion formula together with an ansatz for the form of the thermal 2-t function can be used to *bootstrap* bosonic and fermionic CFTs in arbitrary odd-d dimensions.

The nontrivial dynamics corresponds to a rearrangement of the operator spectrum. The gap equation arises as the condition that certain classes of operators drop out from the spectrum of the nontrivial CFT.

The resulting picture for the operator spectrum corresponds to the well-known large-N CFTs that arise from a generalised Hubbard-Stratonovich transformation (see later).

Further lessons from the gap equation: solutions

- The bosonic gap equation in $d = 3$ reads

$$-m_{\text{th}} = 2\log(1 - e^{-m_{\text{th}}})$$

with the well-known solution (related to the "golden mean")

$$m_{\text{th}}^{(d=3)} = 2\log(\tfrac{1+\sqrt{5}}{2}) \approx 0.96242$$

- In $d = 5$ the bosonic gap equation becomes

$$-\tfrac{1}{6}m_{\text{th}}^3 = \text{Li}_3(e^{-m_{\text{th}}}) + m_{\text{th}}\,\text{Li}_2(e^{-m_{\text{th}}})$$

This has a complex conjugate pair of solutions given numerically by

$$m_{\text{th}}^{(d=5)} \approx 1.17431 \pm 1.19808\,i$$

Gap equations and vacuum structure of CFTs

- In fact, we find that for $\underline{d = 3, 7, 11, \ldots}$ the bosonic gap equation has a **unique real solution for** $m_{\textbf{th}}$ and complex solutions that come in conjugate pairs - except for $d = 3$ where there are no complex solutions. i.e. in $d = 7$ we find a real and a pair of complex conjugate solutions.
- For $\underline{d = 5, 9, 13, \ldots}$ **we do not find any real solutions**, and the gap equation only has pairs of complex conjugate solutions. I.e. $d = 5$ we only find the solutions above, while in $d = 9$ we find four complex conjugate pairs of solutions. Notice also that $m_{\textbf{th}} = 0$ **is never a solution of the bosonic gap equations.**

Gap equations and vacuum structure of CFTs

- The fermionic gap equations in $d = 3, 5$ are given respectively by

$$-\tilde{m}_{\text{th}}^2 = 2\tilde{m}_{\text{th}} \log(1 + e^{-\tilde{m}_{\text{th}}}),$$
$$-\tfrac{1}{6}\tilde{m}_{\text{th}}^4 = \tilde{m}_{\text{th}}\, \text{Li}_3(-e^{-\tilde{m}_{\text{th}}}) + \tilde{m}_{\text{th}}^2\, \text{Li}_2(-e^{-\tilde{m}_{\text{th}}})$$

- For $d = 3$ and $\tilde{m}_{\text{th}} \neq 0$ The fermionic gap equation has only a pair of complex conjugate imaginary solutions $\tilde{m}_{\text{th}}^{(d=3)} = \pm 2\pi i/3$. For $d = 5$ it has a pair of opposite real solutions, as well as a pair of complex conjugate imaginary ones which can be found numerically.
- This pattern continues to higher dimensions, namely for $\underline{d = 7, 11, 15, \ldots}$ **there is no real solution to the corresponding fermionic gap equation**, while for $\underline{d = 9, 13, 17, \ldots}$ **there is always a pair of opposite real solutions and an increasing number of complex conjugate ones.**

Gap equations and vacuum structure of CFTs

The above pattern for the solutions of bosonic and fermionic gap equations for all *odd-d* fits nicely with a renormalization-group understanding of universality classes of scalars and fermions in general dimensions.

