Multiplane Lensing: Theory and Applications

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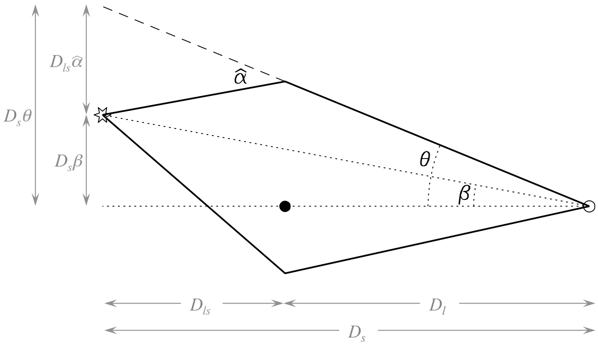
Standard geometry



Euclidean geometry in a plane:

$$D_s\beta = D_s\theta - D_{ls}\hat{\alpha}(\theta)$$

$$\beta = \theta - \alpha(\theta) \qquad \text{where } \alpha = \frac{D_{ls}}{D_s}\hat{\alpha}$$

Extend to FRW cosmology by using angular diameter distances.

Standard theory

Scaled gravitational potential:

$$\nabla^2 \phi = 2\kappa$$

Time delay:

$$\tau(\boldsymbol{x};\boldsymbol{x}_s) = \frac{1+z_l}{c}\,\frac{D_l D_s}{D_{ls}}\left[\frac{1}{2}\left|\boldsymbol{x}-\boldsymbol{x}_s\right|^2 - \phi(\boldsymbol{x})\right]$$

Fermat's principle $\nabla_{\boldsymbol{x}}\tau = 0$ gives lens equation:

$$\boldsymbol{x}_s = \boldsymbol{x} - \nabla\phi(\boldsymbol{x})$$

Distortion/magnification from Jacobian:

$$\boldsymbol{\mu} = \left(\frac{\partial \boldsymbol{x}_s}{\partial \boldsymbol{x}}\right)^{-1} = \begin{bmatrix} 1-\phi_{xx} & -\phi_{xy} \\ -\phi_{xy} & 1-\phi_{yy} \end{bmatrix}^{-1}$$

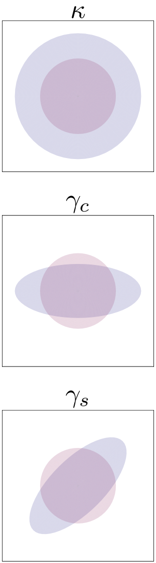
Convergence and shear

Matrix of second derivatives:

$$
\begin{aligned}
\mathbf{\Gamma} &= \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{bmatrix} \\
&= \begin{bmatrix} \kappa + \gamma_c & \gamma_s \\ \gamma_s & \kappa - \gamma_c \end{bmatrix} \\
&= \kappa\,\mathbf{I} + \begin{bmatrix} \gamma_c & \gamma_s \\ \gamma_s & -\gamma_c \end{bmatrix}
\end{aligned}
$$

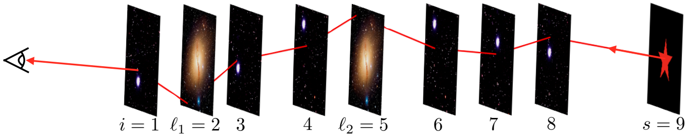
where

$$
\begin{aligned}
\kappa &= \frac{1}{2}\left(\phi_{xx} + \phi_{yy}\right) \\
\gamma_c &= \frac{1}{2}\left(\phi_{xx} - \phi_{yy}\right) \\
\gamma_s &= \phi_{xy}
\end{aligned}
$$



Multiple scattering

Light rays probably suffer multiple deflections:



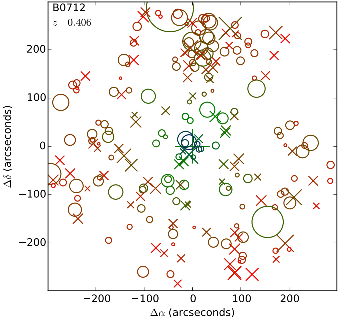
(Figure: McCully et al. 2014)

How can we handle "external" effects?

