Conformal Dimensions in the Sub-Leading Large Charge Sector of the O(4) Model

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Focus Week on
QM Systems at Large Quantum Numbers
http://www.reffert.itp.unibe.ch/largecharge/about/

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Conformal Dimensions: Sub-Leading Sector

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Introduction

Conformal field theories are characterized by dimensionless numbers.

Example: $$\left\langle \mathcal{O}_Q(y)\, \mathcal{O}_Q^\dagger(x) \right\rangle \sim \left(\frac{1}{|x-y|}\right)^{2D_Q}$$

D_Q is the conformal dimension of the operator $\mathcal{O}_Q$

Q labels some set of quantum numbers

Usually no small parameters in the theory, so difficult to compute D_Q

Non-perturbative methods: Monte Carlo, bootstrap,….

Perturbative methods: ε-expansion, large N, large Q, …

Large Q Prediction: expansion of the form

$$D_Q = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} + c_0 + \mathcal{O}(1/Q^{1/2})$$

$c_{3/2}, c_{1/2}$ are unkown low energy constants. c_0 is related to the Casimir energy and can be computed analytically

Hellerman, Orlando, Reffert, Watanabe, JHEP 12 (2015) 71.
Alvarez-Gaume, Loukas, Orlando, Reffert, JHEP 4 (2017) 59.

Q: How well does this approach work in practice?

A: Our goal is to compute D_Q using a Monte Carlo method and check!

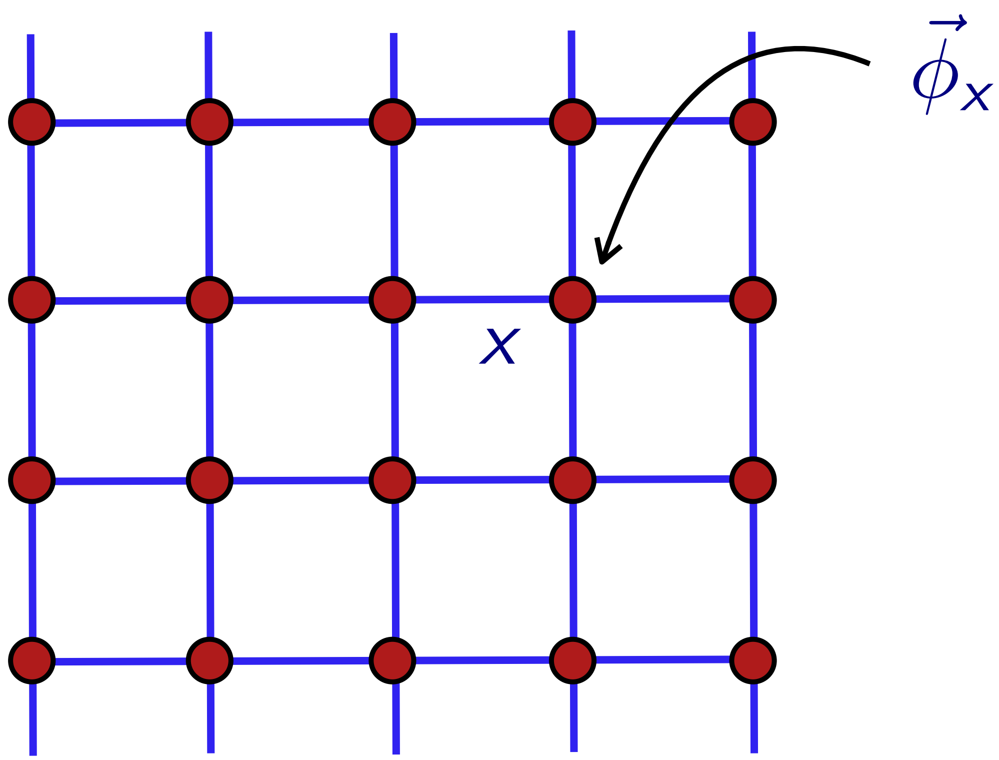
Challenge: Computing D_Q using Monte Carlo methods suffers from severe signal to noise ratio problems with conventional methods for large Q.

related to the sign problem that arise due to chemical potential

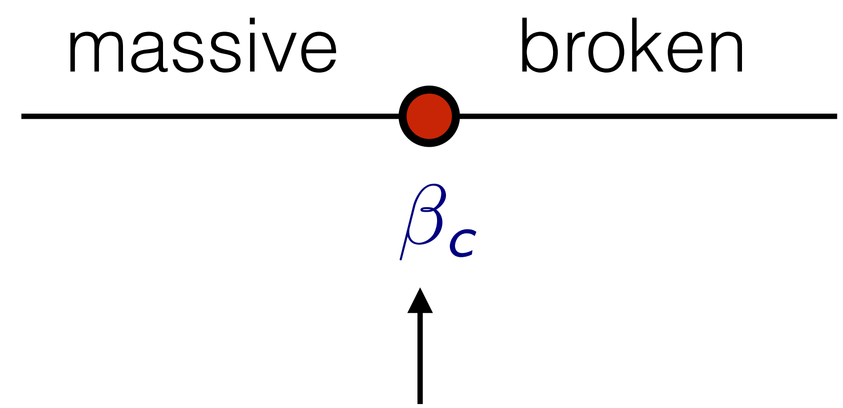
Traditional O(4) model: Large Charge Predictions

Traditional model

$$Z = \int [d\vec{\phi}]\, e^{\beta \sum_{x,\alpha} \vec{\phi}_x \cdot \vec{\phi}_{x+\alpha}}$$



3d lattice



Conformal Field Theory at the Wilson-Fisher fixed point

Global charges are now labeled with

$$Q \equiv (j_L, j_R)$$

since O(4) ~ SU(2) x SU(2).

Conformal Dimensions $D(j_L, j_R)$

$$\left\langle O_x^{(j_L,j_R)}\ O_y^{(j_L,j_R)} \right\rangle \sim \left(\frac{1}{|x-y|}\right)^{2D(j_L,j_R)}$$

Constructing $O_x^{(j_L,j_R)}$ requires the use of group theory.

Start with $O_x^{(1/2,1/2)} \equiv \vec{\phi}_x$

$$\phi_x^\alpha \phi_x^\beta \equiv (1/2, 1/2) \otimes (1/2, 1/2)$$

$$\equiv (0,0) \oplus (1,0) \oplus (0,1) \oplus (1,1)$$

$$O_x^{(1,0)} \equiv \sum_{\alpha\beta} \vec{C}_{\alpha\beta}\ \phi_x^\alpha\ \phi_x^\beta$$

Clebsch-Gordon Coefficients

Generally

$$O_x^{(j_L,j_R)} = \hat{P}_{(j_L,j_R)}\left([\vec{\phi}_x]^k\right)$$

Large charge predictions:

Leading Sector: $j_L = j_R = j$ Large j

$$D(j,j) \;=\; \sqrt{\frac{2j^3}{\pi}}\left(c_{3/2} + \frac{2\pi}{j}c_{1/2} + \mathcal{O}(1/j^2)\right) + c_0$$

$$c_0 = -0.094\ldots$$

Alvarez-Gaume, Loukas, Orlando, Reffert, JHEP 4 (2017) 59.

Subleading Sector: $j_L = j_R + 1 = j$ Large j

$$D(j,j-1) \;=\; D(j,j) \;+\; \tilde{\Delta}(j)\,, \qquad \tilde{\Delta}(j) \;=\; \frac{\lambda^2}{j^{1/2}} \;+\; \mathcal{O}(1/j^{3/2})$$

Alvarez-Gaume, Orlando, Reffert, arXiv:2008.03308.

λ^2 is one new low energy constant.

Interesting prediction

$$\lim_{j\to\infty} \tilde{\Delta}(j) \;=\; 0$$

Sign Problems:

Computing conformal dimensions for large charges, using the traditional model suffers from sign problems.

$$\left\langle O_x^{(j_L,j_R)} \; O_y^{(j_L,j_R)} \right\rangle \sim \left(\frac{1}{|x-y|} \right)^{2D(j_L,j_R)}$$

At large charges

$D(j_L, j_R) \to \infty$

cancellations from fluctuations

small

A new approach is desirable!

Qubit Regularized O(4) model: Loop Gas

From a quantum mechanical perspective, the traditional lattice model has an infinite dimensional Hilbert space at every lattice site.

Basis states:

completeness relation:

Position: $|\vec{\phi}_x\rangle$

$$\int [d\vec{\phi}_x] \;\; |\vec{\phi}_x\rangle\langle\vec{\phi}_x| \;=\; I$$

Momentum: $|\ell, m_1, m_2\rangle_x$

$$\sum_{\ell=0,1/2,1,3/2,\ldots} |\ell, m_1, m_2\rangle\langle\ell, m_1, m_2| \;=\; I$$

Qubit Regularization:
Lattice models with a finite dimensional Hilbert space
at every lattice site.

