Analytic Langlands correspondence for complex curves

Edward Frenkel

Universty of California, Berkeley

May 31, 2021

Introduction

I will talk about a joint work with Pavel Etingof and David Kazhdan:

(1) *An analytic version of the Langlands correspondence for complex curves*, **arXiv:1908.09677**, in the Dubrovin Memorial Volume published by AMS;

(2) *Hecke operators and analytic Langlands correspondence for curves over local fields*, **arXiv:2103.01509**;

(3) *Analytic Langlands correspondence for PGL_2 on $\mathbb{P}^1$ with parabolic structures over local fields*, to appear soon.

Motivated by a suggestion of R.P. Langlands and results of J. Teschner, we propose an analytic version of the Langlands correspondence for complex curves.

In mathematics, Langlands correspondence can be formulated in 3 different scenarios, in the framework of André Weil's *Rosetta Stone*:

Number Fields *Curves over* $\mathbb{F}_q$ *Curves over* $\mathbb{C}$

Exploration of S-duality in 4D supersymmetric gauge theory (and mirror symmetry in 2D QFT) may be viewed as the 4th column.

In 2006, A. Kapustin and E. Witten linked S-duality to the geometric/categorical Langlands correspondence for curves over $\mathbb{C}$. This has inspred a great deal of research in this area.

In recent seminar talks, D. Gaiotto and E. Witten have given a very nice S-duality and "brane quantization" interpretation of some of the results in the analytic Langlands correspondence that I am going to talk about today (see also Edward Witten's talk at this conference later today).

Consider first the *unramified Langlands correspondence* for a curve $X/\mathbb{F}_q$ and a connected reductive algebraic group G (split over $\mathbb{F}_q$):

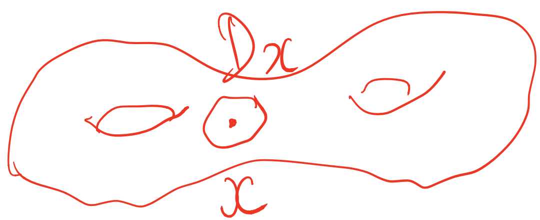
Let $\mathrm{Bun}_G(\mathbb{F}_q)$ be the set of isomorphism classes of principal G-bundles on X. Suppose G is *simple*. If we choose $x \in X(\mathbb{F}_q)$, then

$$\mathrm{Bun}_G(\mathbb{F}_q) \simeq G(\mathbb{F}_q[X\backslash x])\backslash G(\mathbb{F}_q((t_x)))/G(\mathbb{F}_q[[t_x]])$$

This is a discrete countable set with a natural measure assigning $[\mathcal{P}] \mapsto 1/|\operatorname{Aut}(\mathcal{P})|$ (well-defined because $\operatorname{Aut}(\mathcal{P})$ are finite groups).

We use this measure to define a Hermitean inner product on $\mathbb{C}$-valued functions on $\mathrm{Bun}_G(\mathbb{F}_q) \longrightarrow$ Hilbert space $\mathcal{H}_G$ of L^2 functions on $\mathrm{Bun}_G(\mathbb{F}_q)$.

Over $\mathbb{C}$:



transition function is in $G(\mathbb{F}_q((t_x)))$

4/44

Langlands Correspondence for curves over a finite field

Automorphic side: Joint spectrum of the commuting Hecke operators $H_{x,\lambda}$ acting on $\mathcal{H}_G$ (these are labeled by $x \in |X|$, $\lambda \in {}^L P^+$)

Galois side: unramified homomorphisms from the Galois group $\mathrm{Gal}(\overline{F}/F)$ to the Langlands dual group ${}^L G$ (here $F = \mathbb{F}_q(X)$)

(more precisely, from the Weil group of F; plus more data in general)

In other words, joint eigenvalues of the Hecke operators $H_{x,\lambda}$ on a specific eigenvector in $\mathcal{H}_G$ are encoded by a specific homomorphism $\sigma : \mathrm{Gal}(\overline{F}/F) \to {}^L G$ (namely, Hecke eigenvalues correspond to the images under σ of the Frobenius conjugacy classes associated to $x \in X$), and this sets up a one-to-one correspondence.

Now suppose that X is a curve over $\mathbb{C}$ (a compact Riemann surface).

In this case, we also have $\mathrm{Bun}_G(\mathbb{C})$, the set of isomorphism classes of principal G-bundles on X, and

$$\mathrm{Bun}_G(\mathbb{C}) \simeq G(\mathbb{C}[X\backslash x])\backslash G(\mathbb{C}((t_x)))/G(\mathbb{C}[[t_x]])$$

However, it is *not* possible to define integration measure on $\mathrm{Bun}_G(\mathbb{C})$ in the same way as over $\mathbb{F}_q$ because the groups $\mathrm{Aut}(\mathcal{P})$ of automorphisms of G-bundles can now be infinite.