"Direct" approach:

- find galaxies and groups along the line of sight
- build them into 3-d lens models

(Williams et al. 2006; Momcheva et al. 2006; Wong et al. 2011; McCully thesis; McCully et al. in prep.)



"Corrective" approach:

- fit for external shear
- calibrate external convergence statistically
- apply $\kappa_{\rm ext}$ as posterior correction (mass sheet transformation)

(papers by Collett, Fassnacht, Greene, Hilbert, Suyu, et al.)

Multiplane lensing: Formalism

Lens equation: trace light ray "backwards"

$$\boldsymbol{x}_j = \boldsymbol{x}_1 - \sum_{i=1}^{j-1} \beta_{ij} \boldsymbol{\alpha}_i(\boldsymbol{x}_i)$$

Time delay:

$$T = \sum_{i=1}^{s-1} \tau_{i\,i+1} \left[\frac{1}{2} |\boldsymbol{x}_{i+1} - \boldsymbol{x}_i|^2 - \beta_{i\,i+1} \phi_i(\boldsymbol{x}_i) \right]$$

Note:

$$\beta_{ij} = \frac{D_{ij} D_s}{D_j D_{is}} \qquad \text{and} \qquad \tau_{ij} = \frac{1+z_i}{c} \frac{D_i D_j}{D_{ij}}$$

(e.g., Schneider et al. 1992; Petters et al. 2001; McCully et al. 2014)

Tidal approximation

Most planes are perturbative, so treat them with a Taylor series:

$$\phi_i(\boldsymbol{x}) \;=\; \cancel{\phi_{i0}} \;+\; \cancel{\boldsymbol{\alpha}_{i0}^t \boldsymbol{x}} \;+\; \frac{1}{2}\boldsymbol{x}^t \boldsymbol{\Gamma}_{i0}\boldsymbol{x} \;+\; \ldots$$

Then:

$$\begin{aligned} \boldsymbol{\alpha}_i(\boldsymbol{x}) &= \boldsymbol{\Gamma}_{i0}\boldsymbol{x} \;+\; \ldots \\ \boldsymbol{\Gamma}_i(\boldsymbol{x}) &= \boldsymbol{\Gamma}_{i0} \;+\; \ldots \end{aligned}$$

If we drop higher-order terms, this defines the tidal approximation.

Single plane

Suppose we have one main lens galaxy (ℓ) plus many tidal terms, all in one plane:

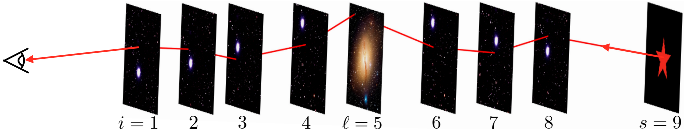
$$\begin{aligned} \boldsymbol{x}_s &= \boldsymbol{x} - \sum_{i\neq\ell} \boldsymbol{\Gamma}_i \boldsymbol{x} - \alpha_\ell(\boldsymbol{x}) \\ &= (\mathbf{I} - \boldsymbol{\Gamma}_{\rm tot})\,\boldsymbol{x} - \alpha_\ell(\boldsymbol{x}) \end{aligned}$$

Remarks:

- external convergence and shear go into the $\boldsymbol{x}$ term
- convergence adds as a scalar, but shear adds as a tensor:

$$\boldsymbol{\Gamma}_{\rm tot} = \left(\sum_{i\neq\ell} \kappa_i\right)\mathbf{I} + \sum_{i\neq\ell} \begin{bmatrix} \gamma_{ci} & \gamma_{si} \\ \gamma_{si} & -\gamma_{ci} \end{bmatrix}$$

