Operator spectrum of NRCFTs at large charge

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Based on [Orlando, VP, Reffert '20], [VP '21]

[Hellerman, Orlando, VP, Reffert, Swanson, *to appear*]

Introduction

Classical expansion of Δ_Q

Intermezzo

Casimir energy in $d = 3$

Conclusion

Context: the relativistic $O(2)$ model

It is now well-established that [Hellerman, Orlando, Reffert, Watanabe '15] [Cuomo '20]

$$\Delta_Q = Q^{\frac{d+1}{d}}\left[\alpha_1 + \frac{\alpha_2}{Q^{\frac{2}{d}}} + \frac{\alpha_3}{Q^{\frac{4}{d}}} + \dots\right] + Q^0\left[\beta_0 + \frac{\beta_1}{Q^{\frac{2}{d}}} + \frac{\beta_2}{Q^{\frac{4}{d}}} + \dots\right] + \dots,$$

in $(d+1)$-dimensions. The second line comes from the one-loop Casimir energy, based on the spectrum

$$\omega_l = \sqrt{\frac{l(l+d-1)}{R^2 d}} + \mathcal{O}(Q^{-\frac{2}{d}}),$$

with multiplicity $\frac{(2l+d-1)\Gamma(l+d-1)}{\Gamma(l+1)\Gamma(d)}$ on the d-sphere.

Context: the relativistic $O(2)$ model

Typically,

$$\Delta_Q^{(d=2)} = \alpha_1 Q^{\frac{3}{2}} + \alpha_2 \sqrt{Q} - 0.0937 + \dots,$$

and

$$\Delta_Q^{(d=3)} = \alpha_1 Q^{\frac{4}{3}} + \alpha_2 Q^{\frac{2}{3}} - \frac{1}{48\sqrt{3}} \log Q + \alpha_3 + \dots$$

Today's results

In the nonrelativistic case with Schrödinger symmetry, I'll show that

[VP '21]

$$\Delta_Q = Q^{\frac{d+1}{d}}\left[a_1 + \frac{a_2}{Q^{\frac{2}{d}}} + \frac{a_3}{Q^{\frac{4}{d}}} + \dots\right] + Q^{\frac{2d-1}{3d}}\left[b_1 + \frac{b_2}{Q^{\frac{2}{3d}}} + \frac{b_3}{Q^{\frac{4}{3d}}} + \dots\right] + Q^{\frac{d-3}{3d}}\left[c_1 + \frac{c_2}{Q^{\frac{2}{3d}}} + \frac{c_3}{Q^{\frac{4}{3d}}} + \dots\right] + \dots$$

Does *not* include quantum corrections, and some b_i's contain $\log Q$-terms when d is even. Dispersion relation [Kravec, Pal '18]

$$\omega_{n,l} = \sqrt{\frac{4n}{d}(n+l+d-1)+l} + \mathcal{O}(Q^{-\frac{2}{3d}}),$$

with multiplicity $\frac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(l+1)\Gamma(d-1)}$ on the $(d-1)$-sphere.

Today's results

Typically, [Kravec, Pal '18] [Orlando, VP, Reffert, '20] [Hellerman, Swanson '20] [VP '21]

$$\Delta_Q^{(d=2)} = d_1 Q^{\frac{3}{2}} + d_2 \sqrt{Q} \log Q + d_3 \sqrt{Q} + d_4 Q^{\frac{1}{6}} - 0.2942 + \dots,$$

and [Son, Wingate '05] [Kravec, Pal '18] [Orlando, VP, Reffert, '20] [VP '21] [Hellerman, Orlando, VP, Reffert, Swanson, *to appear*]

$$\Delta_Q^{(d=3)} = d_1 Q^{\frac{4}{3}} + d_2 Q^{\frac{2}{3}} + d_3 Q^{\frac{5}{9}} + d_4 Q^{\frac{1}{3}} + d_5 Q^{\frac{1}{9}} + \frac{1}{3\sqrt{3}} \log Q + d_6 + \dots$$

Outline

- Classical expansion of Δ_Q
  - Leading-order effective action
  - Subleading operators
  - Structure of the expansion
- Intermezzo
- Casimir energy in $d = 3$
  - Finding the ζ-function
  - Renormalizing the Casimir energy
- Conclusion

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Leading-order effective action

State-operator correspondence: couple to external harmonic trap.

[Werner, Castin '05] [Nishida, Son '07] [Goldberger, Khandker, Prabhu '14]

The leading-order nlsm Lagrangian reads ($\hbar = m = \omega = 1$)

$$\mathcal{L}_{LO} = c_0 U^{\frac{d}{2}+1},$$

(relativistic: $\mathcal{L}_{LO} = c_0(\partial\chi)^{d+1}$) where

$$U = \dot{\chi} - \frac{1}{2}r^2 - \frac{1}{2}(\partial_i\chi)^2$$

(imposed by *general coordinate invariance* [Son, Wingate '05]).