For this reason, the Langlands correspondence for curves over $\mathbb{C}$ has been traditionally formulated in terms of sheaves rather than functions. It is usually referred to as *geometric* or *categorical*.

(V. Drinfeld, G. Laumon, A. Beilinson, ...)

Instead of functions on Bun_G, one considers the derived category of D-modules on Bun_G, and instead of Hecke operators one considers Hecke functors on this category.

Physics counterpart: category of A-branes on the *Hitchin moduli space* $\mathcal{M}_H(G, X)$ (with respect to ω_K) with 't Hooft line operators (Kapustin–Witten).

On the Langlands dual side:

Derived category (sutiably modified) of coherent sheaves on the moduli stack $\mathrm{Loc}_{{}^L G}$ of flat ${}^L G$-bundles on X (Arinkin–Gaitsgory).

Physics counterpart: category of B-branes on the *Hitchin moduli space* $\mathcal{M}_H({}^L G, X)$ (with respect to complex structure I) with Wilson line operators (Kapustin–Witten).

So, the *categorical* Langlands correspondence appears as a kind of non-abelian Fourier–Mukai transform (Belinson-Drinfeld).

This point of view has been the prevailing wisdom for how Langlands correspondence should be interpreted in the 3rd column of the Rosetta stone ("curves over $\mathbb{C}$") for the past 30 years.

However, it turns out that there is also a rich *analytic* Langlands correspondence for complex curves (i.e. *function-theoretic* instead of *sheaf-theoretic*).

Moreover, the two versions (categorical & analytic) complement each other. We can use each of them to gain new insights about the other.

Analogy: correlation functions in 2D conformal field theory are single-valued bilinear combinations of (multi-valued) conformal and anti-conformal blocks.

Namely, it is possible to associate to Bun_G of $X/\mathbb{C}$ (and more generally X/F, where F is a local field) a natural Hilbert space $\mathcal{H}_G$ and define analogues of the Hecke operators acting on a dense subspace of $\mathcal{H}_G$. We conjecture that they give rise to mutually commuting normal compact operators on $\mathcal{H}_G$.

In the case $F = \mathbb{C}$, these Hecke operators commute with the global holomorphic differential operators on Bun_G introduced by Beilinson and Drinfeld, as well as their complex conjugates.

We conjecture that the joint spectrum of this commutative algebra (properly understood) can be identified with the set of ${}^L G$-opers on X whose monodromy is in the *split real form* of ${}^L G$, up to conjugation (these play the role of the Galois representations).

This statement may be viewed as an analytic Langlands correspondence for complex curves.

The spectral problem may be viewed as a quantum integrable system.

Basic definitions:

X – smooth projective irreducible curve over $\mathbb{C}$

$S \subset X(\mathbb{C})$ – finite subset

K_X – canonical line bundle on X

G – connected simple algebraic group over $\mathbb{C}$

${}^L G$ – the Langlands dual group

$\mathrm{Bun}_G = \mathrm{Bun}_G(X, S)$ – algebraic stack of pairs $(\mathcal{F}, r_S)$, where $\mathcal{F}$ is a G-bundle on X and r_S is a B-reduction of $\mathcal{F}|_S$

$\mathrm{Bun}_G^\circ = \mathrm{Bun}_G^\circ(X, S) \subset \mathrm{Bun}_G(X, S)$ – substack of those stable pairs $(\mathcal{F}, r_S)$ whose group of automorphisms is the center $Z(G)$ of G

Assumption:

$\mathrm{Bun}_G^\circ(X,S)$ is *open and dense* in $\mathrm{Bun}_G(X,S)$, i.e. one of the following cases:

1. the genus of X is greater than 1, and S is arbitrary;
2. X is an elliptic curve and $|S| \geq 1$;
3. $X = \mathbb{P}^1$ and $|S| \geq 3$.

The stack $\mathrm{Bun}_G^\circ(X,S)$ is a $Z(G)$-gerbe over a smooth algebraic variety $Bun_G^{\mathrm{rs}}(X,S)$ (coarse moduli space).

For our purposes, $Bun_G^{\mathrm{rs}}(X,S)$ is a good replacement for $\mathrm{Bun}_G^\circ(X,S)$ because all objects we need descend to $Bun_G^{\mathrm{rs}}(X,S)$.

Hilbert space

K_{Bun} – the canonical line bundle on Bun_G.

