Real-analytic Birkhoff sections

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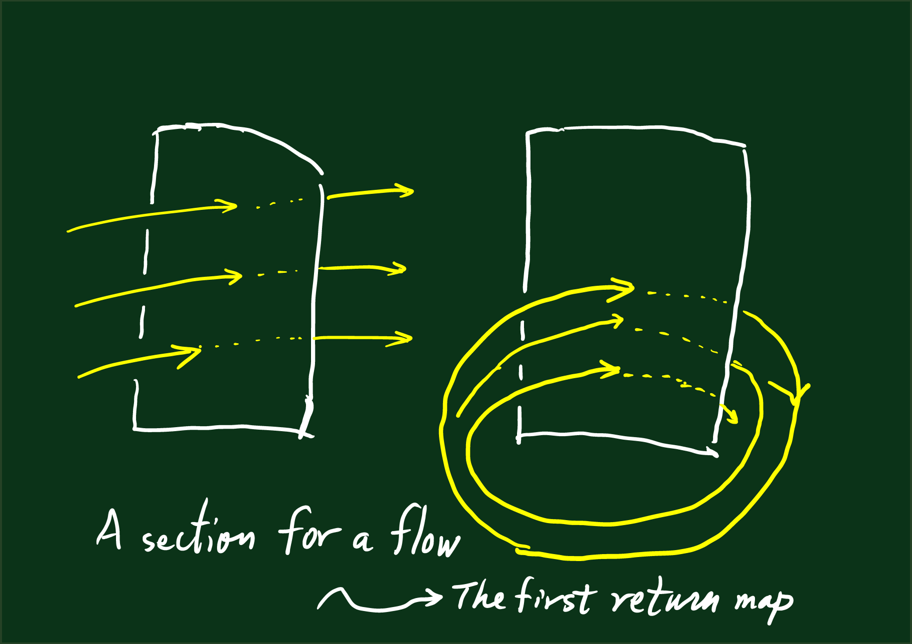
Section for a flow ①

- A flow is defined as the solution of an automonous ordinary differential equation on a manifold, or the time t integral of a vector field.
- A flow is given by a map $\varphi : \mathbf{R} \times M \longrightarrow M$ such that $\varphi(0, x) = x$ and $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$. By writing $\varphi(t, x)$ by $\varphi_t(x)$, $\varphi_t : M \longrightarrow M$ and the conditions are $\varphi_0 = \mathrm{id}$ and $\varphi_s \circ \varphi_t = \varphi_{s+t}$, i.e., A flow is an action of the group $\mathbf{R}$.
- It is important to note that ${\varphi_t}^{-1} = \varphi_{-t} : M \longrightarrow M$ and flows are always invertible.
- The problem of dynamical system is usually the asymptotic behavior of the flow. For example, to ask whether an orbit is asymptotic to a fixed point, a closed orbit, or other type of invariant set.
- A codimension-1 submanifold transverse to a flow is called a section for the flow.

Section for a flow ②



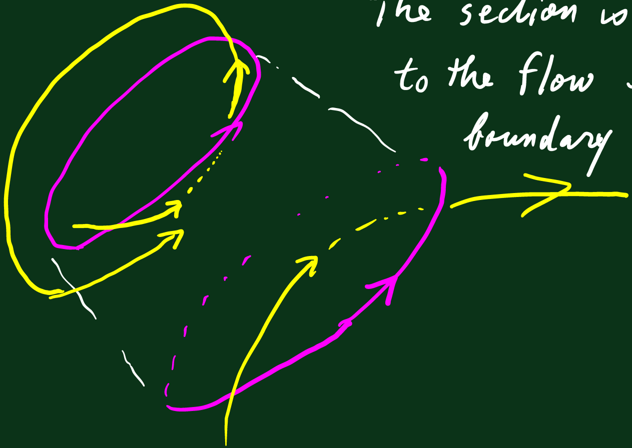
Section for a flow ③

Sections translate problems on flows to problems of local diffeomorphisms.

Boundaries of sections pose problems.

