# The O(N) model at large charge and the quartic/cubic equivalence

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Based on: O. Antipin, JB, F. Sannino, Z. W. Wang and C. Zhang, arXiv:2107.02528[hep-th]
**More on the cubic versus quartic interaction equivalence in the** O(N) **model**

# Outline

I will discuss the O(N) CFT in $4<d<6$ dimensions and use the large-charge expansion to test a proposed dual description of this conformal theory.

# The quartic O(N) model

Consider the massless quartic O(N) theory in d dimensions

$$\mathcal{S} = \int d^d x \left( \frac{(\partial\phi_i)^2}{2} + \frac{(4\pi)^2 g_0}{4!} (\phi_i\phi_i)^2 \right)$$

Evidence for an infrared fixed point in 2<d<4.

Just below d=4, the O(N) CFT is weakly coupled:

$$d = 4 - \epsilon \longrightarrow g^* = \frac{3\epsilon}{8+N}$$

$d = 3$ Superfluid $He^4$, Magnets, Superconductors, ..

# The quartic O(N) model

Consider the massless quartic O(N) theory in d dimensions

$$\mathcal{S} = \int d^d x \left( \frac{(\partial \phi_i)^2}{2} + \frac{(4\pi)^2 g_0}{4!} (\phi_i \phi_i)^2 \right)$$

Above d=4 the quartic interaction becomes RG irrelevant at the Gaussian fixed point: free field theory in the IR.

In this range of dimensions, the model is usually studied in the 1/N expansion.

At large N the model flows to an interacting fixed point in the UV and the theory is renormalizable order by order in the 1/N expansion.

(G. Parisi 1975)

# The O(N) model in 4<d<6

The scaling dimension at the UV FP of the $\Phi_i\Phi_i$ operator is

$$\Delta_{\phi_i\phi_i} = 2 + \mathcal{O}\left(\frac{1}{N}\right)$$

and violates the unitarity bound (d/2-1) when d>6.

<u>The UV FP may be unitary in 4<d<6</u>

Non-SUSY 5D CFT

AdS/CFT: O(N) model in d dimensions/higher-spin theories on $AdS_{d+1}$

(I. R. Klebanov and A. M. Polyakov 2002)

# The O(N) model in 4<d<6

However… there are rigorous results about the triviality of the quartic interaction in d>4.

(M. Aizeman 1981 - O. J. Rosten 2009 - R. Percacci and G. P. Vacca 2014)

**The UV FP is metastable**

The instability is realized by instantonic effects which give rise to complex CFT data.

(S. Giombi, R. Huang, I. R. Klebanov, S. Pufu, G. Tarnopolsky 2020)

Instantons are suppressed at large N. When the imaginary part of the CFT data is negligible the UV FP can be studied via the conformal bootstrap.

(J. B. Bae and S. J. Rey 2014 - S. M. Chester, S. S. Pufu and R. Yacoby 2015)

# The cubic O(N) model

Conjecture: in 4<d<6 the UV fixed point of the quartic O(N) model is equivalent to the IR fixed point of the cubic O(N) model below

(L. Fei, S. Giombi, I. R. Klebanov 2014)

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 + \frac{1}{2}(\partial\eta)^2 + \frac{g_0}{2}\eta(\phi_a)^2 + \frac{h_0}{6}\eta^3$$

N+1 fields and cubic interactions. Weakly coupled just below d=6.

This model is usually investigated in d=6-ε via the ε-expansion.

The cubic FP exists above $N_{\rm crit} \approx 400$ (5 loops Padé)

(M. Kompaniets and A. Pikelner 2021)

$$g^* \equiv \sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1 + \frac{22}{N} + \frac{726}{N^2} + \ldots + \mathcal{O}\left(\epsilon\right)\right) ,$$

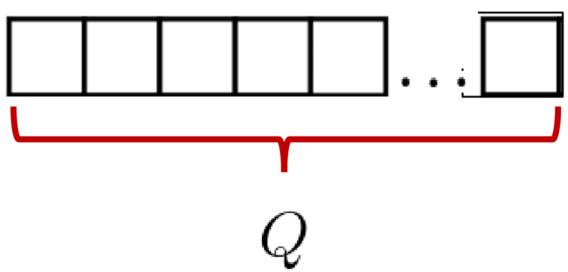
$$h^* \equiv 6\sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1 + \frac{162}{N} + \frac{68766}{N^2} + \ldots + \mathcal{O}\left(\epsilon\right)\right) .$$

Matching of various scaling dimensions and OPE coefficients considering the overlapping terms between the 1/N and the ε-expansion.