Simplest qubit regularization involves a five dimensional Hilbert space:

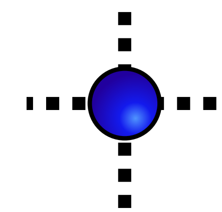
$$|\ell = 0, m_1 = 0, m_2 = 0\rangle_x$$

$$|s, \mathbf{r}\rangle$$

Fock vacuum

$(j_L \ = \ 0, \ j_R \ = \ 0)$ sector

(Monomers)



(0,0)

$$|\ell = 1/2, m_1 = \pm 1/2, m_2 = \pm 1/2\rangle$$

$$|q_L^z, q_R^z, \mathbf{r}\rangle, \quad q_L^z, q_R^z = 1/2, -1/2$$

O(4) vector

$(j_L \ = \ 1/2, \ j_R \ = \ 1/2)$ sector

vector particles

(1/2,1/2) (-1/2,-1/2)

(1/2,-1/2) (-1/2,1/2)

labeled by (q_L^z, q_R^z)

Partition Function: Loop Gas

$$Z = \sum_{[n,b]} U^{N_m([n,b])}$$

$N_m([n,b])$ number of monomers in the configuration [n,b]

Can again study using worm algorithms.

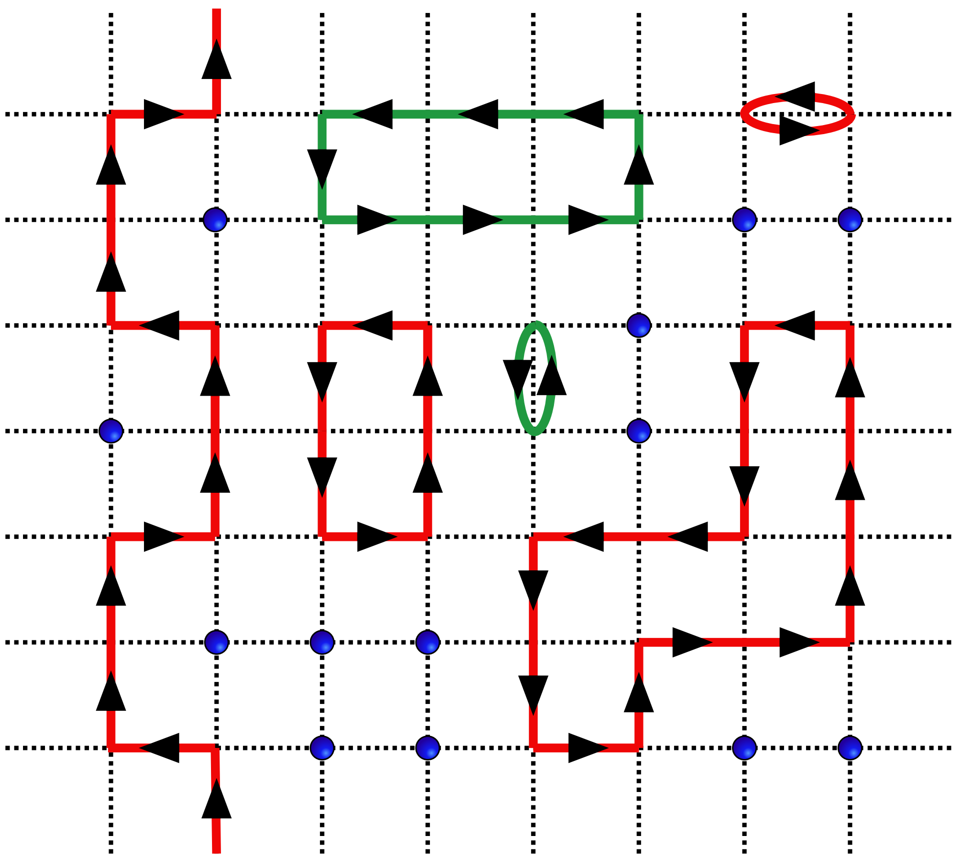
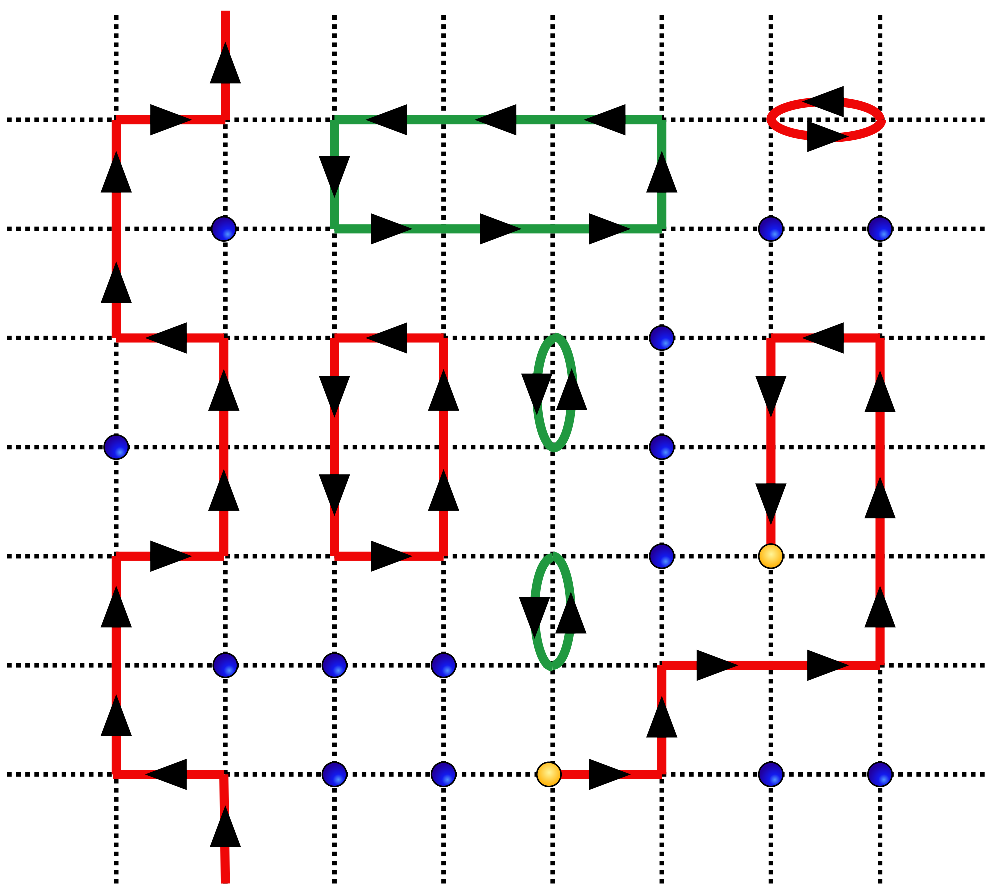


Illustration of a [n,b] configuration

Correlation Functions



Inserting sources/sinks

Observables

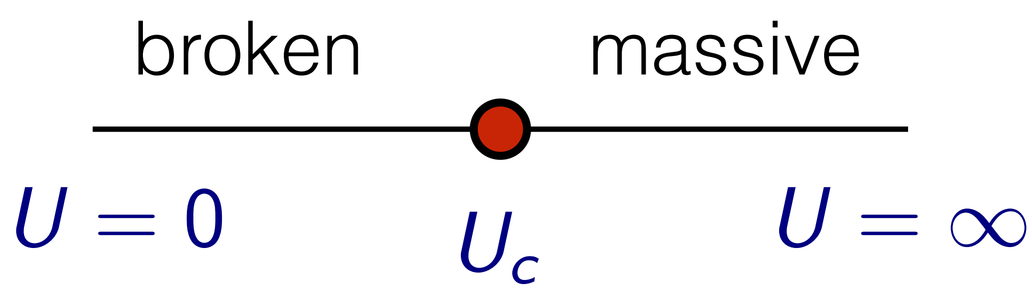
Order Parameter Suceptibility

$$\chi = \frac{1}{ZL^d}\sum_{\mathbf{r},\mathbf{r}'}\int_0^\beta dt\ \mathrm{Tr}\Big(e^{-(\beta-t)H}a_{\mathbf{r},\mathbf{m}}e^{-tH}a^\dagger_{\mathbf{r}',\mathbf{m}}\Big)$$