For simply-connected G, Beilinson and Drinfeld have constructed a square root $K_{\mathrm{Bun}}^{1/2}$ of K_{Bun}. For a general G, their construction sometimes requires a choice of a square root of the canonical line bundle K_X on X. If so, we will make such a choice (however, the line bundle $\Omega_{\mathrm{Bun}}^{1/2}$ below does not depend on this choice).

We'll use the same notation for the restriction of this $K_{\mathrm{Bun}}^{1/2}$ to Bun_G^{rs}.

Given a holomorphic line bundle $\mathcal{L}$ on a variety Y, let

$|\mathcal{L}| := \mathcal{L} \otimes \overline{\mathcal{L}}$ be the corresponding C^∞ line bundle.

Set $\Omega_{\mathrm{Bun}}^{1/2} := |K_{\mathrm{Bun}}^{1/2}|$ – the line bundle of half-densities on Bun_G^{rs}.

Hilbert space

Let V_G – space of smooth compactly supported sections of $\Omega^{1/2}_{\rm Bun}$ over $Bun_G^{\rm rs}$, and let

$\langle \cdot, \cdot \rangle$ – positive-definite Hermitian form on V_G given by

$$\langle v, w \rangle := \int_{Bun_G^{\rm rs}} v \cdot \overline{w}, \qquad v, w \in V_G$$

$\mathcal{H}_G$ – the Hilbert space completion of V_G

What kind of operators could act on the Hilbert space $\mathcal{H}_G$?

1. holomorphic differential operators;
2. anti-holomorphic differential operators;
3. Hecke (integral) operators.

Challenges: Differential operators are unbounded. It is a highly non-trivial task to define their self-adjoint (or normal) extensions, which is necessary to be able to make sense of the notion of their joint spectra on $\mathcal{H}_G$ (and there could be different choices).

Hecke operators are also initially defined on a dense subspace of $\mathcal{H}_G$. But we conjecture that they extend by continuity to normal compact operators on the entire $\mathcal{H}_G$. If one proves this, one gets a good spectral problem for both Hecke & differential operators since one can show that they commute (in the sense we'll discuss later).

Holomorphic differential operators

Consider the case of simply-connected G and $|S| = \emptyset$ (so $g > 1$). Let $\mathcal{D}_G$ be the sheaf of algebraic (hence holomorphic) differential operators acting on the line bundle $K_{\mathrm{Bun}}^{1/2}$ on Bun_G.

$D_G := \Gamma(\mathrm{Bun}_G, \mathcal{D}_G)$ – global holomorphic diff. operators on $K_{\mathrm{Bun}}^{1/2}$

Theorem 1 (Beilinson & Drinfeld)

$D_G \simeq \mathrm{Fun}\,\mathrm{Op}_{{}^LG}(X)$, *where* $\mathrm{Op}_{{}^LG}(X)$ *– space of* ${}^LG$*-opers on* X.

Definition. An ${}^LG$-oper on a curve X is a holomorphic ${}^LG$-*bundle* with a holomorphic *connection* ∇ and a *reduction to a Borel subgroup* ${}^LB$ which is in a special relative position with ∇.

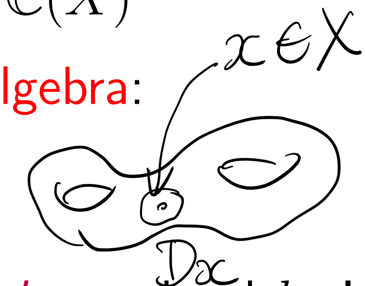
Example (to be discussed later). A PGL_2-oper on X is a *projective connection*, i.e. a second-order holomorphic differential operator of the form $\partial_z^2 - v(z)$: $K_X^{-1/2} \to K_X^{3/2}$.

Beilinson and Drinfeld derived their theorem from a **local result**:

Fix $x \in X$, and let $F_x \simeq \mathbb{C}((t))$ – completion of $F = \mathbb{C}(X)$

$\mathfrak{g}$ – simple Lie algebra, and $\widehat{\mathfrak{g}}_x$ – affine Kac–Moody algebra:

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}}_x \to \mathfrak{g} \otimes \mathbb{C}((t)) \to 0$$



Let $V_k(\mathfrak{g})$ be the corresponding *chiral (or vertex) algebra* at level k. It is generated by the holomorphic Kac–Moody currents $J^a(z)$.

$k = -h^\vee$ is called the critical level.

Theorem 2 (Boris Feigin & E.F.)

Let $Z(V_{-h^\vee}(\mathfrak{g}))$ be the center of the chiral algebra $V_{-h^\vee}(\mathfrak{g})$. Then

$$Z(V_{-h^\vee}(\mathfrak{g})) \simeq \operatorname{Fun} \operatorname{Op}_{{}^L G}(D_x)$$

where D_x is the disc around $x \in X$ and $\operatorname{Op}_{{}^L G}(D_x)$ is the space of ${}^L G$-opers on D_x.