# Fixed-charge operators

We consider the scaling dimension $\Delta_Q$ of the lowest-lying operators carrying total charge Q.

These operators have classical scaling dimension Q and transform in the Q-indices traceless symmetric O(N) representations.




$$Q = 1 \to \phi_i \qquad\qquad Q = 2 \to \phi_i\phi_j - \frac{\delta_{ij}}{N}\phi^2$$

These operators represent anisotropic perturbations in O(N)-invariant systems. $\Delta_Q$ defines a set of crossover (critical) exponents measuring the stability of the system (e.g. magnets) against anisotropic perturbations (e.g. crystal structure).

# The large-charge sector

The contribution to $\Delta_Q$ coming from the exponentiation of the diagrams with the leading charge-scaling at every loop order matches between the quartic and the cubic theory.

$$\Delta_Q = 2Q - \frac{3\epsilon Q^2}{N}$$

(G. Arias-Tamargo, D. Rodriguez-Gomez, J. G. Russo 2020)

Computation of $\Delta_Q$ in the quartic model for arbitrary d in the double-scaling limit

$$N \to \infty \, , Q \to \infty \, , J \equiv Q/N \text{ fixed.}$$

$$\Delta_Q = \sum_{k=-1} \frac{1}{N^k} F_k(J)$$

$F_{-1}$ is known.

(S. Giombi and J. Hyman 2020)

# Complex scaling dimensions

At large charge the instantons contribution are suppressed as $e^{-Q}$

(G. Arias-Tamargo, D. Rodriguez-Gomez, J. G. Russo 2020)

**is the large-charge sector of the theory stable?**

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Above a critical value of the charge, the scaling dimensions become complex.

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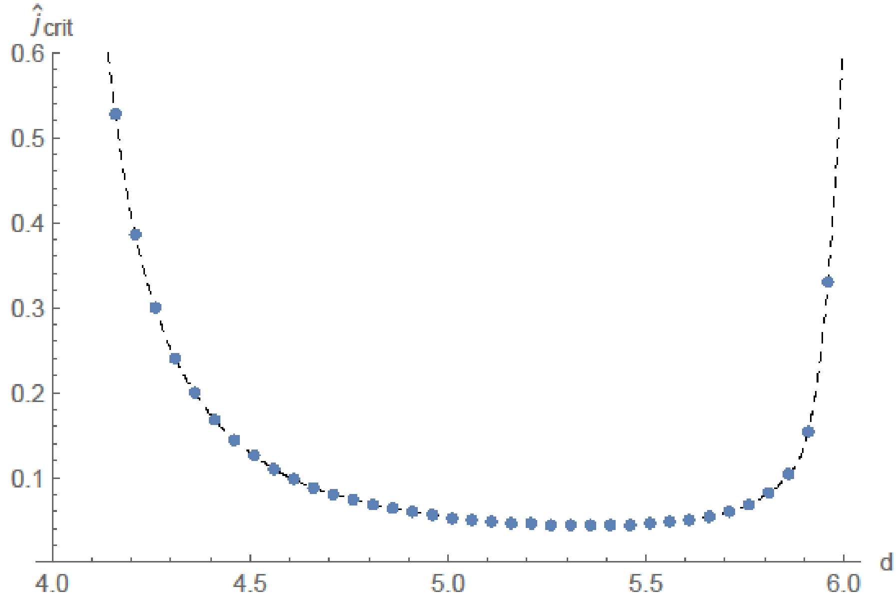
# The critical charge

Above a critical value of the charge the scaling dimensions become complex.

Numerical estimation of the critical charge at the leading order in the double scaling limit

$$N \to \infty\ , Q \to \infty\ , J \equiv Q/N \text{ fixed.}$$




(S. Giombi and J. Hyman 2020)

# The cubic model at large charge

We consider the cubic model in d=6-ε and the double-scaling limit

$$\epsilon \to 0 \ , Q \to \infty \ , \mathcal{A} \equiv Q\epsilon \ \text{fixed.}$$

$$\Delta_Q = \sum_{k=-1} \frac{1}{Q^k} \Delta_k(\mathcal{A})$$

We compute $\Delta_{-1}$ and $\Delta_0$ and compare with the known large N results in the quartic theory.

# Comparison

**Cubic**

$$\Delta_Q = \sum_{k=-1} \frac{1}{Q^k} \Delta_k(\mathcal{A})$$

$$\mathcal{A} \equiv Q\epsilon$$

We compute $\Delta_{-1}$ and $\Delta_0$.

**Quartic**

$$\Delta_Q = \sum_{k=-1} \frac{1}{N^k} F_k(J)$$

$$J \equiv Q/N$$

$F_{-1}$ is known.