One "main" plane, many tidal planes



Let the main lens galaxy be in plane ℓ. The mapping to this plane has a sum over foreground tidal planes:

$$\boldsymbol{x}_\ell = \boldsymbol{x}_1 - \sum_{i=1}^{\ell-1} \beta_{i\ell} \boldsymbol{\Gamma}_i \boldsymbol{x}_i = \boldsymbol{x}_1 - \sum_{i=1}^{\ell-1} \beta_{i\ell} \left(\boldsymbol{x}_1 - \sum_{j=1}^{i-1} \beta_{ji} \boldsymbol{\Gamma}_j \boldsymbol{x}_j \right)$$

All terms on RHS are linear in $\boldsymbol{x}$, so we can write

$$\boldsymbol{x}_\ell = \mathbf{B}_\ell \boldsymbol{x}_1 \qquad \text{where} \qquad \mathbf{B}_j = \mathbf{I} - \sum_{i=1}^{j-1} \beta_{ij} \boldsymbol{\Gamma}_i \mathbf{B}_i$$

(e.g., Schneider et al. 1992; McCully et al. 2014)

One "main" plane

Work through remaining planes, obtain lens equation:

$$\boldsymbol{x}_s = \mathbf{B}_s \boldsymbol{x}_1 - \mathbf{C}_{\ell s} \boldsymbol{\alpha}_\ell (\mathbf{B}_\ell \boldsymbol{x}_1)$$

Propagation between source, main lens, and observer is described by **tensors** that are sums over tidal planes:

$$\mathbf{B}_j = \mathbf{I} - \sum_{i=1, i\neq\ell}^{j-1} \beta_{ij} \boldsymbol{\Gamma}_i \mathbf{B}_i \qquad \text{(observer to } j\text{)}$$

$$\mathbf{C}_{\ell j} = \beta_{\ell j} \mathbf{I} - \sum_{i=\ell+1}^{j-1} \beta_{ij} \boldsymbol{\Gamma}_i \mathbf{C}_{\ell i} \qquad \text{(lens to } j\text{)}$$

(e.g., Schneider et al. 1992; McCully et al. 2014)

Foreground planes: Nonlinear effects

Single main plane lens equation:

$$\boldsymbol{x}_s = \mathbf{B}_s \boldsymbol{x}_1 - \mathbf{C}_{\ell s} \boldsymbol{\alpha}_\ell(\mathbf{B}_\ell \boldsymbol{x}_1)$$

Note:

$$\mathbf{C}_{\ell s}^{-1} \boldsymbol{x}_s = \mathbf{C}_{\ell s}^{-1} \mathbf{B}_s \boldsymbol{x}_1 - \boldsymbol{\alpha}_\ell(\mathbf{B}_\ell \boldsymbol{x}_1)$$

- first term is effectively a tidal term
- in second term, having $\mathbf{B}_\ell$ inside $\boldsymbol{\alpha}_\ell$ creates nonlinear effects

Can we transform away the nonlinear effects? Recall $\boldsymbol{x}_\ell = \mathbf{B}_\ell \boldsymbol{x}_1$:

$$\mathbf{C}_{\ell s}^{-1} \boldsymbol{x}_s = \mathbf{C}_{\ell s}^{-1} \mathbf{B}_s \mathbf{B}_\ell^{-1} \boldsymbol{x}_\ell - \boldsymbol{\alpha}_\ell(\boldsymbol{x}_\ell)$$

Looks nice in coordinates in the main lens plane — but we don't observe those coordinates! *(Bar-Kana 1996; Schneider 1997; CRK 2003; McCully et al. 2014)*

Lens galaxy at $z_\ell = 0.3$.