Example. At the critical level, the Sugawara current

$$S(z) = \frac{1}{2}\sum_a : J^a(z) J_a(z) := \sum_{n\in\mathbb{Z}} S_n z^{-n-2}$$

commutes with the KM currents $J^a(z)$.

Let $G = SL_2, {}^LG = PGL_2$.

Then the center $Z(V_{-2}(\mathfrak{sl}_2))$ is equal to $\mathbb{C}[\partial_z^m S(z)]_{m\geq 0} \simeq \mathbb{C}[S_n]_{n\leq -2}$

On the other hand, PGL_2-oper on D_x is the same as a *projective connection*, i.e. a second-order holomorphic differential operator of the form $\partial_z^2 - v(z)$: $K_X^{-1/2} \to K_X^{3/2}$, where $v(z) = \sum_{n\leq -2} v_n z^{-n-2}$

So, $\mathrm{Fun}\,\mathrm{Op}_{PGL_2}(D_x) = \mathbb{C}[v_n]_{n\leq -2}$

The isomorphism $Z(V_{-2}(\mathfrak{sl}_2)) \simeq \mathrm{Fun}\,\mathrm{Op}_{PGL_2}(D_x)$

sends $S_n \mapsto v_n$

$$\mathrm{Bun}_G \simeq G(\mathbb{C}[X\backslash x])\backslash G(F_x)/G(\mathcal{O}_x)$$

$\widehat{\mathfrak{g}}_x$

$\widehat{\mathfrak{g}}_x$ acts (from the right) on sections of a $G(\mathcal{O}_x)$-equivariant line bundle on $G(X\backslash x)\backslash G(F_x)$, which descends to a square root $K_{\mathrm{Bun}}^{1/2}$ of the canonical line bundle on Bun_G. Central element $\mathbf{1} \mapsto -h^\vee$.

Hence $Z(V_{-h^\vee}(\mathfrak{g})) \to D_G$, global hol. diff. operators on $K_{\mathrm{Bun}}^{1/2}$.

Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} Z(V_{-h^\vee}(\mathfrak{g})) & \xrightarrow[\text{FF}]{\sim} & \mathrm{Fun}\,\mathrm{Op}_{{}^LG}(D_x) \\ \downarrow & & \downarrow \\ D_G & \xrightarrow[\text{BD}]{\sim} & \mathrm{Fun}\,\mathrm{Op}_{{}^LG}(X) \end{array}$$

Anti-holomorphic differential operators

Complex conjugates of elements of D_G are global anti-holomorphic differential operators acting on $\overline{K}^{1/2}_{\mathrm{Bun}}$.

They generate a commutative algebra $\overline{D}_G$.

$\overline{D}_G \simeq \mathrm{Fun}\, \overline{\mathrm{Op}}_{^L G}(X)$

$\mathcal{A}_G := D_G \otimes \overline{D}_G$ is a commutative algebra acting on C^∞ sections of the line bundle $\Omega^{1/2}_{\mathrm{Bun}} = K^{1/2}_{\mathrm{Bun}} \otimes \overline{K}^{1/2}_{\mathrm{Bun}}$ on Bun^{rs}_G.

Let $\widetilde{V}_G$ be the space of smooth sections of $\Omega^{1/2}_{\mathrm{Bun}}$ on $Bun^{\mathrm{vs}}_G \subset Bun^{\mathrm{rs}}_G$, the moduli space of *very stable G-bundles* (i.e. those $\mathcal{F}$ which do not admit non-zero $\phi \in \Gamma(X, \mathfrak{g}_{\mathcal{F}} \otimes K_X)$ taking nilpotent values everywhere).

“Doubling” of the quantum Hitchin system

Given a homomorphism $\Lambda : \mathcal{A}_G \to \mathbb{C}$, denote by $\widetilde{V}_{G,\Lambda}$ the corresponding eigenspace of $\mathcal{A}_G$ in $\widetilde{V}_G$.

$\Lambda = (\chi, \mu)$, where $\chi \in \mathrm{Op}_{^LG}(X), \mu \in \overline{\mathrm{Op}}_{^LG}(X)$.

If f is a non-zero element of $\widetilde{V}_{G,(\chi,\mu)}$, then it satisfies two systems of differential equations:

(1) $P \cdot f = \chi(P) f, \qquad P \in D_G$

(2) $Q \cdot f = \mu(Q) f, \qquad Q \in \overline{D}_G$

System (1) is known as the *quantum Hitchin system*.

System (2) is its anti-holomorphic analogue.