Winding Number Susceptibility

$$\rho_s = \frac{1}{L^{d-2}\beta}\langle\ (Q_w)^2\ \rangle$$

Phase Diagram



Near the critical point we expect

$$\chi / L^{2-\eta} = f((U - U_c) L^{1/\nu})$$

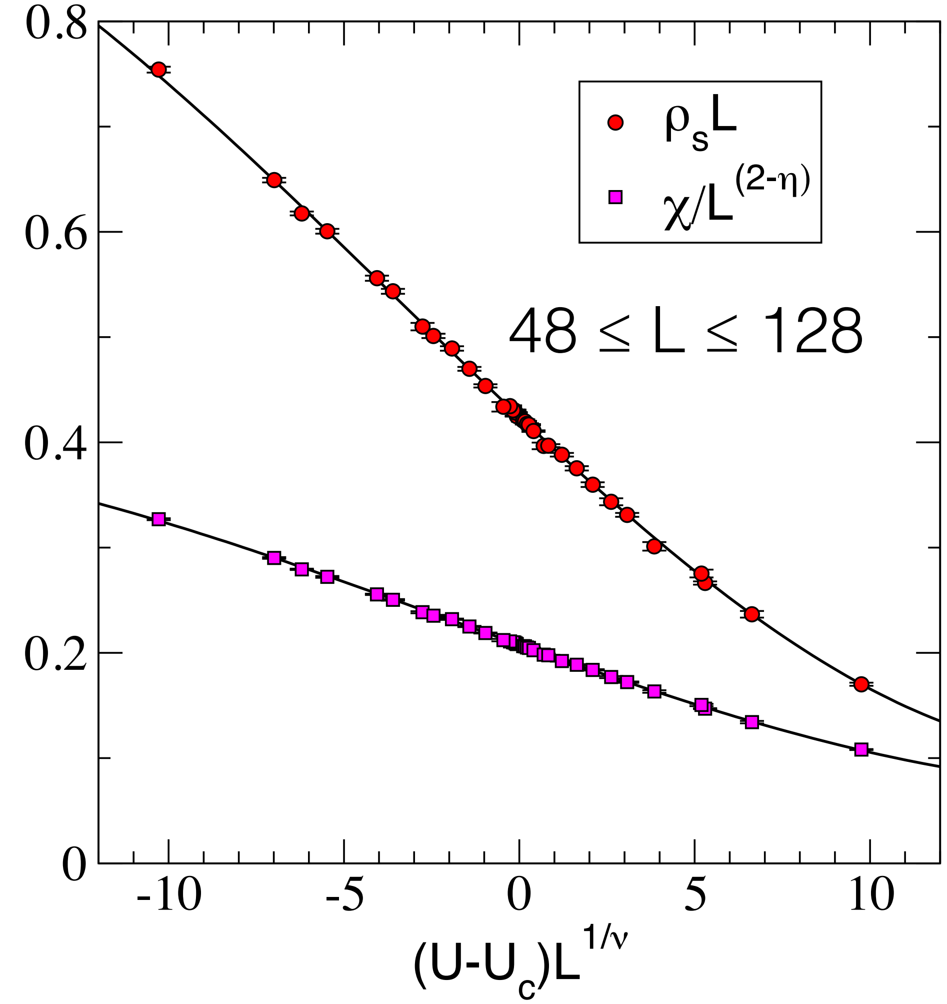
$$\rho_s L = g((U - U_c) L^{1/\nu})$$

Traditional Model Results

Pelisetto, Vicari Phys. Repts. (2002)

$$\nu = 0.749(2), \eta = 0.0365(10)$$

Qubit Model Results



$$U_c = 1.655394(3)$$

$$\nu = 0.746(3), \eta = 0.0353(10)$$

Conformal Dimensions: Leading sector

To compute the conformal dimensions we need to compute

$$C_{j_L,j_R} = \langle \overline{\mathcal{O}}_{j_L,j_R} \; \mathcal{O}_{(j_L,j_R)} \rangle \sim \left(\frac{1}{L}\right)^{2D(j_L,j_R)}$$

separated by L/2

To construct the correct sources and sinks it is helpful to use the fermionic reperesentation of the qubit regularized model.

Fermionic Representation

Introduce four-Grassmann variables per lattice site:

$$\psi_x = \begin{pmatrix} \psi_{1,x} \\ \psi_{2,x} \end{pmatrix}, \quad \overline{\psi} = (\overline{\psi}_{1,x}, \overline{\psi}_{2,x})$$

Introduce a 2 x 2 fermion bilinear matrix on each site: $(M_x)_{ab} = \psi_{a,x}\overline{\psi}_{b,x}$

The partition function of the model:

$$Z \;=\; \int \prod_{a,x} [d\overline{\psi}_{a,x} d\psi_{a,x}] \; e^{-S(\psi,\overline{\psi})}$$

Action: $$S \;=\; -\sum_{\langle x,y\rangle} \mathrm{Tr}(M_x \; M_y) \;-\; \frac{U}{2} \sum_x \mathrm{Det}(M_x)$$

Action is invariant under SU(2) x SU(2) symmetry:

$$M(x) \to R\ M(x)\ L^{\dagger}, \quad \text{x odd}$$

$$M(x) \to L\ M(x)\ R^{\dagger}, \quad \text{x even}$$

Action:

$$S = -\sum_{\langle xy \rangle} \mathrm{Tr}\big(M(x)M(y)\big) - U \sum_{x} \mathrm{Det}\, M(x)$$

We can find conformal dimensions through appropriate correlation functions:

$$C_{j_L,j_R} = \frac{1}{Z} \int \prod_{a,x} [d\overline{\psi}_{a,x} d\psi_{a,x}]\ e^{-S(\psi,\overline{\psi})}\ \overline{\mathcal{O}}_{j_L,j_R}\ \mathcal{O}_{j_L,j_R}$$

$$= \langle\ \overline{\mathcal{O}}_{j_L,j_R}\ \mathcal{O}_{j_L,j_R}\ \rangle$$

sink

source

Constructing $\mathcal{O}_{j_L,j_R}$ and $\overline{\mathcal{O}}_{j_L,j_R}$ requires some group theory!

Let us be a bit more explicit and define

$$\mathcal{O}_{j_L,j_R} \equiv \mathcal{O}_{(j_L,m_L);(j_R,m_R)} \qquad \overline{\mathcal{O}}_{j_L,j_R} \equiv \overline{\mathcal{O}}_{(j_L,m_L);(j_R,m_R)}$$

Single site sources:

$$\mathcal{O}_{(1/2,1/2);(1/2,1/2)} = -i\psi_{1,x}\overline{\psi}_{2,x} \qquad \mathcal{O}_{(1/2,-1/2);(1/2,-1/2)} = i\psi_{2,x}\overline{\psi}_{1,x}$$

$$\mathcal{O}_{(1/2,1/2);(1/2,-1/2)} = \begin{cases} i\psi_{1,x}\overline{\psi}_{1,x} & \text{even site} \\ -i\psi_{2,x}\overline{\psi}_{2,x} & \text{odd site} \end{cases} \qquad \mathcal{O}_{(1/2,-1/2);(1/2,1/2)} = \begin{cases} -i\psi_{2,x}\overline{\psi}_{2,x} & \text{even site} \\ i\psi_{1,x}\overline{\psi}_{1,x} & \text{odd site} \end{cases}$$

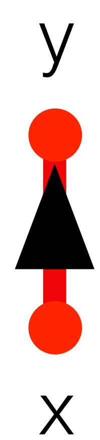
Single site sinks:

$$\overline{\mathcal{O}}_{(1/2,1/2);(1/2,1/2)} = -\mathcal{O}_{(1/2,-1/2);(1/2,-1/2)}$$

$$\overline{\mathcal{O}}_{(1/2,1/2);(1/2,-1/2)} = \mathcal{O}_{(1/2,-1/2);(1/2,1/2)}$$