(S. Giombi and J. Hyman 2020)

The overlapping terms between the two expansions are

- $\alpha_j Q \left(\frac{Q\epsilon}{N}\right)^j, \quad j \geq 0$ Match between $\Delta_{-1}$ and $F_{-1}$
- $\beta_j \left(\frac{Q\epsilon}{N}\right)^j, \quad j \geq 0$ Match between $\Delta_0$ and $F_{-1}$

# Semiclassical computation

We use the state-operator correspondence on the cylinder and compute $\Delta_Q$ in a semiclassical expansion around the fixed-charge ground state.

$$\Delta_Q = \sum_{k=-1} \frac{1}{Q^k} \Delta_k(\mathcal{A})$$

To simplify the computation, we use that $\Delta_Q$ depends only on the total charge and not on the values taken by the individual O(N) charges. Thus we fix only one charge to Q.

$$< Q|e^{-HT}|Q > \underset{T\to\infty}{=} \tilde{\mathcal{N}} e^{-E_Q T} = \tilde{\mathcal{N}} e^{-\frac{\Delta_Q}{R} T}$$

# The ground state

Introducing N/2 complex field variables as

$$\varphi_j = \frac{1}{\sqrt{2}}(\phi_{2j-1} + i\phi_{2j}) \qquad \varphi_1 = \frac{1}{\sqrt{2}}\rho e^{i\chi}$$

The classical solution to the EOM reads

$$\rho = f \ , \quad \chi = -i\mu\tau \ , \quad \eta = v$$

$$\varphi_i = 0 \qquad i = 2, \ldots, N/2$$

where

- $\mu^2 - m^2 = g_0 v$
- $\frac{g_0}{2} f^2 + \frac{h_0}{2} v^2 + m^2 v = 0$
- $\frac{Q}{\Omega_{d-1} R^{d-1}} = \mu f^2$

# The leading order $\Delta_{-1}$

The leading order $\Delta_{-1}$ of the semiclassical expansion is given by the classical energy on the cylinder and reads

$$Q\frac{\Delta_{-1}}{R} = -\frac{f^2\mu^2}{2} + \frac{g_0 v f^2}{2} + \frac{h_0 v^3}{6} + \frac{m^2 f^2}{2} + \frac{m^2 v^2}{2} + \frac{Q\mu}{\Omega_{d-1}R^{d-1}}$$

In the "small-charge" regime we obtain

$$Q\epsilon \ll 1$$

$$\begin{aligned} Q\Delta_{-1} = 2Q &- \frac{\epsilon Q^2}{N}\left(3 + \frac{132}{N} + \frac{5808}{N^2} + \dots\right) \\ &- \frac{Q^3\epsilon^2}{N^2}\left(45 + \frac{9000}{N} + \dots\right) - \frac{Q^4\epsilon^3}{N^3}\left(1350 + \frac{495720}{N} + \dots\right) \\ &- \frac{Q^5\epsilon^4}{N^4}\left(\frac{213597}{4} + \frac{28653588}{N} + \dots\right) + \dots \end{aligned}$$

The red terms match between the cubic and the quartic model.

# The NLO $\Delta_0$

$$Q\epsilon \ll 1$$

$$Q\Delta_0 = -Q\epsilon\left[\frac{1}{2} + \mathcal{O}\left(\frac{1}{N}\right)\right] + \frac{(Q\epsilon)^2}{N}\left[\frac{7}{4} + \mathcal{O}\left(\frac{1}{N}\right)\right] + \frac{(Q\epsilon)^3}{N^2}\left[\frac{3}{4}(48\zeta(3) + 31) + \mathcal{O}\left(\frac{1}{N}\right)\right] + \frac{(Q\epsilon)^4}{N^3}\left[\frac{27}{2}(128\zeta(3) + 40\zeta(5) + 41) + \mathcal{O}\left(\frac{1}{N}\right)\right] + \dots$$

The red terms match between the cubic and the quartic model.

Considering a “large-charge” double-scaling limit it is possible to compare an infinite series of terms between the two theories.

# The large-charge regime

$$Q\epsilon \gg 1$$

$$\Delta_Q = -e^{\pm i4\pi/5}\frac{5N}{3}(2\epsilon)^{1/5}J^{6/5}+e^{\pm i\pi/5}\frac{5N}{6}(2\epsilon)^{-1/5}J^{4/5} - e^{\pm 3i\pi/5}\frac{N}{9}(2\epsilon)^{-3/5}J^{2/5}+\mathcal{O}\left(J^0\right)$$

$$J \equiv Q/N$$

The above matches the quartic model result and illustrates the emergence of complex CFT data in the large-charge limit.