Superfluid GS: $\langle\chi\rangle = \mu \cdot t$, where $\mu =$ chemical potential. Then,

$$\langle U\rangle = \mu - \frac{1}{2}r^2 \equiv \mu \cdot z,$$

where $z \equiv 1 - \frac{r^2}{2\mu} \equiv 1 - \frac{r^2}{R_{cl}^2}$, with $R_{cl} \equiv \sqrt{2\mu}$.

Leading-order effective action

Ground-state charge density:

$$\langle \rho \rangle = \langle \frac{\partial \mathcal{L}_{LO}}{\partial \dot{\chi}} \rangle = \langle \frac{\partial \mathcal{L}_{LO}}{\partial U} \rangle \sim \langle U^{\frac{d}{2}} \rangle \sim (\mu \cdot z)^{\frac{d}{2}},$$

i.e. particles confined in a (classically) spherical cloud of radius R_{cl}.

Thus, μ depends on the charge as ($\zeta =$ constant)

$$\mu = \zeta Q^{\frac{1}{d}}.$$

Ground-state energy [Kravec, Pal '18] [Orlando, VP, Reffert, '20] [VP '21]

$$\Delta_Q = \frac{d}{d+1} \zeta Q^{\frac{d+1}{d}}.$$

Remark on the dimensionless z-coordinate

The GS preserves spherical symmetry $\rightarrow$ use $z \equiv 1 - \frac{r^2}{2\mu}$, with

$$(\partial_i f(\vec{x}))(\partial_i g(\vec{x})) = \frac{2(1-z)}{\mu} f'(z) g'(z),$$

$$\nabla^2 f(\vec{x}) = \frac{2}{\mu}\left[(1-z) f''(z) - \frac{d}{2} f'(z)\right],$$

$$\int_{cloud} \mathrm{d}^d x\, f(\vec{x}) = \frac{(2\pi\mu)^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^1 \mathrm{d}z\, (1-z)^{\frac{d-2}{2}} f(z),$$

where primes refer to derivatives with respect to z and f, g are spherically invariant functions.

Subleading operators

Besides U and $(\partial_i U)^2$, the only operator with a nonzero VEV is

$$Z = \nabla^2 A_0 - \frac{1}{d}(\nabla^2 \chi)^2,$$

with $\langle Z \rangle = d$. Therefore, in the bulk, all operators are of the form

$$\mathcal{O}^{(m,n)}_{bulk} \equiv (\partial_i U)^{2m} Z^n U^{\frac{d}{2}+1-(3m+2n)},$$

where m and n are integers. Using Eq. (8), we see that

$$\begin{aligned}\int_{cloud} \mathrm{d}^d x \, \langle \tilde{\mathcal{O}}^{(m,n)}_{bulk} \rangle &\sim \mu^{d+1-2(m+n)} \cdot \int_0^1 \mathrm{d}z \, (1-z)^{\frac{d}{2}-1+m} z^{\frac{d}{2}+1-(3m+2n)} \\ &\sim \mu^{d+1-2(m+n)} \cdot \frac{\Gamma\left(\frac{d}{2}+m\right)\Gamma\left(\frac{d}{2}+2-(3m+2n)\right)}{\Gamma\left(d+2-2(m+n)\right)}.\end{aligned}$$

(Classical) structure of the large-charge expansion of Δ_Q

Generic contribution

$$\int_{cloud} \mathrm{d}^d x \, \langle \tilde{\mathcal{O}}^{(m,n)}_{bulk} \rangle \sim \mu^{d+1-2(m+n)} \cdot \Gamma\left(\frac{d}{2} + 2 - (3m+2n)\right)$$

- If Γ-function is finite $\rightarrow$ expansion in $\mu^{-2} \sim Q^{-\frac{2}{d}}$ starting at $\mu^{d+1} \sim Q^{\frac{d+1}{d}}$, as in the relativistic case.
- In particular, when d is odd and $d+1 = 2(m+n) \rightarrow$ Q^0-term. This hints at a pole in the Casimir energy.

(Classical) structure of the large-charge expansion of Δ_Q

Generic contribution

$$\int_{cloud} \mathrm{d}^d x \, \langle \tilde{\mathcal{O}}_{bulk}^{(m,n)} \rangle \sim \mu^{d+1-2(m+n)} \cdot \Gamma\left(\frac{d}{2} + 2 - (3m+2n)\right)$$

- When d is even and $d+4 = 6m+4n \to$ pole.
- Let $d+4 = 6m+4n-2\epsilon$, so that

$$\int_{cloud} \mathrm{d}^d x \, \langle \tilde{\mathcal{O}}_{bulk}^{(m,n)} \rangle \sim \mu^{\frac{2d-1-2n}{3}} \left[-\frac{1}{\epsilon} + \frac{2}{3} \log \mu - \gamma_E + \mathcal{O}(\epsilon) \right].$$

i.e. a *classical* non-universal $\mu^{\frac{2d-1-2n}{3}} \log \mu \sim Q^{\frac{2d-1-2n}{3d}} \log Q$.