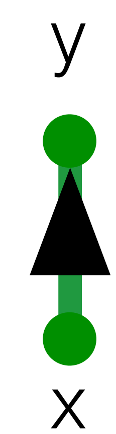
Mapping to worldlines

$\overline{\mathcal{O}}_{(1/2,1/2);(1/2,1/2)}$



$\mathcal{O}_{(1/2,1/2);(1/2,1/2)}$

$\overline{\mathcal{O}}_{(1/2,1/2);(1/2,-1/2)}$

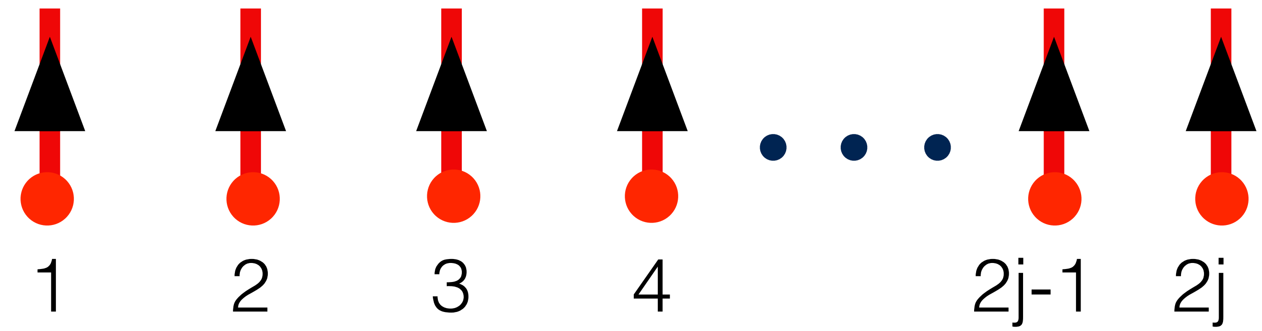


$\mathcal{O}_{(1/2,1/2);(1/2,-1/2)}$

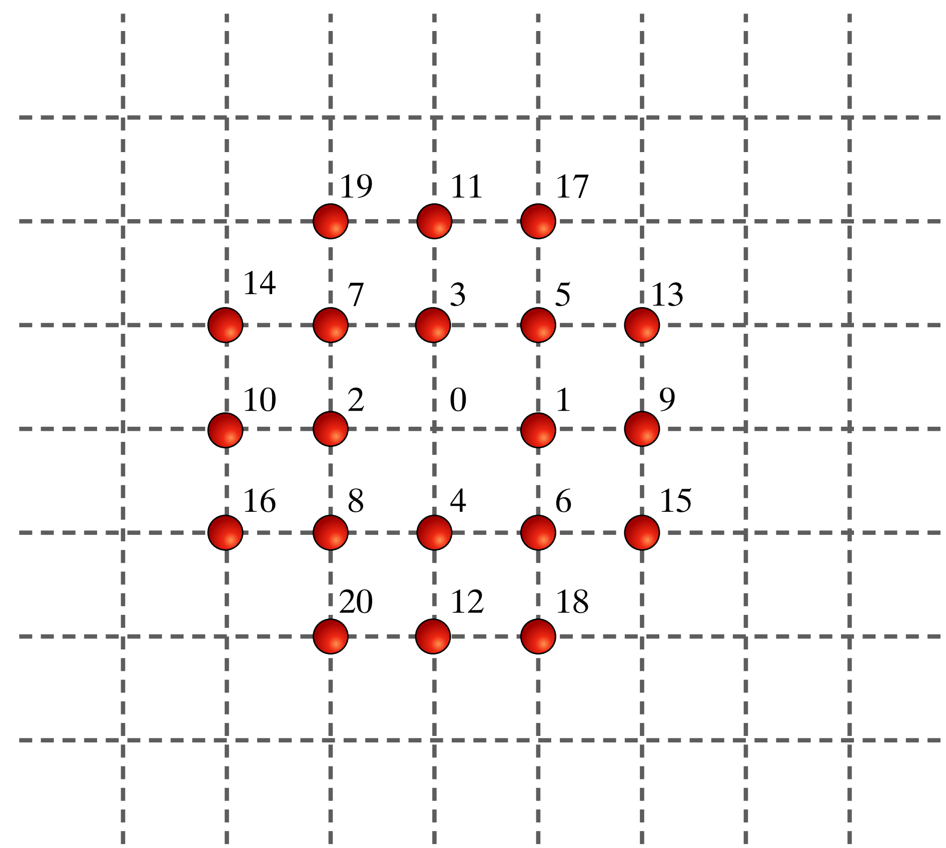
We can build higher $\mathcal{O}_{(j_L,m_L);(j_R,m_R)}$ by creating more red and and green world lines sources.

Example:

$$\mathcal{O}_{(j,j),(j,j)} \equiv$$



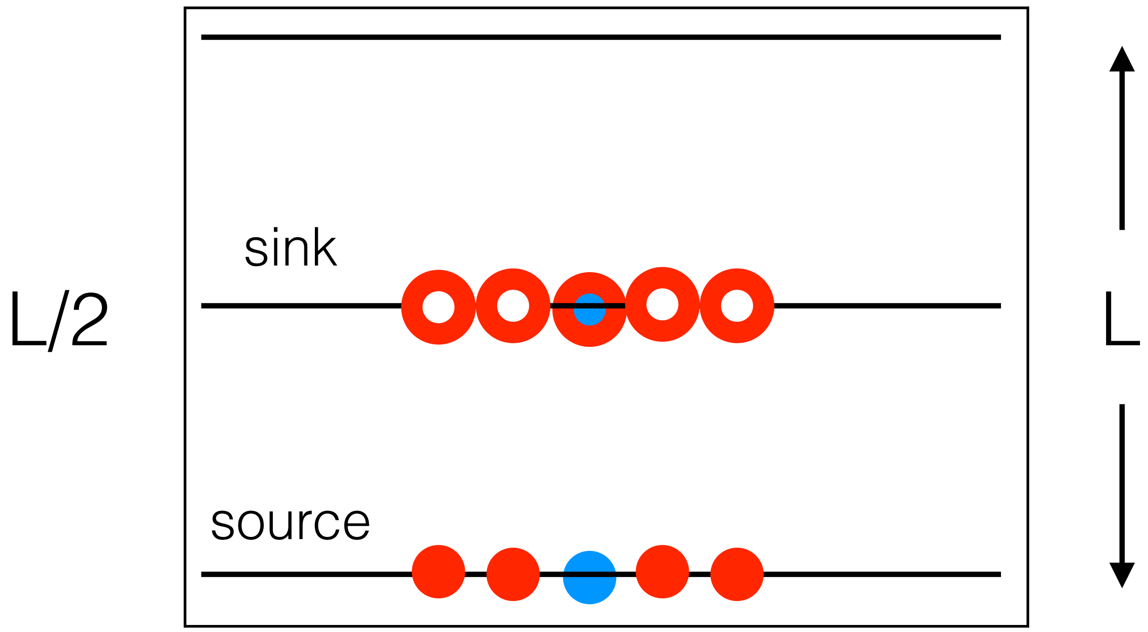
We need to spread out the individual sources/sinks over a spatial region on the lattice!



2j = 20

Source is placed on the t=0 slice and the sink at t = L / 2 slice

Monte Carlo Algorithm: Leading Sector



L x L box

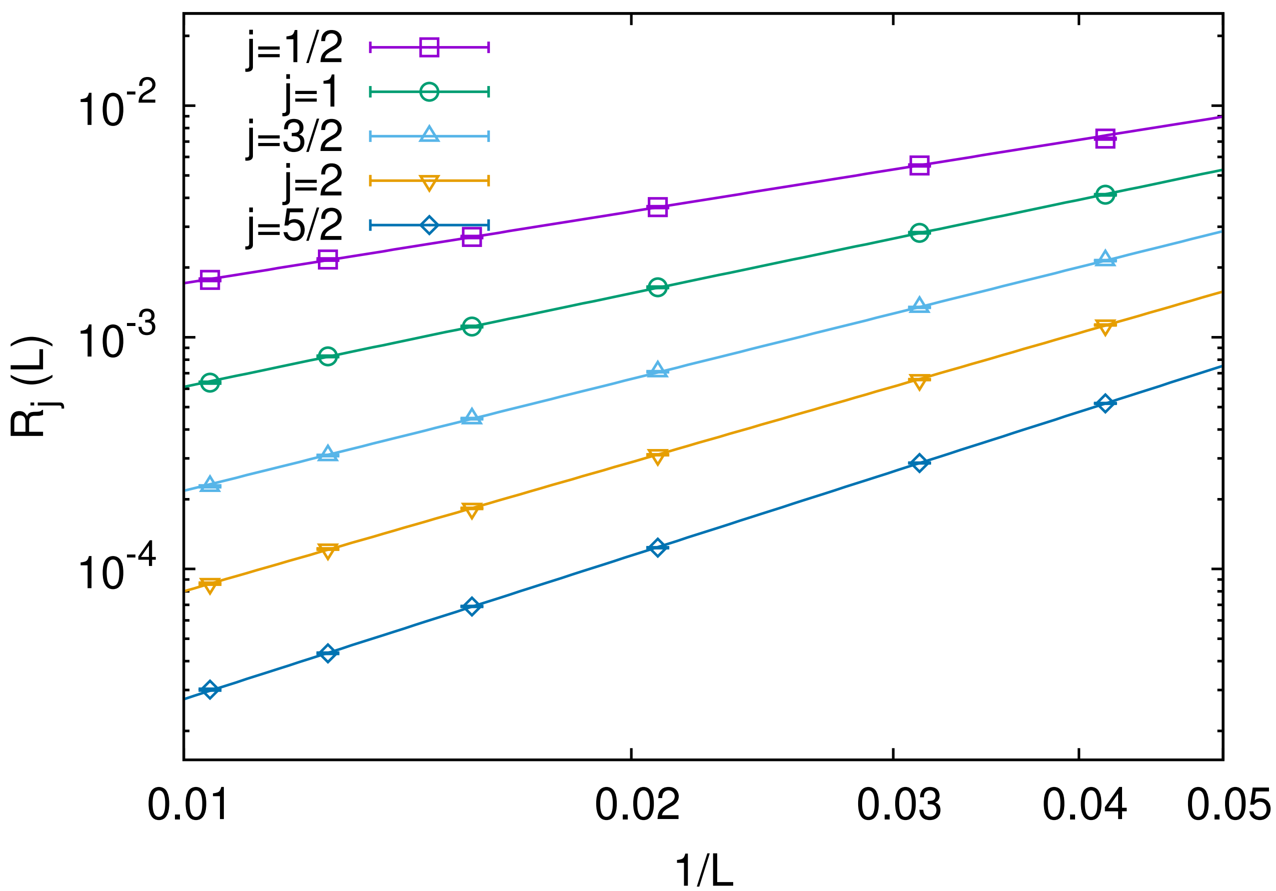
Scaling: $C_{j_L,j_R} \sim A_{j_L,j_R} \left(\frac{1}{L}\right)^{2D(j_L,j_R)}$