The corresponding left $\mathcal{D}_G$-module

$$\Delta_\chi := \mathcal{D}_G \underset{D_G}{\otimes} \mathbb{C}_\chi$$

was introduced and studied by Beilinson and Drinfeld, who have proved that Δ_χ is a Hecke eigensheaf corresponding to the ${}^LG$-oper χ under the geometric/categorical Langlands correspondence.

Moreover, they have shown that the restriction of Δ_χ to Bun_G^{vs} is a vector bundle with a projectively flat connection (of a rank that grows exponentially with the genus of X).

Local sections of Δ_χ over Bun_G^{vs} are local holomorphic solutions of system (1). They are *multi-valued* and the monodromy is rather complicated, which is why there is no natural way in general to attach to a given χ a specific holomorphic half-form. (Even if there were single-valued solutions, it wouldn't be clear which one to choose.) Instead, we attach a whole $\mathcal{D}_G$-module on Bun_G to χ.

Likewise, to $\mu \in \overline{\mathrm{Op}}_{^LG}(X)$ we attach an anti-holomorphic D-module $\overline{\Delta}_\mu$ whose local sections on Bun_G^{vs} are local anti-holomorphic solutions of system (2), also multi-valued.

However, if we look for *smooth solutions* of systems (1) and (2) simultaneously, it is possible that for some χ and μ there will be a single-valued solution, which can be written locally in bilinear form

$f = \sum_{i,j} a_{ij} \; \phi_i(\mathbf{z})\overline{\psi}_j(\overline{\mathbf{z}})$

$\{\phi_i\}$ – local sections of Δ_χ

$\{\overline{\psi}_j\}$ – local sections of $\overline{\Delta}_\mu$.

This actually implies that $\dim \widetilde{V}_{G,(\chi,\mu)} < \infty$.

Moreover, if Δ_χ is irreducible and has regular singularities (for $G = SL_n$, this follows from the results of Dennis Gaitsgory) and $\widetilde{V}_{G,(\chi,\mu)} \neq 0$, then $\dim \widetilde{V}_{G,(\chi,\mu)} = 1$.

Conjecture 3

1. All $\widetilde{V}_{G,(\chi,\mu)} \subset \mathcal{H}_G$
2. There is an orthogonal decomposition

   $$\mathcal{H}_G = \widehat{\bigoplus}_{(\chi,\mu)} \widetilde{V}_{G,(\chi,\mu)}$$

3. If $\widetilde{V}_{G,(\chi,\mu)} \neq 0$, then $\mu = \tau(\overline{\chi})$, where τ is the Chevalley involution on ${}^L G$ and $\chi \in \mathrm{Op}_{{}^L G}(X)_{\mathbb{R}}$.

Definition. $\mathrm{Op}_{{}^L G}(X)_{\mathbb{R}}$ is the set of ${}^L G$-opers on X such that the *monodromy representation* $\rho_\chi : \pi_1(X, p_0) \to {}^L G(\mathbb{C})$ is isomorphic to its complex conjugate, i.e. $\rho_\chi \simeq \overline{\rho}_\chi$.

We expect that $\mathrm{Op}_{{}^L G}(X)_{\mathbb{R}}$ is a *discrete subset* of $\mathrm{Op}_{{}^L G}(X)$. This is known for $G = SL_2$ (G. Faltings).

For $G = SL_2$, Conjecture 3 implements ideas of J. Teschner.

Remark.

We expect that $\mathrm{Op}_{{}^LG}(X)_{\mathbb{R}}$ coincides with the set of all ${}^LG$-opers on X with *real monodromy*, i.e. such that the image in ${}^LG(\mathbb{C})$ of the monodromy representation

$$\rho_\chi : \pi_1(X, p_0) \to {}^LG$$

associated to χ is contained, up to conjugation, in the split real form ${}^LG(\mathbb{R})$ of ${}^LG(\mathbb{C})$.

This is known for $G = SL_2$ and we can prove it for general G in the case when there is at least one point with Borel reduction (i.e. $|S| \neq \emptyset$).

Hecke operators

Proving Conjecture 3 directly is a daunting task. This is where the third set of operators on $\mathcal{H}_G$ – integral Hecke operators – comes in handy.

Though they are also initially defined on a dense subspace of $\mathcal{H}_G$ (like diff. operators), we conjecture that, unlike the differential operators, they extend to (mutually commuting) continuous operators on the entire $\mathcal{H}_G$, which are moreover *normal* and *compact* with trivial common kernel.

If so, then by a general result of functional analysis, $\mathcal{H}_G$ decomposes into a (completed) direct sum of mutually orthogonal *finite-dimensional eigenspaces* of the Hecke operators. Moreover, we can show that they commute with the differential operators, and so the Compactness Conjecture can be used to prove Conjecture 3.