(Classical) structure of the large-charge expansion of Δ_Q

Edge counterterms [Hellerman, Swanson '20] [VP '21]

$$Z^n_{edge} \equiv Z^n \cdot \delta(U) \cdot (\partial_i U)^{\frac{d+4(1-n)}{3}}$$

with contribution

$$\Delta_Q \ni \mu^{\frac{2n-1-2n}{3}} \sim Q^{\frac{2n-1-2n}{3d}}.$$

The corresponding Wilsonian coefficient κ_n thus gets renormalized:

$$\kappa_n = \kappa_n^{ren.} + \frac{cst}{\epsilon}.$$

On top of taking care of edge divergences, counterterms trigger an expansion in $\mu^{-\frac{2}{3}} \sim Q^{-\frac{2}{3d}}$ starting at $\mu^{\frac{2d-1}{3}} \sim Q^{\frac{2d-1}{3d}}$.

(Classical) structure of the large-charge expansion of Δ_Q

Moreover, since $\mu = \zeta Q^{\frac{1}{d}}\left[1 + \mathcal{O}\left(Q^{-\frac{d+2}{3d}}\right)\right]$, we get

$$
\begin{aligned}
\Delta_Q &= \mu^{d+1}\left[a_1 + \frac{a_2}{\mu^2} + \frac{a_3}{\mu^4} + \dots\right] \\
&\quad + \mu^{\frac{2d-1}{3}}\left[b_1 + \frac{b_2}{\mu^{\frac{2}{3}}} + \frac{b_3}{\mu^{\frac{4}{3}}} + \dots\right] + \dots \\
&= Q^{\frac{d+1}{d}}\left[a_1 + \frac{a_2}{Q^{\frac{2}{d}}} + \frac{a_3}{Q^{\frac{4}{d}}} + \dots\right] \\
&\quad + Q^{\frac{2d-1}{3d}}\left[b_1 + \frac{b_2}{Q^{\frac{2}{3d}}} + \frac{b_3}{Q^{\frac{4}{3d}}} + \dots\right] \\
&\quad + Q^{\frac{d-3}{3d}}\left[c_1 + \frac{c_2}{Q^{\frac{2}{3d}}} + \frac{c_3}{Q^{\frac{4}{3d}}} + \dots\right] + \dots
\end{aligned}
$$

where some $b_n = cst + cst \cdot \log Q$ when d is even.

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Experiments

Trapped gases were observed in the lab long before the state-operator correspondence was found!

Cf. e.g. reviews [Dalfovo, Giorgini, Pitaevskii, Stringari '98] [Giorgini, Pitaevskii, Stringari '08]

BCS-BEC crossover and the unitary Fermi gas

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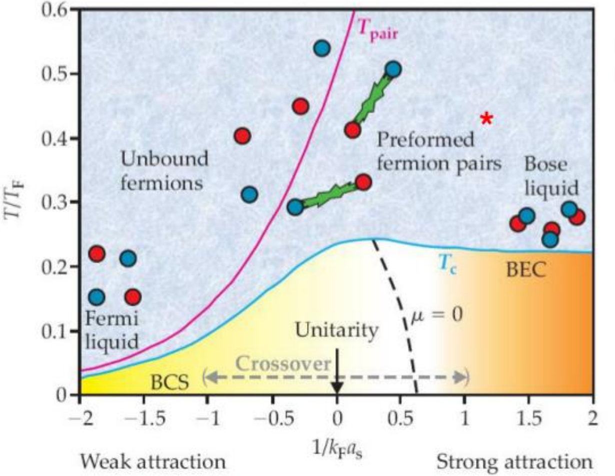

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Pairing pseudogap:
MR, Trivedi, Moreo, Scalettar PRL (92)
Trivedi & MR, PRL (95)

Based on
Sa de Melo, MR & Engelbrecht, PRL (1993)

from: Sa de Melo, Phys.Today (Oct. 2008)

BEC in a harmonic trap

Ground-state energy of the BEC for a large number Q of trapped particles [Dalfovo, Giorgini, Pitaevskii, Stringari '98]

$$E_0 = d_1 Q^{\frac{7}{5}} + d_2 Q^{\frac{3}{5}} \log Q + \dots$$

The $\log Q$-term is obtained by regularizing edge divergences.

Can we understand this better now?

