Large quantum number expansion in O(2N) vector model and Resurgence

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Talk for "IPMU focus week", Bern, September 2021
Based on the arXiv:2102.12488v2

Outline

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What is the Large quantum number expansion?

1. It is limited to theories with global symmetries.
2. It allows the analytic treatment of otherwise inaccessible strogly coupled systems.

The idea

- Study subsectors of the theory with fixed quantum number Q.
- In each sector, a large Q is the controlling parameter in a perturbative expansion.

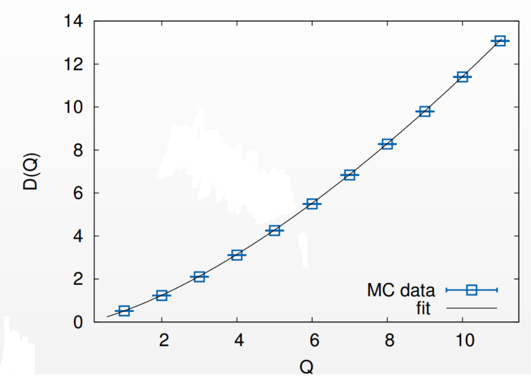
In this talk

- Consider the $O(2N)$ vector model in $2+1$ dimensions.
- Fine tuned, in the IR it flows to a conformal fixed point.
- Use large charge to compute the scaling dimension of the lowest primary operator.

$$\Delta_Q = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi} c_{1/2} Q^{1/2} - 0,094 + \mathcal{O}(Q^{-1/2}).$$

What you should remember

1. The 0.094 is a prediction of the theory.
2. The large charge expansion seems to work for small charges!



- **Why**?

- No reason for the large charge expansion to work at small Q from the EFT point of view.
- Can add another controlling parameter, e.g. large N and go beyond the EFT.
- Use the double scaling limit:
  $Q \to \infty,\ N \to \infty,\ Q/(2N) = \hat{q} = \text{constant}$
  to solve the problem exactly.
- The expansion is asymptotic.
- Asymptotic series = non-perturbative phenomena = resurgent asymptotics.

Outline

- Start with the Landau-Ginzburg model for N complex fields on $\mathcal{R} \times \mathcal{M}$

$$S[\phi_i] = \sum_{i=1}^{N} \int dtd\mathcal{M} \left[g^{\mu\nu} (\partial_\mu \phi_i)^* (\partial_\nu \phi_i) + r \phi_i^* \phi_i + \frac{u}{2} (\phi_i^* \phi_i)^2 \right]$$

  The system flows to a Wilson-Fisher fixed point in the IR, i.e. $u \to \infty$, when r is fined tuned to the conformal coupling, i.e. $r = R/8$.

- Work in sector of fixed charge Q.

In the limit:

$$N \to \infty, \qquad Q \to \infty, \qquad \hat{q} = Q/(2N) = \text{fixed}$$

express the free energy as the Legendre transform of a zeta function.

$$f(\hat{q}) = F/(2N) = \sup_{\mu}(m\hat{q} - \omega(\mu)) \qquad \text{Free energy per dof}$$

$$\hat{q} = \frac{d(\omega(\mu))}{d\mu} \qquad \text{Charge}$$

$$\omega(\mu) = -\frac{1}{2}\zeta\Big(-\frac{1}{2}\Big|\mathcal{M}, \mu\Big) \qquad \text{Grand potential}$$

where μ is the chemical potential.

• $\zeta\big(s\big|\mathcal{M},\mu\big)$ is the Hurwitz zeta function of the operator $-\Delta+\mu^2$.
• In the Mellin representation

$$\zeta\big(s\big|\mathcal{M},\mu\big)=\frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}\ e^{-\mu^2 t}\ \mathrm{Tr}\big(e^{\Delta t}\big).$$

• Large $\hat{q}$ is large μ. Can be written in terms of Seeley-DeWitt coefficients:

$$\mathrm{Tr}\Big(e^{\Delta t}\Big)\sim\frac{\mathcal{V}}{4\pi t}\bigg(1+\frac{R}{12}t+\dots\bigg).$$

Outline

- First let's take $\mathcal{M} = T^2$.
- All but the first Seeley-DeWitt coefficients vanish, hence:

$$\mathrm{Tr}\Big(e^{\Delta t}\Big) \sim \frac{L^2}{4\pi t} + \mathcal{O}(e^{-1/t}), \qquad \zeta\big(s\big|\, T^2, \mu\big) = \frac{L^2 \mu^{2(1-s)}}{4\pi(s-1)} + \mathcal{O}(e^{\mu})$$

- Use these to derive:

$$\omega(\mu) = \frac{L^2\mu^3}{12\pi}, \qquad \hat{q} = \frac{L^2\mu^2}{4\pi}, \qquad f(\hat{q}) = \frac{4\sqrt{\pi}}{3L}\hat{q}^{3/2}.$$

- We can do better, closed form expression of the corrections.