Before giving a mathematical definition of the Hecke operators, I'd like to give an informal interpretation of these operators from the point of view of 2D CFT. Consider the case of $G = PGL_2, {}^L G = SL_2$.

When the level k is non-critical, there are natural chiral vertex operators of the Kac-Moody algebra $\widehat{\mathfrak{g}}$ which we denote by $\Phi_{m,n}(z)$, where m, n are positive integers.

Quantum Drinfeld–Sokolov reduction maps them to the more familiar chiral (m, n) vertex operators of the CFT with Virasoro symmetry (they appear naturally in the minimal models and Liouville theory).

There are two extreme cases: the *electric* vertex operators $V_{m,1}(z)$ and the *magnetic* vertex operators $V_{1,n}(z)$.

The latter have a nice limit as $k \to -2$, the critical level, where they become commutative. The simplest is $V_{1,2}(z)$ and it has conformal dimension $-1/2$, i.e. it behaves as a section of the line bundle $K_X^{-1/2}$, where K_X is a canonical line bundle on X.

Chiral vertex operator $V_{1,2}(z)$ has an anti-chiral analogue $\overline{V}_{1,2}(\overline{z})$, which behaves as a section of the line bundle $\overline{K}_X^{-1/2}$ on X.

From these chiral and anti-chiral operators we can cook up a "mixed" vertex operator $\Phi_{1,2}(z, \overline{z})$, which is a section of $K_X^{-1/2} \otimes \overline{K}_X^{-1/2}$.

One way to construct it is to observe that both operators $V_{1,2}(z)$ and $\overline{V}_{1,2}(z)$ have two components (since they correspond to the two-dim. rep. of the dual group ${}^L G = SL_2$). Then $\Phi_{1,2}(z, \overline{z})$ can be constructed as a bilinear combination of these components.

Now the point is that, roughly speaking, the action of the Hecke operator $H_{\omega_1, x}$ corresponds to the insertion of the "mixed" vertex operator $\Phi_{1,2}(z, \overline{z})$ into the correlation functions of "2D CFT at the critical level" at the point $x \in X$ (here I mean a full-fledged 2D CFT, combining chiral and anti-chiral sectors).

From this point of view, it is not surprising that $\Phi_{1,2}(z,\overline{z})$ would satisfy an analogue of the BPZ equation familiar from the study of 2D CFT with Virasoro symmetry.

In fact, this equation has the form of the "universal oper equation"

The reason is that $V_{1,2}(z)$ satisfies the differential equation

$$(\partial_z^2 - S(z))V_{1,2}(z) = 0$$

which is equivalent to the equation ("decoupling of a null-vector")

$$(L_{-1}^2 - S_{-2})|V_{1,2}\rangle = 0$$

This implies that $\Phi_{1,2}(z,\overline{z})$ satisfies the same equation, *plus* its complex conjugate:

$$(\partial_z^2 - S(z))\Phi_{1,2}(z,\overline{z}) = 0, \qquad (\overline{\partial}_z^2 - \overline{S}(\overline{z}))\Phi_{1,2}(z,\overline{z}) = 0.$$

From these local equations, we can derive global differential equations.

To write them down explicitly, pick a point $\chi_0 \in \mathrm{Op}_{SL_2}(X)$ and use it to identify $\mathrm{Op}_{SL_2}(X)$ with $H^0(X, K_X^2)$.

The corresponding projective connection is $P_{\chi_0} = \partial_z^2 - v_0(z)dz^2$ (locally on X).

Pick a basis $\{\varphi_i, i = 1, \ldots, 3g-3\}$ of $H^0(X, K_X^2)$

Then every other projective connection can be written as

$$\partial_z^2 - v_0(z)dz^2 - \sum_{i=1}^{3g-3} a_i \varphi_i, \qquad a_i \in \mathbb{C}$$

We want a universal projective connection with values in D_{PGL_2}, global holomorphic diff. operators on Bun_{PGL_2}.

So, let $\{F_i, i = 1, \ldots, 3g-3\}$ be the dual set of generators of the polynomial algebra $\mathrm{Fun}\,\mathrm{Op}_{SL_2}(X) = D_{PGL_2}$ dual to the basis $\{\varphi_i\}$.

We view each F_i as a global holomorphic diff. operator on Bun_{PGL_2}.