- In the bosonic case the standard lore is that the large-N universality class for scalars in $d = 2k + 1$, $k = 1, 2, \ldots$, is accessible via the ε expansion starting from $d = 2k + 2$.
- The Hubbard–Stratonovich transformation introduces a field σ via the classically marginal interaction $\sigma\phi^2$. σ has dimension $\Delta_\sigma = 2$ in all d, and the scalars ϕ can be integrated out resulting in an effective potential of the general form

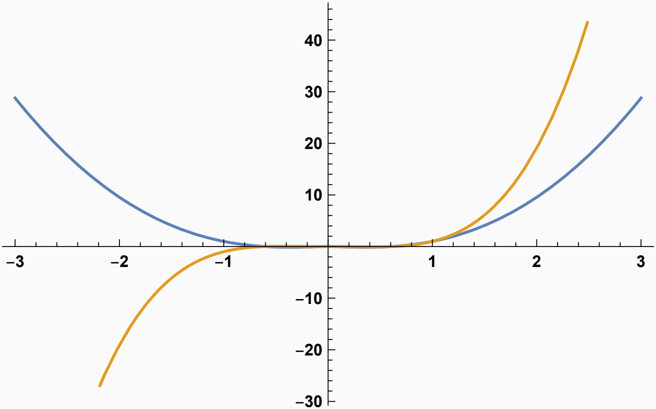
$$V_{\text{eff}}(\sigma) \sim \mathrm{Tr}_d \log(-\partial^2 + \sigma) + g_* \sigma^{\frac{d}{2}} + \cdots$$

with g_* some critical dimensionless coupling.

Gap equations and vacuum structure of CFTs

- For general d the effective potential can also receive contributions from terms involving derivatives of σ, but the term $\sigma^{\frac{d}{2}}$ is universal.
- Performing the $\mathrm{Tr}_d \log$ calculation in $d-\varepsilon$ one finds that for $\underline{d = 4, 8, 12, \ldots}$ there is a resulting contribution of the form $\sigma^{\frac{d}{2}} \log \sigma^2$, which is positive and dominates for large σ. Thus, besides various possible local minima, the effective potential has a **global minimum**.
- On the other hand, for $\underline{d = 6, 10, 14, \ldots}$ the term $\sigma^{\frac{d}{2}}$ leads to an **unbounded potential,** and hence to the absence of a global minimum, regardless of the sign of the $\mathrm{Tr}_d \log$ contribution. This matches exactly the pattern we see for m_{th}: **a real m_{th} implies a global minimum, while a complex m_{th} signals unstable local extrema with nonzero decay width.**

Gap equations and vacuum structure of CFTs



Gap equations and vacuum structure of CFTs

- In the fermionic case our results are consistent with large-N universality classes in $d = 2k+1$, $k = 1, 2, \ldots$ that are accessible via the ε expansion starting from a generalization of the Gross–Neveu–Yukawa model to $d = 2k+2$ [P. Zinn-Justin NPB B367 (1991)]
- The corresponding Hubbard–Stratonovich transformation introduces $\tilde{\sigma}$ via the classically marginal interaction $\tilde{\sigma}\bar{\psi}\psi$. Here $\tilde{\sigma}$ has dimension $\Delta_{\tilde{\sigma}} = 1$ in all d, and one gets an effective potential of the form **the** $\mathrm{Tr}_d \log$ **term enters with the opposite sign**

$$V_{\text{eff}}(\tilde{\sigma}) \sim -\mathrm{Tr}_d \log(\partial\!\!\!/ + \tilde{\sigma}) + \tilde{g}_* \tilde{\sigma}^d + \cdots$$

- $\tilde{\sigma}^d$ gives always a bounded from below contribution (recall d is even). However, the $\mathrm{Tr}_d \log$ term changes sing as $d - \varepsilon$: for $\underline{d = 4, 8, 12, \ldots}$ it gives a negative contribution that dominates at infinity leading to an **unstable vacuum structure**, while for $\underline{d = 6, 10, 14, \ldots}$ it gives a positive contribution that guarantees the **presence of a global minimum**. In either case there can be a number of unstable extrema. This matches the obtained pattern for the $\tilde{m}_{\text{th}}$.

Gap equations and vacuum structure of CFTs