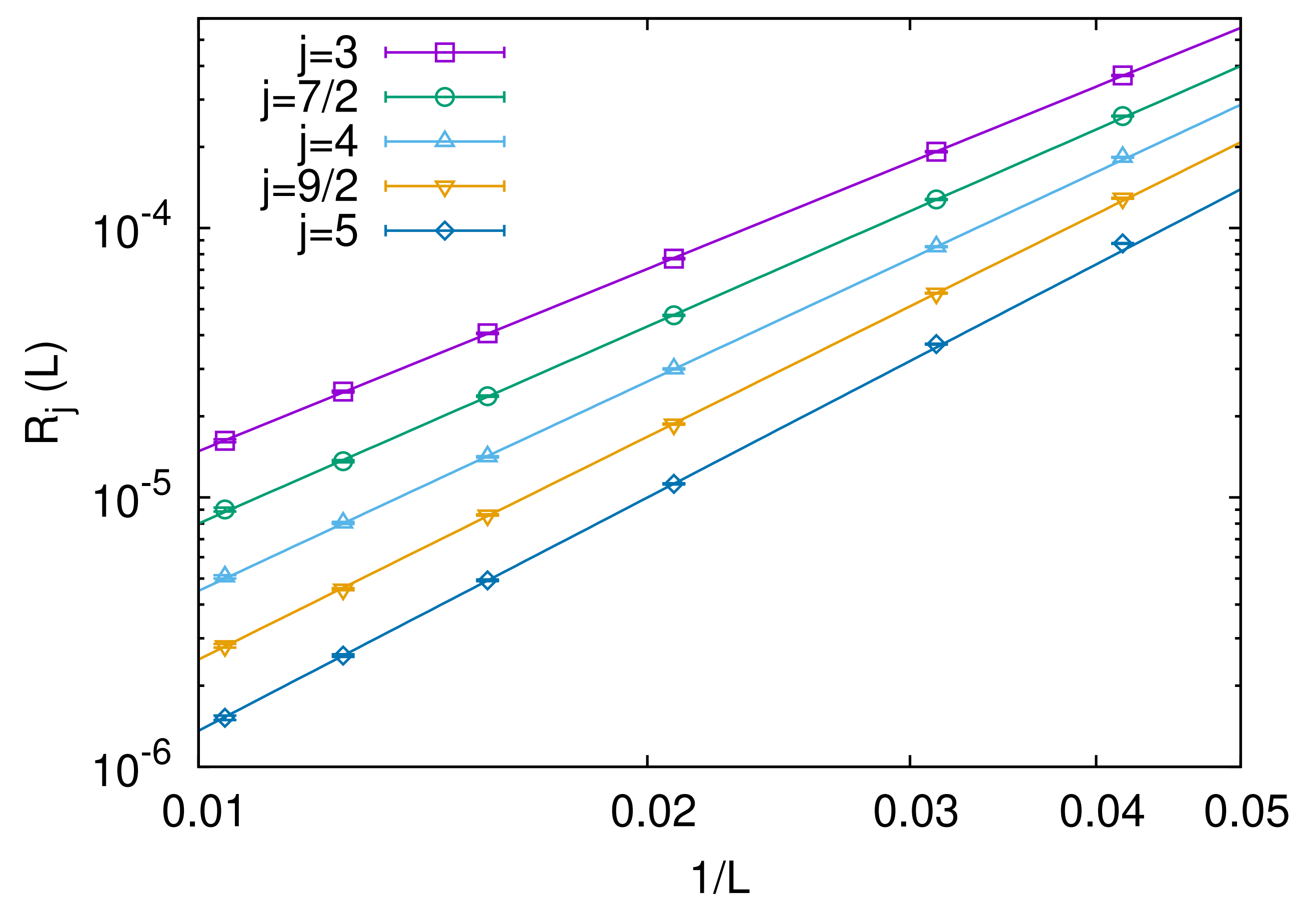
Worm algorithms can efficiently compute

$$R_j = \frac{C_{j,j}}{C_{j-1,j-1}} \sim \frac{A_{j,j}}{A_{j-1,j-1}} \left(\frac{1}{L}\right)^{2\Delta(j)}$$

$$\Delta(j) = D(j,j) - D(j-1,j-1)$$

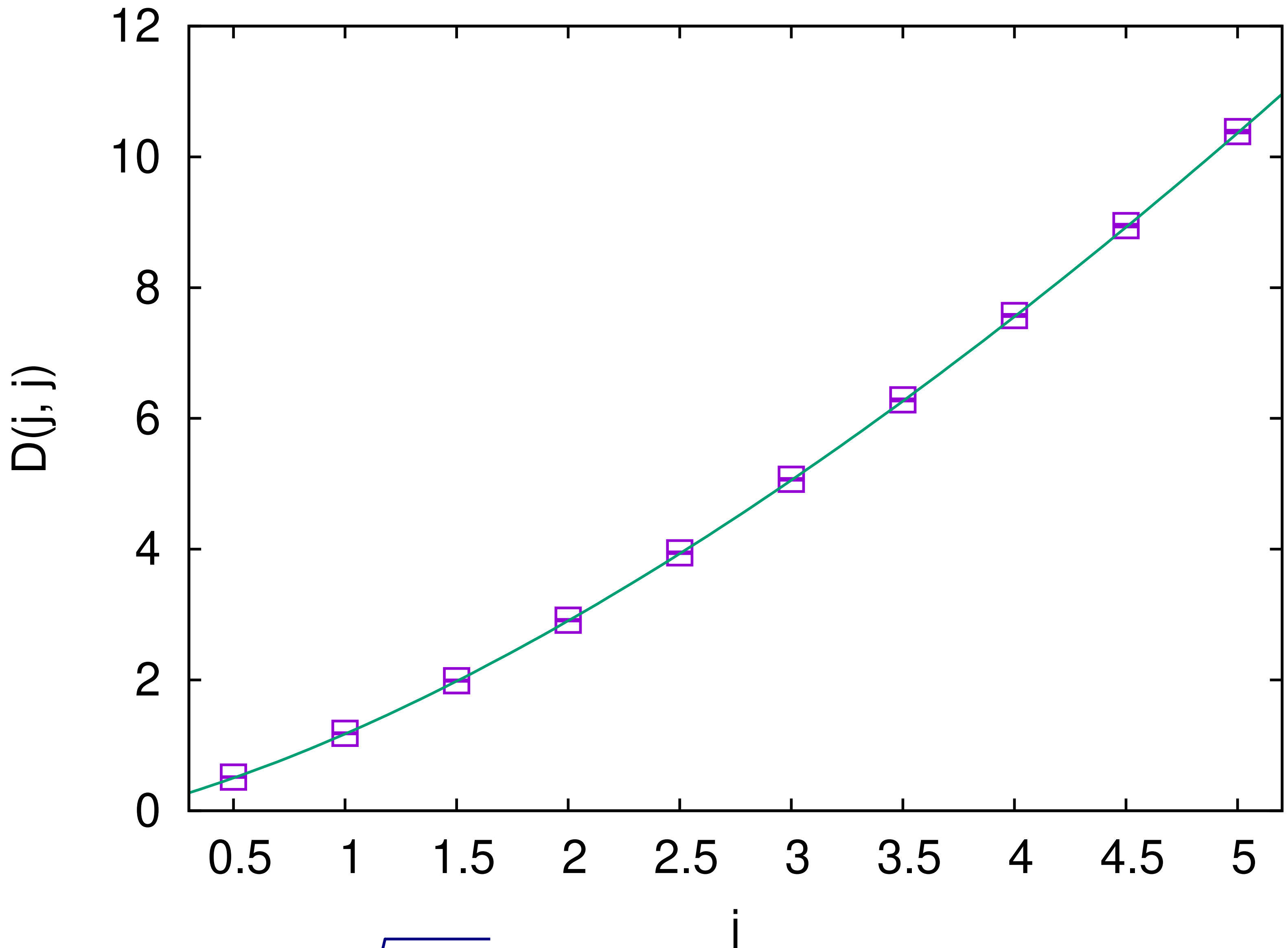
From fitting the ratio as a function of L we can extract $\Delta(j)$


j=1/2
j=1
j=3/2
j=2
j=5/2
10^{-2}
10^{-3}
10^{-4}
R_j (L)
0.01
0.02
0.03
0.04
0.05
1/L


j=3
j=7/2
j=4
j=9/2
j=5
10^{-4}
10^{-5}
10^{-6}
R_j (L)
0.01
0.02
0.03
0.04
0.05
1/L

The leading sector prediction works very well!

Banerjee, SC, Orlando, Reffert, PRL 123, 051603 (2019)



$$D(j,j) \;=\; \sqrt{\frac{2j^3}{\pi}}\left(c_{3/2} + \frac{2\pi}{j}c_{1/2} + \mathcal{O}(1/j^2)\right) + c_0$$

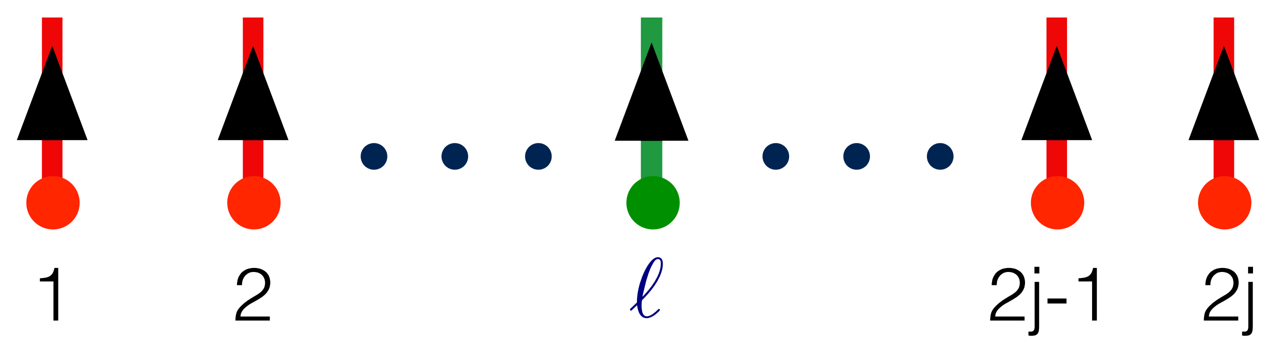
$$c_{3/2} \;=\; 1.068(4), \quad c_{1/2} \;=\; 0.083(3) \quad c_0 = -0.094\ldots$$