Define the universal projective connection (universal SL_2-oper)

$$\sigma = \partial_z^2 - v_0(z)dz^2 - \sum_{i=1}^{3g-3} F_i \otimes \varphi_i \; : \; K_X^{-1/2} \to D_{PGL_2} \otimes K_X^{3/2}$$

Then we have the following equations on the correlation functions:

$$\sigma \cdot \langle \Phi_{1,2}(z,\overline{z}) ... \rangle = 0, \qquad \overline{\sigma} \cdot \langle \Phi_{1,2}(z,\overline{z}) ... \rangle = 0$$

We have proved that our Hecke operators $H_{\omega_1,x}$ satisfy these differential equations. Our proof (in our second paper) is purely global. But the above local-to-global argument can also be made mathematically rigorous, and hence it provides an alternative proof.

These equations allow us to express the eigenvalues of $H_{\omega_1,x}$ in terms of the eigenvalues of the global differential operators (which are given by real SL_2-opers), as we will see below.

Let's now give a mathematical definition of the Hecke operators.

In fact, they can be defined for curves over any local field.

For non-archimedian local fields, these operators were essentially defined by A. Braverman and D. Kazhdan in *Some examples of Hecke algebras for two-dimensional local fields*, Nagoya Math. J. Volume 184 (2006), 57-84.

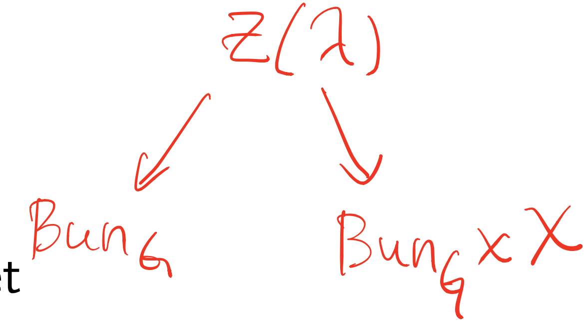
For $G = PGL_2, X = \mathbb{P}^1$, Hecke operators were studied by M. Kontsevich in his paper *Notes on motives in finite characteristic* (2007). In his letters to us (2019) he conjectured compactness of averages of the Hecke operators over sufficiently many points.

The idea that Hecke operators over $\mathbb{C}$ could be used to construct an analogue of the Langlands correspondence was suggested in 2018 by R.P. Langlands, who sought to construct them in the case when $G = GL_2$, X is an elliptic curve, and $S = \emptyset$ (however, for an elliptic curve X we can only define Hecke operators if $|S| \neq \emptyset$).

For a dominant coweight λ of G, denote by

$$q : Z(\lambda) \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times X$$

the *Hecke correspondence* attached to λ. Let



$$p_{1,2} : \mathrm{Bun}_G \times \mathrm{Bun}_G \times X \to \mathrm{Bun}_G, \quad p_3 : \mathrm{Bun}_G \times \mathrm{Bun}_G \times X \to X$$

be the projections, and set $q_i := p_i \circ q$.

The following is due to Beilinson–Drinfeld and Braverman–Kazhdan.

Theorem 4

There exists an isomorphism

$$a : q_1^*(K_{\mathrm{Bun}}^{1/2}) \simeq q_2^*(K_{\mathrm{Bun}}^{1/2}) \otimes \omega_2 \otimes q_3^*(K_X^{-\langle\lambda,\rho\rangle})$$

where ω_2 is the relative canonical bundle along the fibers of $q_2 \times q_3$ and ρ is the half sum of positive roots.

The isomorphism a gives rise to an isomorphism

$$|a| : q_1^*(\Omega_{\text{Bun}}^{1/2}) \simeq q_2^*(\Omega_{\text{Bun}}^{1/2}) \otimes \Omega_2 \otimes q_3^*(|K_X|^{-\langle\lambda,\rho\rangle})$$

where $\Omega_2 := |\omega_2|$ is the relative line bundle of densities along the fibers of $q_2 \times q_3$. Let

$$U_G(\lambda) := \{\mathcal{F} \in Bun_G^{\text{rs}} | (q_2(q_1^{-1}(\mathcal{F})) \subset Bun_G^{\text{rs}}\}$$

This is an open subset of Bun_G^{rs}, which is dense if

$$\dim \text{Bun}_G = \dim G \cdot (g-1) + \dim G/B \cdot |S| \quad (g > 1)$$

is sufficiently large. (For example, for $G = PGL_2, \lambda = \omega_1$, this is so if $\dim \text{Bun}_G > 1$.)

Assume that $U_G(\lambda) \subset Bun_G^{\text{rs}}$ is dense and let $V_G(\lambda) \subset V_G$ be the subspace of half-densities f such that $\text{supp}(f) \subset U_G(\lambda)$.

$Z_{\mathcal{G},x} := (q_2 \times q_3)^{-1}(\mathcal{G} \times x), \qquad \mathcal{G} \in \mathrm{Bun}_G(\mathbb{C}), \quad x \in X(\mathbb{C})$

It is compact and isomorphic to the closure $\overline{\mathrm{Gr}}_\lambda$ of the $G[[z]]$-orbit Gr_λ in the affine Grassmannian of G.