Birkhoff took the boundary consisting of closed orbits.

Section for a flow ④



The section is transverse to the flow except along boundary closed orbits.

Section for a flow ⑤

- Poincaré might be one of the pioneers to take a section to a flow and look at the first return map. Poincaré studied the three body problem at the beginning of 20th century..
- Examples of flows without closed orbits by Paul Schweitzer [1974] and Krystyna Kuperberg [1994] come from the idea of using sections or fat sections (pluging constructions) to modify the given flow.
- In a paper in 1917, Birkhoff introduced a surface of section for a flow of 3-manifold, whose interior is transverse to the flow and whose boundary consists of closed orbits.
- This surface is usually called the Birkhoff section. It is used, for example, to investigate Anosov flows.

Real analytic Birkhoff sections for $U(1)$ actions

I am going to talk about real-analytic Birkhoff sections for $U(1)$ actions which appeared in the study of the group of real-analytic diffeomorphisms.

Let M be a closed connected manifold with real-analytic semi-free $U(1)$ action.

A real-analytic Birkhoff section $S = f^{-1}(0)$ should be a union of (-1)-invariant components $(-1 \in U(1))$

$\rightsquigarrow$ a $U(1)$ equivariant map $M \longrightarrow \mathbf{C}$

$\rightsquigarrow$ an orbitwise $SL(2;\mathbf{R})$ action

$\rightsquigarrow$ the perfectness of $\mathrm{Diff}^{\omega}(M)_0$

We need real-analytic Birkhoff sections for $U(1)$-actions, and we found some interesting feature of them.

Notation

- Let $\mathbf{Diff}^\omega(M)$ denote the group of real-analytic diffeomorphisms of a closed real-analytic manifold M.
- Let $\mathbf{Diff}^\omega(M)_0$ denote the identity component with respect to the C^1 topology.

Theorem [Herman 1974]

$\mathbf{Diff}^\omega(T^n)_0$ is a simple group.

Real-analytic diffeomorphisms ②

Theorem [T 2009]

- If M^n admits a free $U(1)$ action or a special semi-free $U(1)$ action, then $\mathrm{Diff}^\omega(M^n)_0$ is a perfect group.

  Here the isotropy subgroups of a semi-free action are either trivial or the whole group.

  A special semi-free action is that on $N \times U(1)/(\partial N \times U(1) \sim \partial N)$.

- For $n = 2, 3$, if M^n admits a nontrivial $U(1)$ action, then $\mathrm{Diff}^\omega(M^n)_0$ is a perfect group.

[T 2009]: T. Tsuboi, On the group of real analytic diffeomorphisms, *Ann. Scient. Éc. Norm. Sup.* 4^{e} série, 42, (2009), 601 – 651.

Conjectures

Conjecture [Herman 1974]

- $\mathbf{Diff}^{\omega}(M^n)_0$ is a simple group.

Conjecture [T]

- If M^n admits a nontrivial $U(1)$ action, then $\mathbf{Diff}^{\omega}(M^n)_0$ is a perfect group.

Some more evidence

- $\mathbf{C}P^2$, $\mathbf{C}P^n$
- Manifolds with semi-free $U(1)$ actions with fixed points of type $\mathbf{diag}(u, 1, \ldots, 1)$ acting on $\mathbf{C} \times \mathbf{R}^{n-2}$.

Orbitwise rotations

Proposition 9.1 [T 2009]

- If M^n admits a nontrivial $U(1)$ action, then any element of $\mathrm{Diff}^\omega(M^n)_0$ is homologous to an orbitwise rotation.

- Using the above proposition, it is only necessary to show that orbitwise rotations can be written as products of commutators.

- To write as product of commutators, we may use diffeomorphisms mapping orbits to orbits and/or diffeomorphisms mapping each orbit to itself.
- For the moment, we use diffeomorphisms mapping each orbit to itself.