Perturber of mass $10^{12} M_\odot$ contributes line-of-sight shear:

$$\boldsymbol{x}_s = \mathbf{B}_s \boldsymbol{x}_1 - \mathbf{C}_{\ell s} \boldsymbol{\alpha}_\ell(\mathbf{B}_\ell \boldsymbol{x}_1)$$

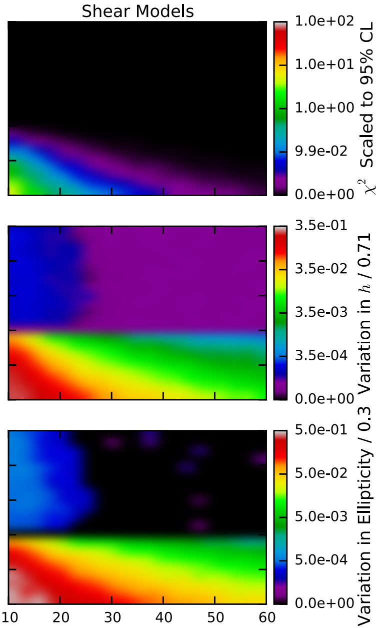
Fit with a simple external shear:

$$\boldsymbol{x}_s = (\mathbf{I} - \boldsymbol{\Gamma})\boldsymbol{x}_1 - \boldsymbol{\alpha}_\ell(\boldsymbol{x}_1)$$

Vary source position and galaxy orientation, and look at scatter in recovered χ^2, h, and e.

- background perturber can be mimicked by external shear
- foreground perturber cannot

(McCully thesis; McCully et al. in prep.)



Lessons: I. Nonlinearity

Even if we use the tidal approximation for all perturbers, there are complications due to nonlinearity.

$$\boldsymbol{x}_s = \mathbf{B}_s \boldsymbol{x}_1 - \mathbf{C}_{\ell s} \boldsymbol{\alpha}_\ell (\mathbf{B}_\ell \boldsymbol{x}_1)$$

Direct approach:

- all planes are built into the lens model
- planes are calibrated from our observations
- LOS matrices are computed self-consistently from a 3-d mass model

Corrective approach:

- three LOS matrices could be calibrated from simulations . . . but they would be correlated
- even then, $\mathbf{B}_\ell$ must be incorporated into the modeling . . . it cannot be applied as a posterior correction

Lessons: II. Whither $\kappa_{\rm ext}$?

We argued that standard external shear/convergence does not fully account for LOS effects in the lens equation.

The same goes for time delays. Strictly speaking, applying $\kappa_{\rm ext}$ does not capture all of the LOS effects in time delays and H_0.

(There **is** a mass sheet transformation [later], but it involves a particular rescaling of the planes. It cannot be used to transform away the planes.)

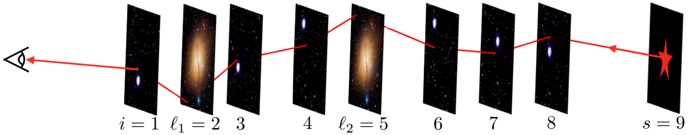
These are *formal* statements. How important are they in practice?

Multiple “main” planes

Perturbers that are massive and/or projected close to the lens:

- higher-order terms are non-negligible
- need to be treated explicitly

Still want to treat other planes as tidal — need hybrid approach.



Arbitrary mixture of “main” and tidal planes. Lens equation:

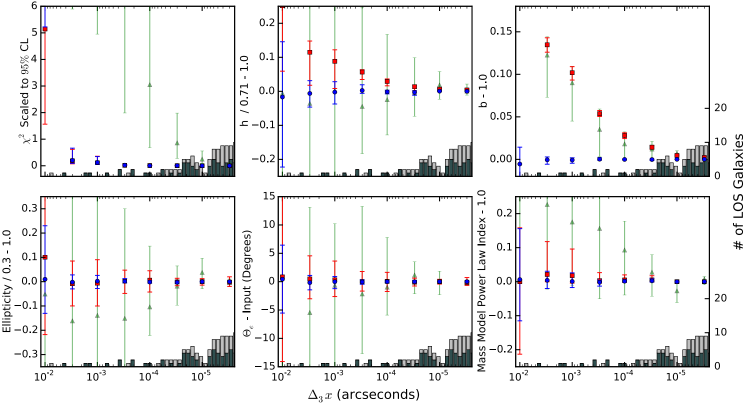
$$\boldsymbol{x}_i = \mathbf{B}_i \boldsymbol{x}_1 - \sum_{\ell \in \{\ell_\mu < i\}} \mathbf{C}_{\ell i} \boldsymbol{\alpha}_\ell(\boldsymbol{x}_\ell)$$

We also worked through time delays. . .