# Complex CFT data

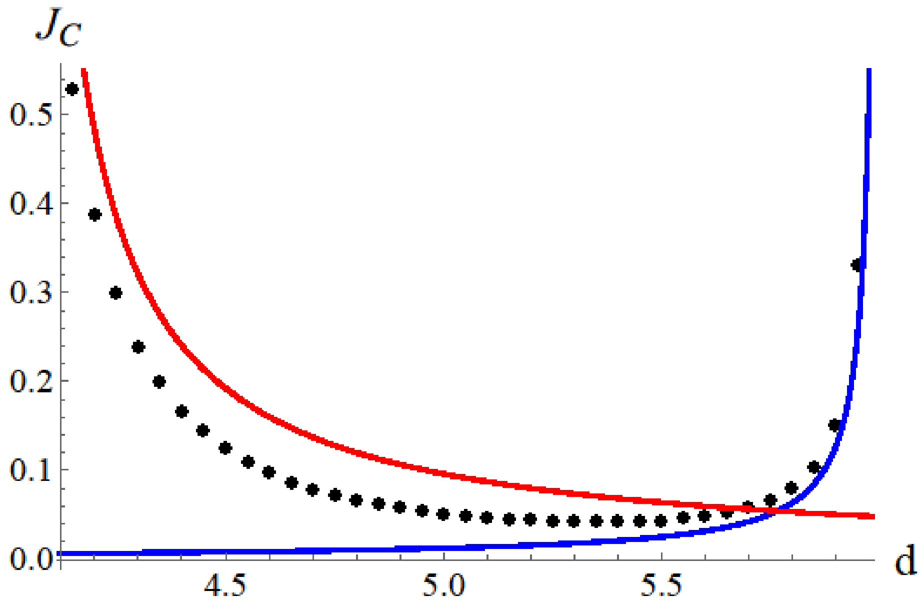
We can use our results to study the onset of complex CFT dynamics above a certain critical value of the charge $Q_c$.

We compute $Q_c$ at the leading order the ε-expansion for the quartic model in $d=4+\varepsilon$ and for the cubic model in $d=6-\varepsilon$.

Above $Q_c$ there are no real solutions to the saddle point equations.

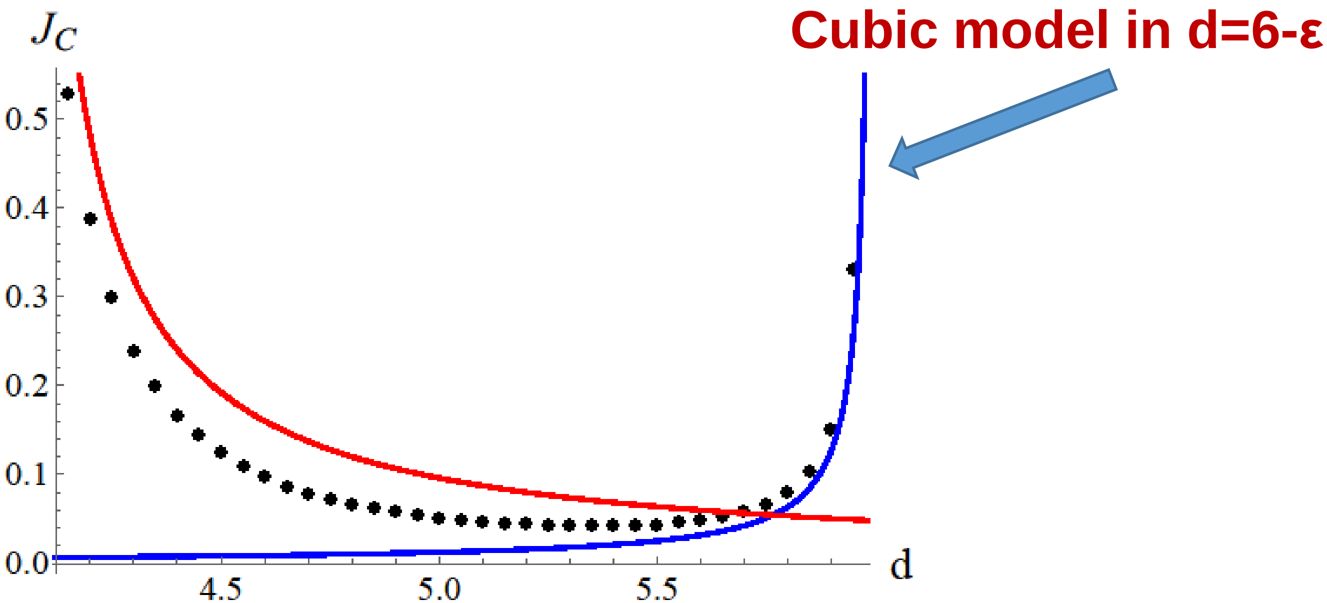
# Complex CFT data

$J_c \equiv Q_C/N$



# Complex CFT data

$$J_c \equiv Q_C/N \qquad Q_c = \frac{N}{90\epsilon}\left(-9+\sqrt{105}\right)\sqrt{\frac{1}{30}\left(15+\sqrt{105}\right)}$$