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Finding the ζ-function

Compute

$$E_{Casimir} \equiv \frac{1}{2} \sum_{n,l=0}^{\infty} \sqrt{\frac{4n}{d}(n+l+d-1)+l} \cdot \frac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(l+1)\Gamma(d-1)},$$

Define instead

$$E(s) \equiv \frac{1}{2} {\sum_{n,l=0}^{\infty}}' \left[\frac{4n}{d}(n+l+d-1)+l\right]^{-s} \cdot \frac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(l+1)\Gamma(d-1)}$$

and focus on $d = 3 - 2\epsilon$.

Finding the ζ-function

Use the binomial expansion:

$$E(s) = \left(\frac{d}{4}\right)^s \sum_{k,k_1,k_2,k_3=0}^{\infty} \binom{-s}{k}\binom{1}{k_1}\binom{-s-k}{k_2}\binom{-s-k}{k_3} \times \left(-\frac{15}{16}\right)^k \left(-\frac{1}{2}\right)^{k_1} \left(-\frac{1}{4}\right)^{k_2} \left(-\frac{3}{4}\right)^{k_3} \times \zeta_{MT}(k_1+2\epsilon-1; s+k+k_2; s+k+k_3),$$

where the Mordell-Tornheim zeta function $\zeta_{MT}(s_1, s_2, s_3)$ is

$$\zeta_{MT}(s_1, s_2, s_3) \equiv \sum_{n,l=1}^{\infty} l^{-s_1} n^{-s_2} (n+l)^{-s_3}.$$

It leads to

$$E_{Casimir} = \frac{1}{2\sqrt{3}\epsilon} + \text{regular}.$$

Renormalizing the Casimir energy

Pick any operator with $m + n = 2$ and Wilsonian coefficient c:

$$c \cdot \int_{cloud} \mathrm{d}^d x \, \langle \tilde{\mathcal{O}}^{(m,n)}_{bulk} \rangle = c\alpha\mu^{-2\epsilon} = c\alpha \left[1 - 2\epsilon \log \mu + \mathcal{O}(\epsilon^2)\right],$$

where α is a constant. Renormalize c to cancel the pole:

$$\lim_{\epsilon \to 0} \epsilon \cdot \left[E_{Casimir} + c \int_{cloud} \mathrm{d}^d x \, \langle \tilde{\mathcal{O}}^{(m,n)}_{bulk} \rangle \right] = 0,$$

i.e.

$$c = c^{ren.} - \frac{1}{2\sqrt{3}\epsilon} \frac{1}{\alpha},$$

where $c^{ren.}$ is regular. Then,

$$\begin{aligned} E_{Casimir} + c \cdot \int_{cloud} \mathrm{d}^d x \, \langle \tilde{\mathcal{O}}^{(m,n)}_{bulk} \rangle &= \frac{1}{\sqrt{3}} \log \mu + Q^0 \times (\text{regular}) \\ &= \frac{1}{3\sqrt{3}} \log Q + Q^0 \times (\text{regular}) \end{aligned}$$

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Summary

$$\Delta_Q = Q^{\frac{d+1}{d}}\left[a_1 + \frac{a_2}{Q^{\frac{2}{d}}} + \frac{a_3}{Q^{\frac{4}{d}}} + \dots\right] + Q^{\frac{2d-1}{3d}}\left[b_1 + \frac{b_2}{Q^{\frac{2}{3d}}} + \frac{b_3}{Q^{\frac{4}{3d}}} + \dots\right] + Q^{\frac{d-3}{3d}}\left[c_1 + \frac{c_2}{Q^{\frac{2}{3d}}} + \frac{c_3}{Q^{\frac{4}{3d}}} + \dots\right] + \dots$$

where some $b_n = cst + cst \cdot \log Q$ when d is even. In particular,

$$\Delta_Q^{(d=2)} = d_1 Q^{\frac{3}{2}} + d_2\sqrt{Q}\log Q + d_3\sqrt{Q} + d_4 Q^{\frac{1}{6}} - 0.2942 + \dots,$$

and

$$\Delta_Q^{(d=3)} = d_1 Q^{\frac{4}{3}} + d_2 Q^{\frac{2}{3}} + d_3 Q^{\frac{5}{9}} + d_4 Q^{\frac{1}{3}} + d_5 Q^{\frac{1}{9}} + \frac{1}{3\sqrt{3}}\log Q + d_6 + \dots,$$

where we included the leading qu. correction, which is universal.

Outlook

- Include spin [Kravec, Pal '19]
- Gravity dual [Son '08] [Balasubramanian, McGreevy '08]
- BCS-BEC crossover
- Non-Abelian $Sp(N)$ at large-N [Veillette, Sheehy, Radzihovsky '06] [Sachdev, Nikolic '06]

Thanks for the attention!