Conformal Dimensions: Sub-leading sector

Let us now construct sources in the sub-leading sector $\mathcal{O}_{(j,j),(j-1,j-1)}$

Consider the “2j” sources constructed as

$\mathcal{O}_\ell \quad =$



$$|(1/2,1/2);(1/2,1/2);\ \ldots.\ ;(1/2,-1/2);\ldots\ ;(1/2,1/2);(1/2,1/2)\rangle$$

We know that $\mathcal{O}_{(j,j),(j,j-1)} = \dfrac{1}{\sqrt{2j}} \displaystyle\sum_{\ell=1}^{2j} \mathcal{O}_\ell$

There are (2j-1) orthogonal superpositions that lead to $\mathcal{O}_{(j,j),(j-1,j-1)}$

Let us denote these as

$$\mathcal{O}^{M}_{(j,j),(j-1,j-1)} = \frac{1}{\sqrt{2j}} \sum_{\ell=1}^{2j} e^{i2\pi(\ell-1)M/(2j)} \mathcal{O}_{\ell}$$

for $M = 1, 2, ..., (2j-1)$

We can similarly construct (2j-1) sinks that transform as $\overline{\mathcal{O}}_{(j,j),(j-1,j-1)}$

$$\overline{\mathcal{O}}^{M}_{(j,j),(j-1,j-1)} = \frac{1}{\sqrt{2j}} \sum_{\ell=1}^{2j} e^{-i2\pi(\ell-1)M/(2j)} \overline{\mathcal{O}}_{\ell}$$

We can use any combination of $\mathcal{O}^{M}_{(j,j),(j-1,j-1)}$ and $\overline{\mathcal{O}}^{M}_{(j,j),(j-1,j-1)}$

to extract the conformal dimensions $D(j, j-1)$

Consider the (2j-1) x (2j-1) matrix of correlation functions

$$C^{MM'}_{j,j-1} \;=\; \left\langle \overline{\mathcal{O}}^{M}_{(j,j),(j-1,j-1)} \mathcal{O}^{M'}_{(j,j),(j-1,j-1)} \right\rangle \quad \sim A^{MM'}_{j,j-1} L^{-2D(j,j-1)}$$

decay at large L

In our calculation we consider the trace of this matrix:

$$C_{j,j-1} \;=\; \frac{1}{(2j-1)} \sum_{M=1}^{2j-1} C^{MM}_{j,j-1}$$

It is easy to simplify this and show that

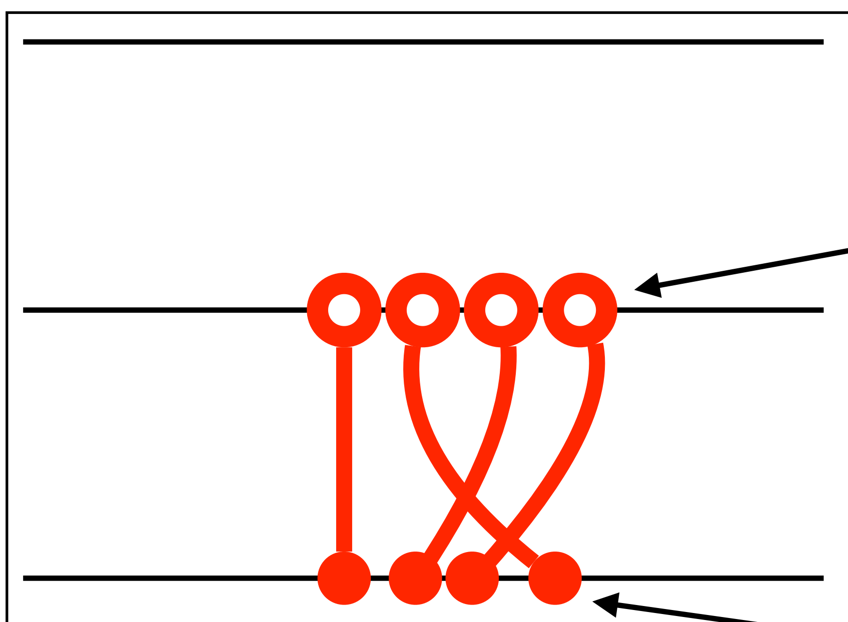
$$C_{j,j-1} \;=\; \sum_{\ell=1}^{2j} \left\{ \frac{1}{2j} \langle \overline{\mathcal{O}}_\ell \mathcal{O}_\ell \rangle - \frac{1}{2j(2j-1)} \sum_{\ell' \neq \ell} \langle \overline{\mathcal{O}}_\ell \mathcal{O}'_\ell \rangle \right\}$$

Using the worm algorithm it is easier to compute $\tilde{R}_j = C_{j,j-1}/C_{j,j}$

which scales as $\tilde{R}_j \sim \frac{A_{j,j-1}}{A_{j,j}} L^{-2\tilde{\Delta}(j)}$ where $\tilde{\Delta}(j) = D(j,j) - D(j,j-1)$

configurations that contribute to $C_{j,j}$

configurations that contribute to $C_{j,j-1}$



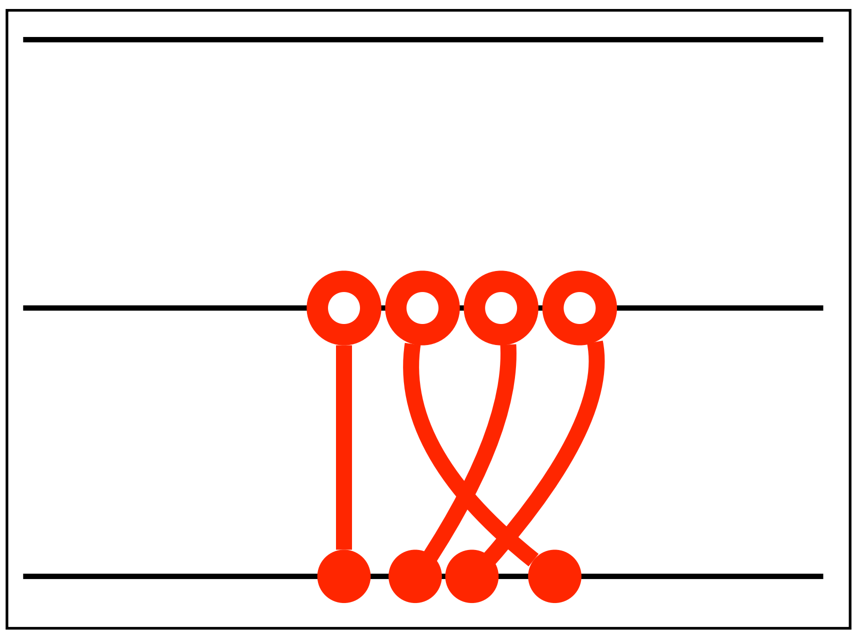
replace with $\overline{\mathcal{O}}_{\ell'}$

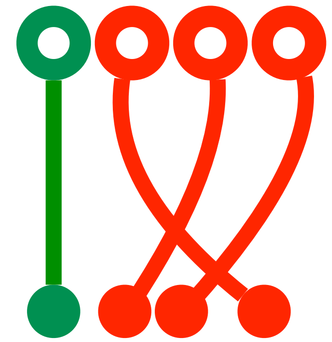
replace with $\mathcal{O}_{\ell}$

Then compute $\tilde{R}_j \equiv \frac{1}{2j}\delta_{\ell,\ell'} - \frac{1}{2j(2j-1)}(1-\delta_{\ell,\ell'})$

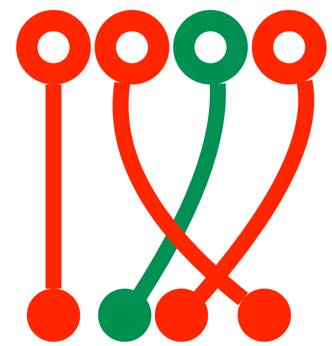
Computing $\tilde{R}_j$ in a configuration