- Through the spectrum:

$$\mathrm{spec}\left(\Delta_{T^2}\right) = \left\{ -\frac{4\pi^2}{L^2}\left({k_1}^2 + {k_2}^2\right) \Big| k_1, k_2 \in \mathbb{Z} \right\}.$$

- The heat kernel trace is

$$\mathrm{Tr}\left(e^{\Delta t}\right) = \sum_{k_1, k_2 \in \mathbb{Z}} e^{-\frac{4\pi^2}{L^2}\left({k_1}^2 + {k_2}^2\right)t} = \left[\theta_3(0, e^{-\frac{4\pi^2}{L^2}t})\right]^2$$

the square of a theta function.

- For t small we can use Poisson resummation

$$\sum_{n\in\mathbb{Z}} h(n) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} d\rho h(\rho) e^{2\pi i k\rho}$$

and expand around $t \to 0^+$.

- Then

$$\mathrm{Tr}\left(e^{\Delta t}\right) = \left[\frac{L}{\sqrt{4\pi t}}\left(1+\sum_{k\in\mathbb{Z}} e^{-\frac{k^2L^2}{4t}}\right)\right]^2 = \frac{L^2}{4\pi t}\left(1+\sum_{k\in\mathbb{Z}^2} e^{-\frac{\|k\|^2L^2}{4t}}\right)$$

- Now we can find the subleading contribution to the grand potential and free energy

$$\begin{aligned} f(\hat{q}) &= \sup_{\mu}(m\hat{q}-\omega(\mu)) = \frac{4\sqrt{\pi}}{3L}\hat{q}^{3/2}\left(1-\sum_k \frac{e^{-\|k\|\sqrt{4\pi\hat{q}}}}{8\|k\|^2\pi\hat{q}}+\dots\right), \\ \omega(\mu) &= -\frac{1}{2}\zeta\left(-\frac{1}{2}\middle|T^2,\mu\right) = \frac{L^2\mu^3}{12\pi}\left(1+\sum_k \frac{e^{-\|k\|\mu L}}{\|k\|^2\mu^2L^2}\left(1+\frac{1}{\|k\|\mu L}\right)\right). \end{aligned}$$

Outline

- Now let's study the sphere of radius r, $\mathcal{M} = S^2$.
- The spectrum is

$$\mathrm{spec}(\Delta_{S^2}) = \left\{ -\frac{\ell(\ell+1)}{r^2} \middle| \ell \in \mathbb{N} \right\},$$

with multiplicity $2\ell + 1$.
- Again use Poisson resummation to write the heat kernel as follows

$$\begin{aligned} \mathrm{Tr}\left[e^{\left(\Delta_{S^2} - \frac{1}{4r^2}\right)t} \right] &= \sum_{\ell=0}^{\infty} (2\ell+1) e^{-\frac{t}{r^2}(\ell+\frac{1}{2})^2} \\ &= \frac{r^2}{t} + \sum_{k\in\mathbb{Z}} (-1)^k \left[\frac{r^2}{t} - \frac{2|k|\pi r^3}{t^{3/2}} F\left(\frac{\pi r |k|}{\sqrt{t}}\right) \right], \end{aligned}$$

where

$$F(z) = e^{-z^2} \int_0^z dt e^{-t^2} = \frac{\sqrt{\pi}}{2} e^{-z^2} \mathrm{erfi}(z).$$

- For small t, we can use the asymptotic expansion of $F(z)$

$$F(z) \sim \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^{n+1}} \left(\frac{1}{z}\right)^{2n+1}.$$

and

$$\mathrm{Tr}\left[e^{\left(\Delta_{S^2}-\frac{1}{4r^2}\right)t}\right] = \frac{r^2}{t} - \sum_{n=1}^{\infty} \frac{(-1)^n(1-2^{1-2n})}{n!\,r^{2n-2}} B_{2n} t^{n-1} \equiv \frac{r^2}{t} \sum_{n=0}^{\infty} \alpha_n \left(\frac{t}{r^2}\right)^n.$$

- The series is asymptotic since the Seeley-DeWitt coefficients diverge like $n!$:

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) \to \alpha_n = \frac{(-1)^{n+1}(1-2^{1-2n})}{n!} B_{2n} \sim \frac{2}{\sqrt{\pi}} \frac{n^{-1/2}}{\pi^{2n}} n!.$$

- We assume that this series can be completed into a resurgent trans-series.