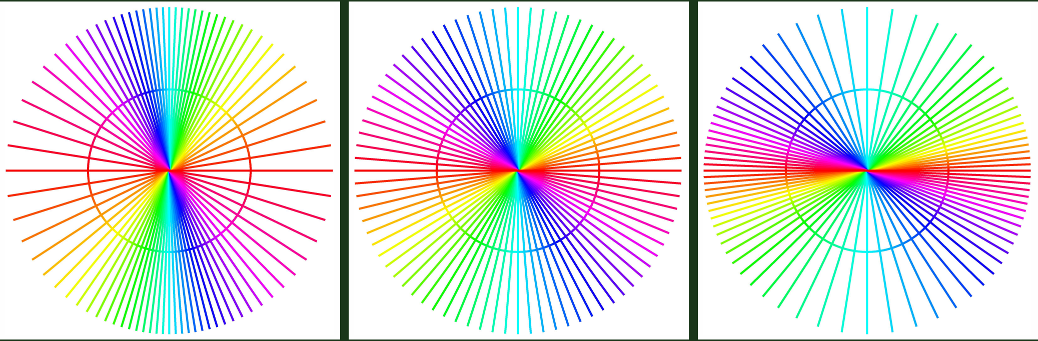
Orbitwise $SL(2;\mathbf{R})$ actions ①

Rotations in $SL(2;\mathbf{R})$

- Rotations are written as products of commutators in $SL(2;\mathbf{R}) \subset \mathrm{Diff}^{\omega}_0(S^1)$ (or in $\widetilde{SL}(2;\mathbf{R}) \subset \mathrm{Diff}^{\omega}_0(S^1)$).
- Explicitly, $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}^2 = \left[\begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} W & Z \\ Z & W \end{pmatrix}\right]\left[\begin{pmatrix} W & Z \\ Z & W \end{pmatrix}, \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}\right]$,
  where $\begin{pmatrix} W & Z \\ Z & W \end{pmatrix} = \begin{pmatrix} X & \frac{2a^2Y}{a^4-1} \\ \frac{2a^2Y}{a^4-1} & X \end{pmatrix} \Big/ \sqrt{\det}$
- If we can define a nice orbitwise $SL(2;\mathbf{R})$ action, so that $A = A_a = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ acts real analytically for some a, then we can use this to write orbitwise rotations as products of commutators.
- When it is necessary to put $a = 1$ at some orbit in M, we can still write an orbitwise rotation as a product of commutators if the rotation angle ($\sim Y$) is divisible by $(a - 1)$.

Orbitwise $SL(2;\mathbb{R})$ actions ②

Hyperbolic actions on the set of rays, i.e., on the circle



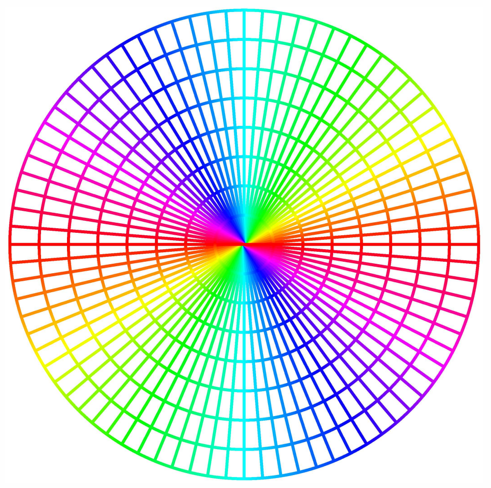
Orbitwise $SL(2;\mathbb{R})$ actions ③

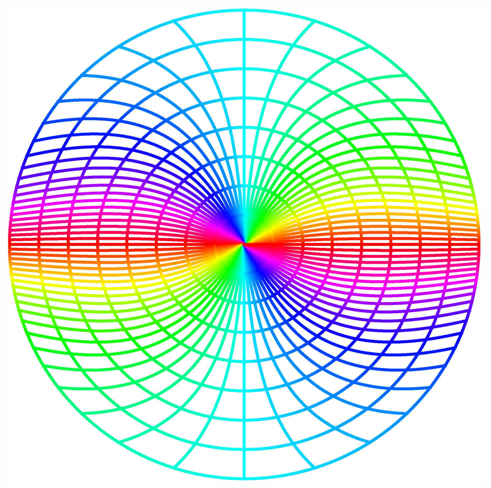
In the complex coordinates $(z, \bar{z})$