Replace every red line at the source with a green line

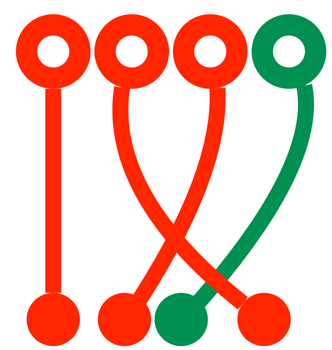




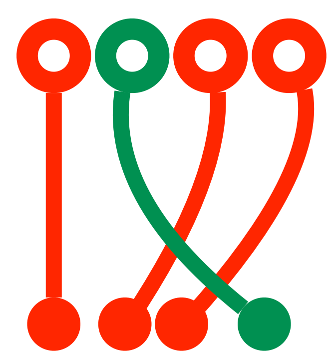
$$\frac{1}{2j}$$



$$-\frac{1}{(2j)(2j-1)}$$



$$-\frac{1}{(2j)(2j-1)}$$

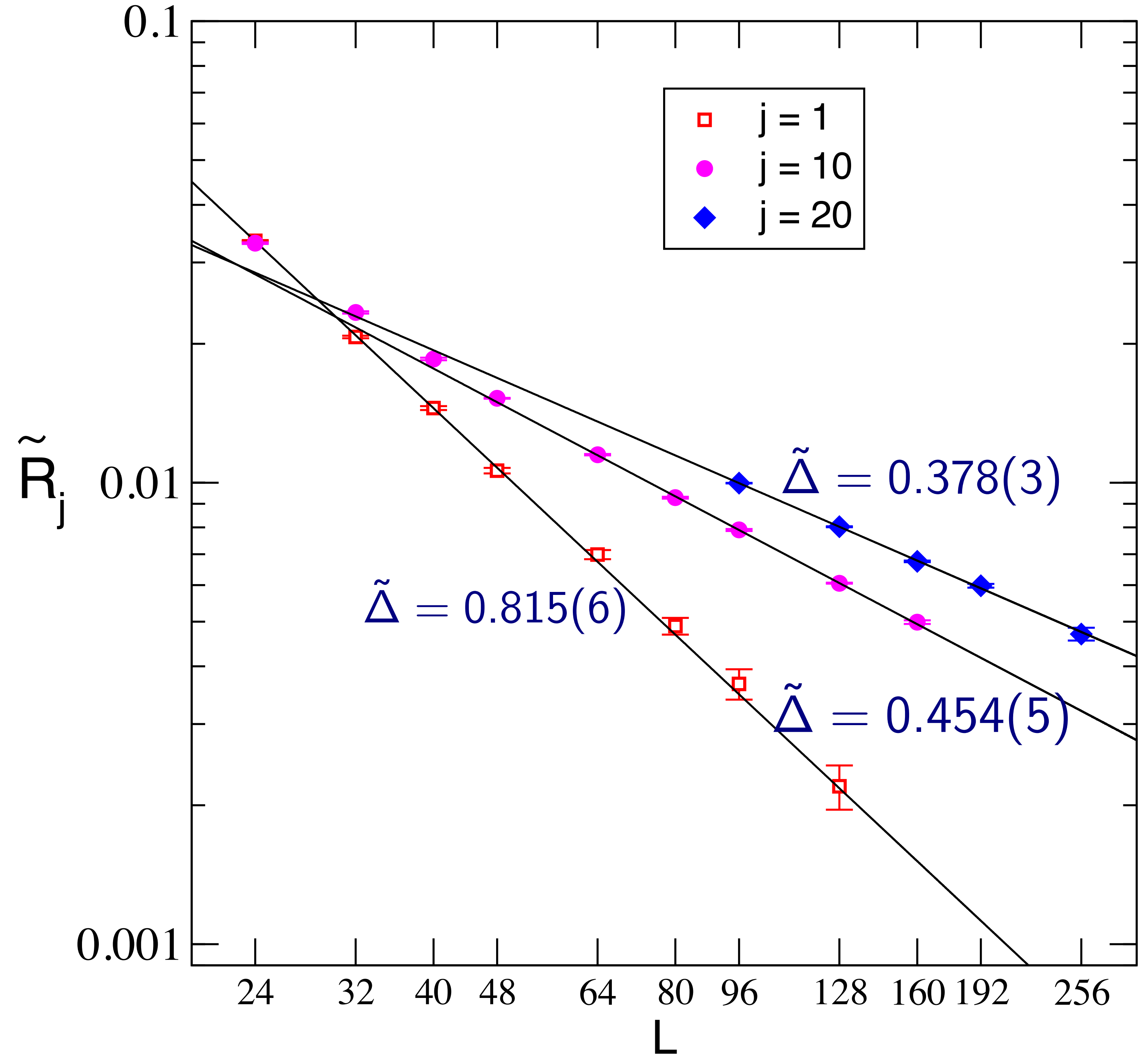


$$-\frac{1}{(2j)(2j-1)}$$

In this example $2j = 4$

$$\tilde{R}_j = 0$$

Results



Fits to the form $\tilde{R}_j \sim \frac{A_{j,j-1}}{A_{j,j}} L^{-2\tilde{\Delta}(j)}$ are shown as solid lines

Results of fit to the form

$$\tilde{R}_j \sim \frac{A_{j,j-1}}{A_{j,j}} L^{-2\tilde{\Delta}(j)}$$

at various values of j.

j	L-range	$A_{j,j-1}/A_{j,j}$	$\widetilde{\Delta}(j)$	χ^2/DOF
1	24 – 128	5.93(24)	0.815(6)	1.16
3/2	32 – 96	2.43(12)	0.746(7)	1.00
2	32 – 96	2.15(14)	0.723(9)	0.52
5/2	32 – 96	1.75(08)	0.685(6)	1.28
3	32 – 96	1.54(08)	0.659(7)	0.93
7/2	32 – 96	1.35(05)	0.633(5)	0.38
4	32 – 96	1.18(04)	0.607(4)	0.40
9/2	40 – 160	1.05(04)	0.586(5)	0.95
5	40 – 160	0.94(04)	0.566(5)	0.90
11/2	48 – 160	0.88(03)	0.553(4)	0.86
6	48 – 160	0.83(03)	0.541(5)	1.40
13/2	64 – 160	0.75(04)	0.525(7)	1.11
7	64 – 160	0.71(03)	0.513(5)	1.18
15/2	64 – 160	0.69(04)	0.506(6)	1.45
8	64 – 160	0.60(03)	0.486(5)	0.77
17/2	64 – 160	0.61(03)	0.484(5)	0.83
9	80 – 160	0.54(04)	0.467(8)	0.98
19/2	80 – 160	0.53(03)	0.463(7)	0.47
10	80 – 160	0.49(02)	0.454(5)	0.62
20	96 – 256	0.32(01)	0.378(3)	0.91