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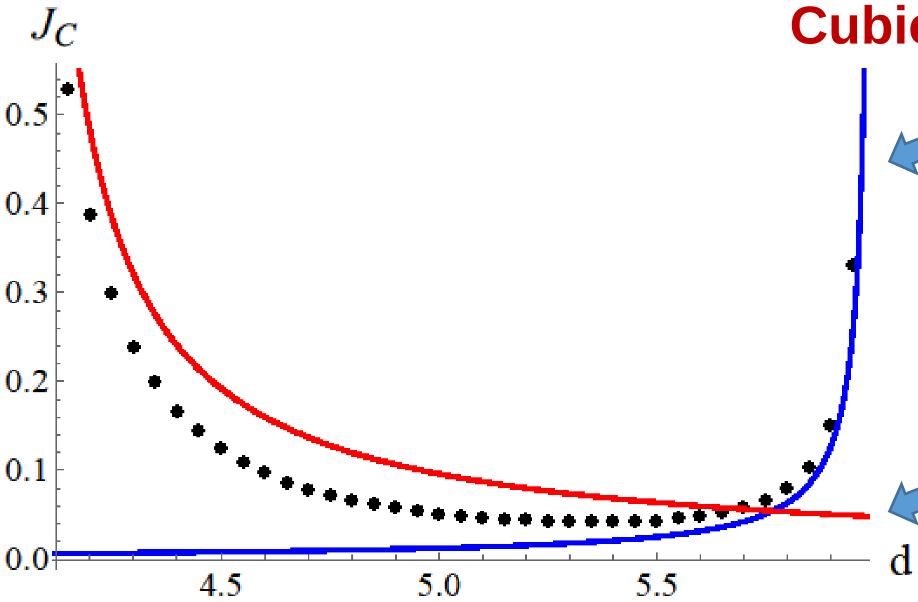


**Cubic model in d=6-ε**

**Quartic model in d=4-ε**

$$Q_c = \frac{N+8}{6\sqrt{3}\epsilon}$$

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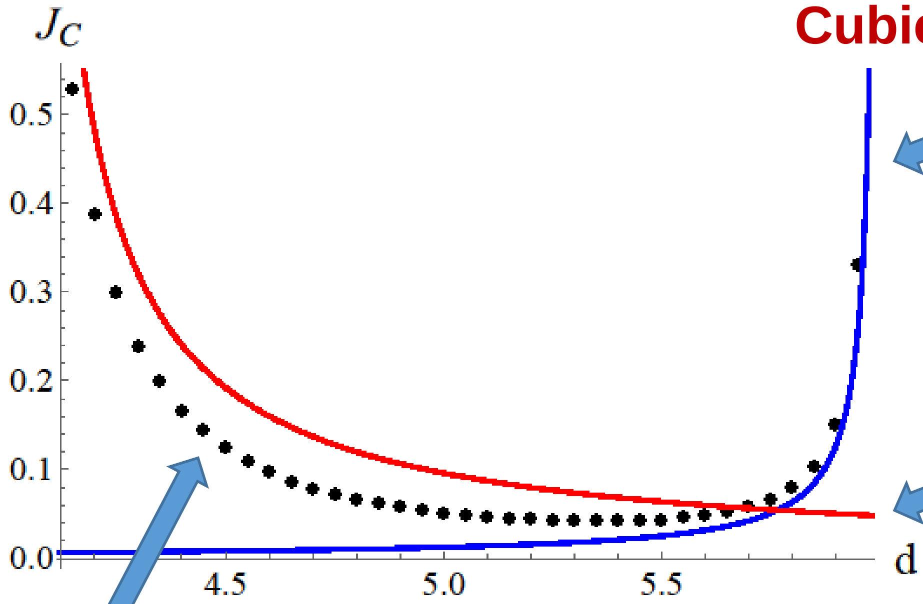
**Cubic model in d=6-ε**




**Quartic model in d=4-ε**

$$Q_c = \frac{N+8}{6\sqrt{3}\epsilon}$$

**Numerical estimation in the quartic model at large N**

(S. Giombi and J. Hyman 2020)