- Put $Az = A(x + yi) = ax + iy/a$. On $U(1) = \{\frac{z}{|z|}\}$, the action of $A = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ on $U(1)$ is written as
  $$A \cdot z = A \cdot (x + yi) = \frac{ax + iy/a}{|ax + iy/a|} = \frac{Az}{|Az|}.$$
- [T 2009 Lemma 3.3]: In the same way, A acts on the concentric circles in $\mathbb{C}$. $A \cdot z = A \cdot (x + yi) = |x + yi| \frac{ax + iy/a}{|ax + iy/a|} = \frac{|z|}{|Az|} \frac{Az}{z} z$,
  where $\frac{Az}{z} = \frac{1}{z}(a\frac{z + \bar{z}}{2} + \frac{1}{a}\frac{z - \bar{z}}{2}) = \frac{1}{2}(a + \frac{1}{a}) + \frac{1}{2}\frac{a + 1}{a}\frac{a - 1}{z\bar{z}}\bar{z}^2$.
  Then if $|z|^2$ divides $a - 1$, A acts real-analytically on the concentric circles in $\mathbb{C}$.

Orbitwise $SL(2;\mathbb{R})$ actions ④

Hyperbolic actions on the concentric circles





Semi-free $U(1)$ actions

- We restrict ourselves to the case of semi-free $U(1)$ action. For, otherwise we also need the action of covering groups of $SL(2;\mathbf{R})$.
- A $U(1)$ action is semi-free if the orbits other than the fixed points are free orbits.
- For a fixed point of the semi-free $U(1)$ action, there is $k \in \mathbf{Z}_{>0}$ such that the $U(1)$ action is described on the coordinates $(z_1,\dots,z_k,x_{2k+1},\dots,x_n) \in \mathbf{C}^k \times \mathbf{R}^{n-2k}$ as
$$u\cdot(z_1,\dots,z_k,x_{2k+1},\dots,x_n) = (uz_1,\dots,uz_k,x_{2k+1},\dots,x_n).$$
Fixed points of this type form a codimension-$2k$ submanifold.
- For example, $\mathbf{C}P^n$ admits the semi-free $U(1)$-action given by
$$u\cdot[z_0:\dots:z_n] = [uz_0:\dots:uz_k:z_{k+1}:\dots:z_n].$$
The fixed point set of this action is $\mathbf{C}P^k \sqcup \mathbf{C}P^{n-k-1}$.

Real-analytic Birkhoff section ①

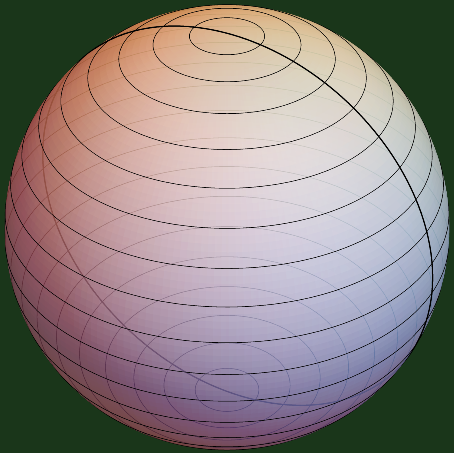
F, S, Z and $f : M \longrightarrow \mathbf{R}$

- Assume that we can find a real-analytic Birkhoff section S for a semi-free $U(1)$ action such that
  - $(-1) \cdot S = S$,
  - $F \cup Z \subset S$, where $F = \mathrm{Fix}(U(1))$,
  - $Z \setminus (F \cap Z)$ is a union of nontrivial $U(1)$ orbits, and
  - $S \setminus (F \cup Z)$ is transverse to (nontrivial) $U(1)$ orbits.
- Assume also that $S = f^{-1}(0)$, where $f : M \longrightarrow \mathbf{R}$ is a real-analytic odd (meaning $((-1)\cdot)^* f = -f$) function.
- It looks strange because f should change sign on $U(1)$ orbits.
- Orbits hit S twice transversely.