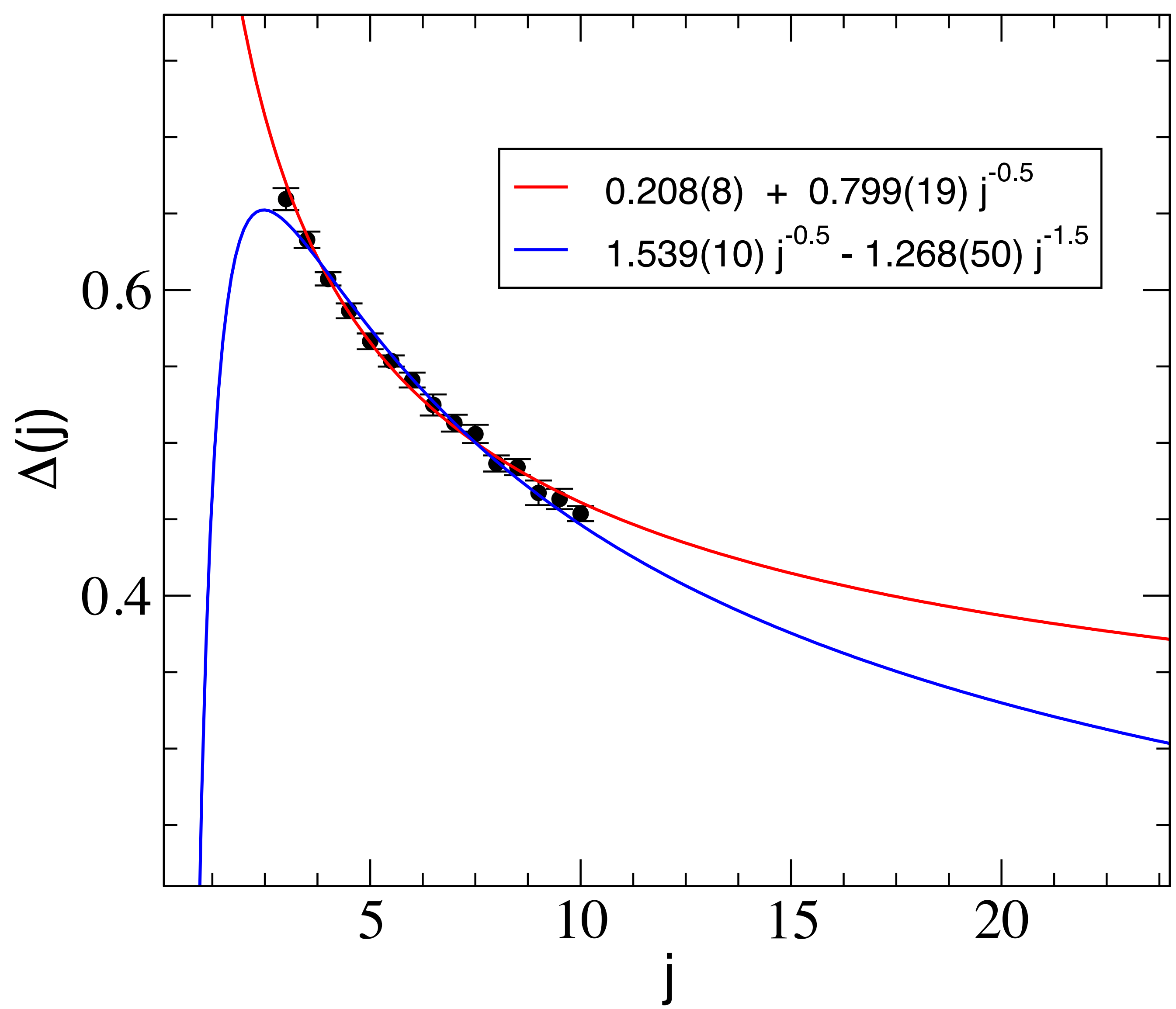
latest result

Large Charge Predictions

to be determined

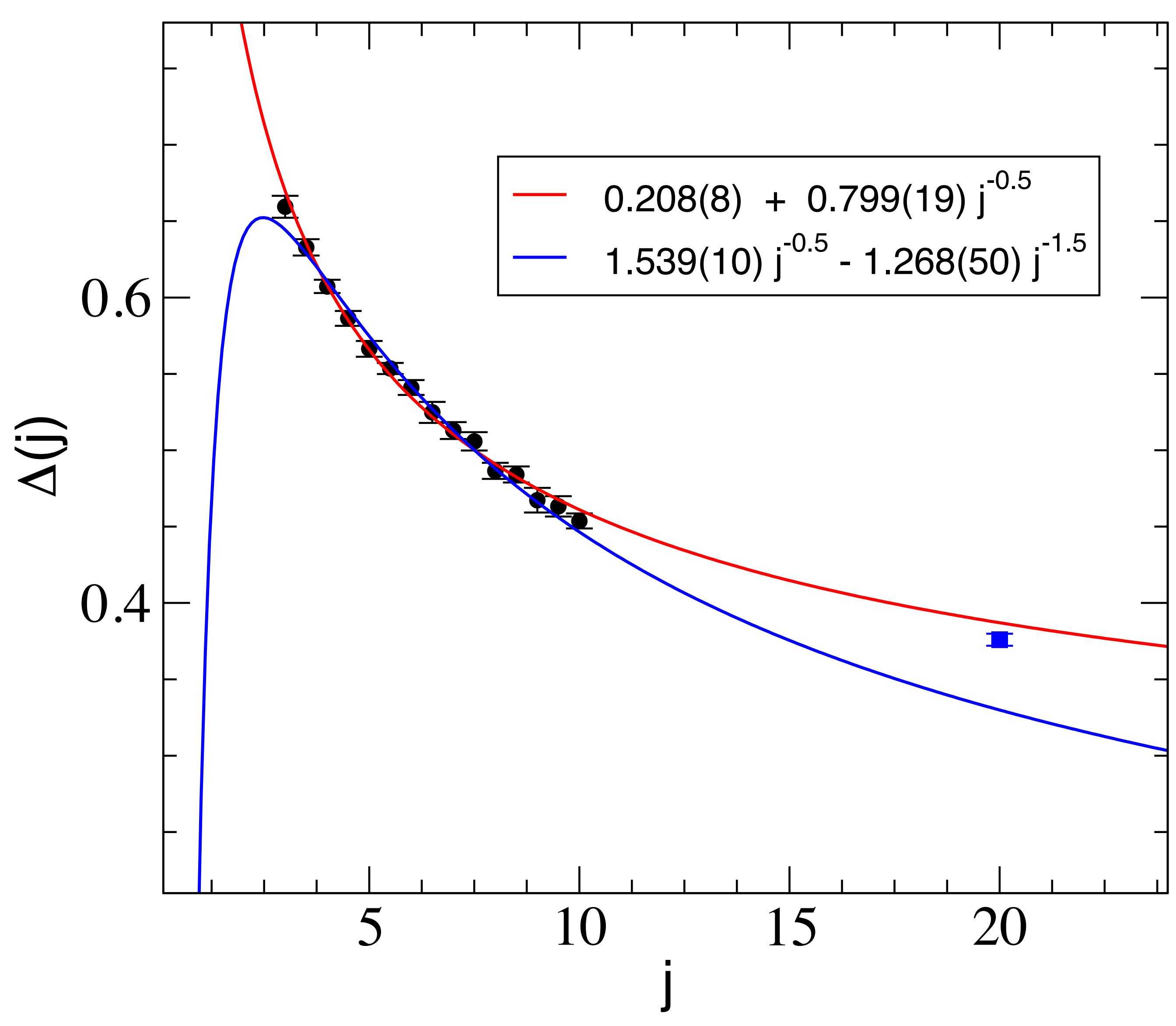
$$\tilde{\Delta}(j) = \tilde{c}_0 + \frac{\lambda^2}{j^{1/2}} + \mathcal{O}(1/j^{3/2})$$

= 0



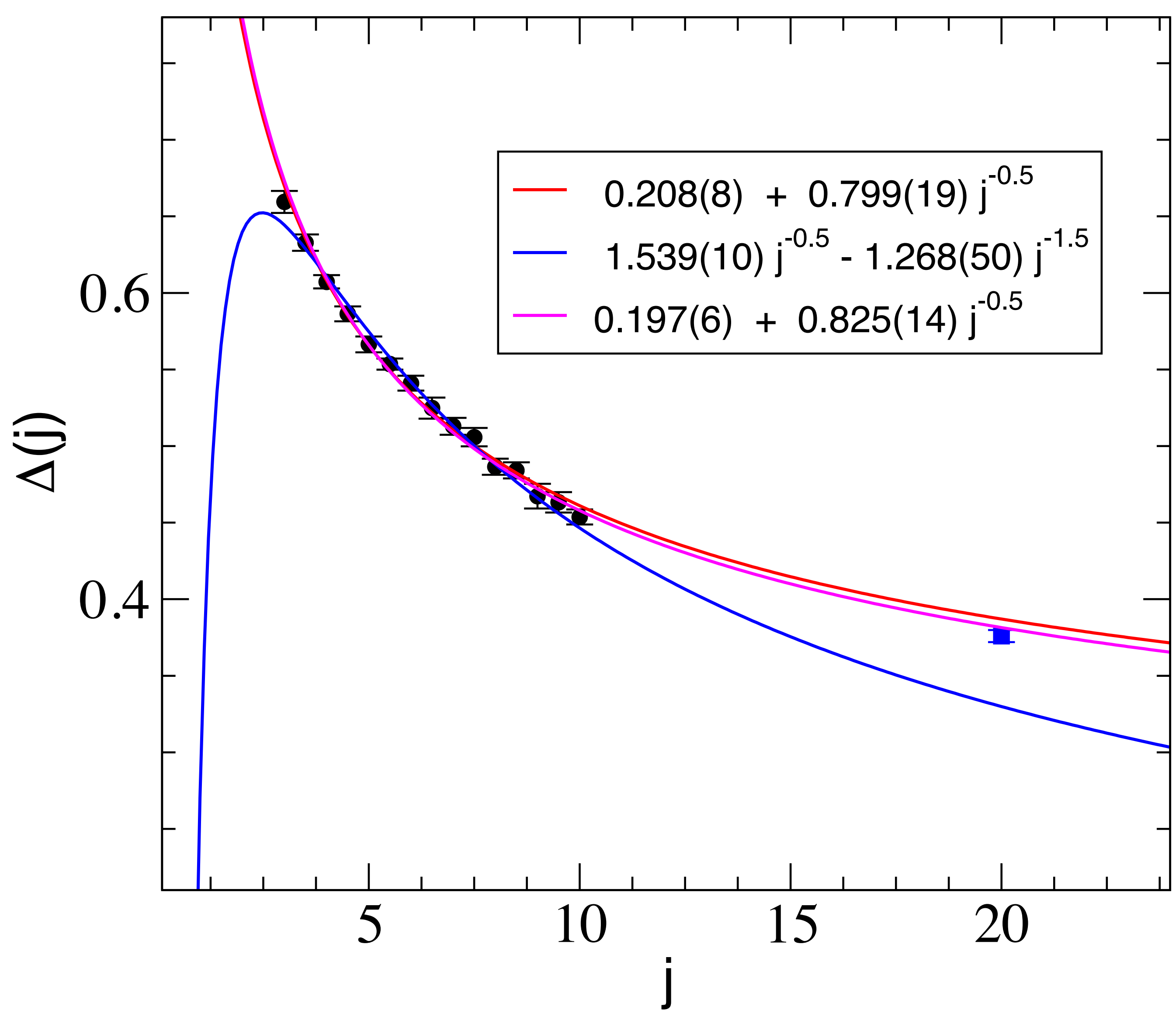
Large Charge Predictions

$$\tilde{\Delta}(j) = \tilde{c}_0 + \frac{\lambda^2}{j^{1/2}} + \mathcal{O}(1/j^{3/2})$$



Large Charge Predictions

$$\tilde{\Delta}(j) = \tilde{c}_0 + \frac{\lambda^2}{j^{1/2}} + \mathcal{O}(1/j^{3/2})$$



Large Charge Predictions

$$\tilde{\Delta}(j) = \tilde{c}_0 + \frac{\lambda^2}{j^{1/2}} + \mathcal{O}(1/j^{3/2})$$

Prediction: $\tilde{c}_0 = 0$ while our result $\tilde{c}_0 = 0.197(6)$

We obtain $\lambda^2 = 0.825(14)$

We cannot of course rule out the possibility that the true large j predictions only begin for much larger values of j than we are currently exploring.

Conclusions

The recent proposal of Q-expansion for CFTs continues to be a promising approach.

In the sub-leading sector, the Q-expansion in the O(4) model seems to suggest the presence of an extra term.

Is the large Q expansion valid only for much larger values of Q in some sectors?

To study the O(4) model we used a drastically simpler formulation of the theory called a “Qubit regularization.”

These are new ways of studying CFTs and QFTs.