• In general a trans-series solution with a small parameter z has the form for $z \to 0$:

$$\Phi(\sigma_k, z) = \Phi^{(0)}(z) + \sum_{k \neq 0} \sigma_k e^{-A_k/z^{1/\beta_k}} z^{-b_k/\beta_k} \Phi^{(k)}(z).$$

• The coefficients of the non-perturbative part are encoded in the large-order behavior of the perturbative series:

$$\alpha_n \sim \sum_k \frac{S_k}{2\pi i} \frac{\beta_k}{A_k^{n\beta_k + b_k}} \sum_{\ell=0}^{\infty} \alpha_\ell^{(k)} A_k^\ell \Gamma(\beta_k n + b_k - \ell),$$

where S_k are Stokes constants.

• In this case we have complete knowledge of the α_n and we write them in the suggestive form:

$$\alpha_n = -\frac{1}{\sqrt{\pi}} \sum_{k \neq 0} (-1)^k \frac{\Gamma(n+\frac{1}{2})}{(\pi k)^{2n}}.$$

• Comparing the two expressions

$$\beta = 1, \quad b_k = \frac{1}{2}, \quad A_k = (\pi k)^2, \quad \frac{S_k}{2\pi i} \alpha_0^{(k)} = (-1)^{k+1} |k| \sqrt{\pi}, \quad \alpha_{>0}^{(k)} = 0.$$

• The series around each exponential are truncated to only one term and the heat trace has to contain the terms

$$\mathrm{Tr}\left[e^{\left(\Delta_{S^2} - \frac{1}{4r^2}\right)t} \right] \supset 2i \left(\frac{\pi r^2}{t} \right)^{\frac{3}{2}} (-1)^{k+1} |k| e^{-(k\pi r)^2/t}.$$

• The result is defined up to a k-dependent complex constant hence resurgence leaves us with an ambiguity in the non-perturbative contribution.

- The ambiguity can be resolved in two ways:
  - A resurgent analysis of the Dawson's function using the Borel resummation.
  - Using a geometric intepretation in terms of worldline instantons.
- For the later, the key is to write the heat trace as a path integral over closed loops

$$\mathrm{Tr}\left[e^{\Delta t}\right] \equiv \int_{x(t)=x(0)} \mathcal{D}x^{\mu} e^{-S[x]},$$

where

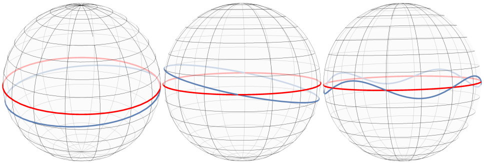
$$S[x] = \frac{1}{4}\int_0^t d\tau g_{\mu\nu}(x)\dot{x}^{\mu}(\tau)\dot{x}^{\nu}(\tau).$$

• In the case of the sphere $x^\mu = (\theta, \phi)$ the EOMs are

$$\begin{aligned} S &= \frac{r^2}{4t}\int_0^1 d\tau \left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right], \\ \ddot{\phi} &+ 2\cot(\theta)\dot{\theta}\dot{\phi} = 0, \\ \ddot{\theta} &- \dot{\phi}^2\sin(2\theta) = 0. \end{aligned}$$

• We can solve them to find the classical solutions and then add fluctuations around them.

• Solving these we can see that there is a zero mode and multiple negative modes.



- In both cases the final, real and unambiguous result is

$$\mathrm{Tr}\left[e^{\left(\Delta_{S^2}-\frac{1}{4r^2}\right)t}\right] = \frac{2}{\sqrt{\pi}}\left(\frac{r^2}{t}\right)^{\frac{3}{2}}\int_{\mathcal{C}_\pm} d\zeta \frac{\zeta e^{-\zeta^2 r^2/t}}{\sin\zeta} \pm i\left(\frac{\pi r^2}{t}\right)^{\frac{3}{2}}\sum_{k\neq 0}(-1)^{k+1}|k|e^{-\frac{k^2\pi^2 r^2}{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{r^2}{t}\right)^{\frac{3}{2}}\mathrm{P.V.}\left[\int_{\mathcal{C}_\pm} d\zeta \frac{\zeta e^{-\zeta^2 r^2/t}}{\sin\zeta}\right]$$

- Now we can write the exact expression of the grand potential and numerically compare it with the convergent small-charge expansion. They agree to at least eight digits!

Outline

- The large-charge expansion of the Wilson–Fisher point is asymptotic.
- In the double-scaling limit $Q \to \infty$, $N \to \infty$ we control the perturbative expansion.
- We have a geometric intepretation of the non-perturbative corrections.
- We can propose an exact form of the grand potential valid for any value of $\hat{q}$.
- The fact that non-perturbative corrections are finite-volume effects motivates us to extend our results to large charge but finite N.
- We conjecture that the large-charge expansion is always asumptotic with an optimal truncation of $N^* = \mathcal{O}(\sqrt{Q})$, consistent with the lattice results.

Thank you!