Example of the real-analytic Birkhoff section S^2

Rotating sphere

- Let $U(1)$ act on $S^2 \in \mathbf{R}^3$ by rotation around z-axis.
- S is the great circle on xz-plane, $F = \{(0, 0, \pm 1)\}$, and $f = y$.
- In this example, the map $h = (i\cdot)^* f + if : S^2 \longrightarrow \mathbf{C}$ is $U(1)$-equivariant, that is, for $p \in S^2$, $h(u \cdot p) = u\, h(p)$.



Example of the real-analytic Birkhoff section S^3

Hopf fibration

- $U(1)$ acts on $S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ diagonally.
- This gives the Hopf fibration $S^3 \longrightarrow \mathbf{C}P^1$.
- $S = \{\mathrm{Im}(z_2) = 0\} = f^{-1}(0)$, where $f(z_1, z_2) = \mathrm{Im}(z_2)$ (a great sphere).
- $Z = \{z_2 = 0\}$ (a circle).
- $h = (i\cdot)^* f + if = i\, z_2 : S^3 \longrightarrow \mathbf{C}$.
- http://www.tsuboiweb.matrix.jp/showroom/public_html/animations/gif/hopf30/hopf30.html

Real-analytic Birkhoff section ②

- The defining function f can be taken so that $h = (i\cdot)^* f + if : M \longrightarrow \mathbf{C}$ is $U(1)$ equivariant.
- This means that $U(1)$ orbits in $M \setminus (F \cup Z)$ are mapped to orbits of concentric circles in $\mathbf{C}$, S is mapped to the real line in $\mathbf{C}$, and $F \cup Z$ is mapped to $0 \in \mathbf{C}$.
- The $SL(2;\mathbf{R})$ action on the orbits in $M \setminus (F \cup Z)$ can be induced from the $SL(2;\mathbf{R})$ action on the concentric circles in $\mathbf{C}$.
- We can show that the action of $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ on $M \setminus (F \cup Z)$ can be extended to $F \cup Z$ if $|h|^2 = |f|^2 + |(i\cdot)^* f|^2$ divides a.

Examples

- Thus the problem left to settle for the semi-free $U(1)$ action case is to find the Birkhoff section $S = f^{-1}(0)$.
- For free $U(1)$ actions, $F = \emptyset$, Z is a codimension-2 submanifold dual to the Euler class.
- For special semi-free $U(1)$ actions on $M = N \times U(1)/(\partial N \times U(1) \sim \partial N)$, S is the double DN of N, $F = \partial N$ and $Z = \emptyset$..
- For $\mathbf{C}P^n$, we can find $S = f^{-1}(0)$.
- For the case of $U(1)$ actions with fixed points of type $\mathbf{diag}(u, 1, \ldots, 1)$ acting on $\mathbf{C} \times \mathbf{R}^{n-2}$, we managed to find $S = f^{-1}(0)$.

Work in progress and tasks left

"Theorem"

- For semi-free $U(1)$ actions, we can find the Birkhoff section $S = f^{-1}(0)$.
- For a real-analytic manifold M, with semi-free $U(1)$ action, $\mathrm{Diff}^{\omega}(M)_0$ is a perfect group.

- We use the Oka principle formulated by Henri Cartan for real analytic functions. It is a homotopy principle developed later by Gromov and others.
- In fact, Cieliebak and Eliashberg wrote a book titled " From Stein to Weinstein and Back", whose Section 5.7 seems to contain necessary propositions.
- For semi-free $U(1)$ actions, there are still several things to be clarified.
- We hope to find real-analytic Birkhoff sections $S = f^{-1}(0)$ for other $U(1)$ actions.

THANKS

Thank you very much for your attention.