(McCully et al. 2014; also Schneider 2014)

Quantifying the importance of LOS effects

- Objective criterion for deciding whether tidal approximation is adequate, based on strength of higher-order terms.
- Scatter/bias in model results for different ways of treating external effects?



(McCully thesis; McCully, Wong, et al. in prep.)

Mass sheet transformation: Traditional

Single plane: add uniform sheet and rescale galaxy

$$\boldsymbol{\alpha}'(\boldsymbol{x}) = \lambda\boldsymbol{\alpha}(\boldsymbol{x}) + (1-\lambda)\boldsymbol{x}$$

This leads to a rescaling of the source plane:

$$\boldsymbol{x}_s' \; = \; \boldsymbol{x} - \boldsymbol{\alpha}'(\boldsymbol{x}) \; = \; \lambda\boldsymbol{x} - \lambda\boldsymbol{\alpha}(\boldsymbol{x}) \; = \; \lambda\boldsymbol{x}_s$$

Such a rescaling cannot be observed unless we know absolute magnifications.

Differential time delays also rescale by λ.

(Falco et al. 1985; and much more)

Generalized (I)

One main plane plus many tidal planes. Required transformation:

$$\boldsymbol{\alpha}'_\ell(\boldsymbol{x}_\ell) = \lambda\boldsymbol{\alpha}_\ell(\boldsymbol{x}_\ell) + (1-\lambda)\mathbf{C}_{\ell s}^{-1}\mathbf{B}_s\mathbf{B}_\ell^{-1}\boldsymbol{x}_\ell$$

Differential time delays again rescale by λ.

Note: general case has shear along with a mass sheet.

Key quantity is **effective tidal tensor**,

$$\mathbf{\Gamma}_{\text{eff}} = \mathbf{I} - \mathbf{C}_{\ell s}^{-1}\mathbf{B}_s\mathbf{B}_\ell^{-1}$$

In small-shear limit,

$$\mathbf{\Gamma}_{\text{eff}} \approx \sum_{i=1,i\neq\ell}^{N} (1-\beta)\mathbf{\Gamma}_i$$

(McCully et al. 2014; Schneider 2014)

Generalized (II)

Peter Schneider (1409.0015) found a transformation that applies to multiple main planes, but it has a "curious behavior":

$$\begin{aligned} \boldsymbol{x}'_j &= \nu_j \boldsymbol{x}_j \\ \boldsymbol{\alpha}'_j(\boldsymbol{x}'_j) &= \lambda \boldsymbol{\alpha}_j(\boldsymbol{x}_j) + \mathbf{G}_j \boldsymbol{x}_j \end{aligned}$$

where

	ν_j	$\mathbf{G}_j$
j odd	1	$(1-\lambda)\mathcal{D}_j$
j even	λ	$(1-1/\lambda)\mathcal{D}_j$

With $0 < \lambda < 1$, we add positive density to odd planes and negative density to even planes.

Differential time delays again rescale by λ.

Lessons

Foreground perturbers create nonlinear effects.

- Image positions in the lens plane are not the same as image position on the sky.

3-d tidal approximation.

- $\mathbf{B}_\ell$ (foreground), $\mathbf{C}_{\ell s}$ (background), $\mathbf{B}_s$ (all)
- effective shear/convergence, $\boldsymbol{\Gamma}_{\rm eff} = \mathbf{I} - \mathbf{C}_{\ell s}^{-1}\mathbf{B}_s\mathbf{B}_\ell^{-1}$

Perturbers that are massive and/or projected close to the lens create higher-order terms.

- These perturbers need to be treated explicitly.

Our hybrid framework can handle an arbitrary combination of exact and tidal perturbers.