The results of Braverman–Kazhdan imply that for any $f \in V_G(\lambda)$ and $x \in X(\mathbb{C})$, the restriction of the pull-back $q_1^*(f)$ to $Z_{\mathcal{G},x}$ is a well-defined measure with values in the line $|\Omega_{\mathrm{Bun}}|_{\mathcal{G}}^{1/2} \otimes |K_X|_x^{-\langle \lambda,\rho \rangle}$.

Hence for any $f \in V_G(\lambda)$, the integral

$$(\widehat{H}_\lambda(x) \cdot f)(\mathcal{G}) := \int_{Z_{\mathcal{G}}^x(F)} q_1^*(f)$$

is absolutely convergent for all $\mathcal{G} \in Bun_G^{\mathrm{rs}}(\mathbb{C})$ and belongs to the space V_G of compactly supported smooth sections on $Bun_G^{\mathrm{rs}}(\mathbb{C})$.

Therefore this integral defines a *Hecke operator*

$$\widehat{H}_\lambda(x) : V_G(\lambda) \to V_G \otimes |K_X|_x^{-\langle \lambda,\rho \rangle}$$

Thus, we obtain an operator

$$\widehat{H}_\lambda(x) : V_G(\lambda) \to \mathcal{H}_G \otimes |K_X|_x^{-\langle \lambda, \rho \rangle}$$

Conjecture 5 (Compactness Conjecture)

1. For any identification $(K_X^{1/2})_x \cong \mathbb{C}$, the corresponding operators $V_G(\lambda) \to \mathcal{H}_G$ extend to a family of commuting compact normal operators on $\mathcal{H}_G$, which we denote by $H_\lambda(x)$.
2. $H_\lambda(x)^\dagger = H_{-w_0(\lambda)}(x)$.
3. $\bigcap_{\lambda,x} \mathrm{Ker} H_\lambda(x) = \{0\}$.

In our third paper, we have proved this conjecture for $G = PGL_2$ and $X = \mathbb{P}^1$ (with $|S| \geq 3$).

From now on we **assume** that Compactness Conjecture holds.

Let $\mathbb{H}_G$ be the commutative algebra generated by operators $H_\lambda(x), \lambda \in \check{P}^+, x \in X$. Denote by $\mathrm{Spec}(\mathbb{H}_G)$ its spectrum.

Corollary 6

There is an orthogonal decomposition

$$\mathcal{H}_G = \widehat{\bigoplus}_{s \in \mathrm{Spec}(\mathbb{H}_G)} \mathcal{H}_G(s)$$

where $\mathcal{H}_G(s), s \in \mathrm{Spec}(\mathbb{H}_G)$, are the finite-dimensional joint eigenspaces of $\mathbb{H}_G$ in $\mathcal{H}_G$.

At the moment, we only have a (conjectural) description of $\mathrm{Spec}(\mathbb{H}_G)$ for $F = \mathbb{C}$ (and, in some cases, for $F = \mathbb{R}$).

Recall that in the case $F = \mathbb{C}$, we also have the algebra $\mathcal{A}_G = D_G \otimes \overline{D}_G$ of differential operators.

Observe that $\mathcal{A}_G$ acts on the space $V_G^\vee$ of distributions on Bun_G^{rs}, and $\mathcal{H}_G$ is naturally realized as a subspace of $V_G^\vee$. Hence we can apply elements of $\mathcal{A}_G$ to vectors in the eigenspaces $\mathbb{H}_G(s)$ of the Hecke operators, viewed as *distributions*.

Conjecture 7

Every $\mathbb{H}_G(s)$ is an eigenspace of $\mathcal{A}_G$.

Corollary 8

The spectrum of $\mathcal{A}_{PGL_2}$ on $\mathcal{H}_{PGL_2}$ (properly defined) is contained in the set of (χ, μ), such that $\mu = \overline{\chi}$, and $\chi \in \mathrm{Op}_{SL_2}(X)_{\mathbb{R}}$.

We have proved this for $G = PGL_2$ and $X = \mathbb{P}^1$ (in our third paper).

The case of $G = PGL_2$, so ${}^L G = SL_2$

Consider SL_2-opers on X (following Beilinson and Drinfeld):

$$\mathrm{Op}_{SL_2}(X) = \bigsqcup_{\gamma \in \theta(X)} \mathrm{Op}^{\gamma}_{SL_2}(X)$$

where $\theta(X)$ is the set of isomorphism classes of square roots of K_X.

Pick a square root $K_X^{1/2}$ of K_X. An SL_2-oper in the corresponding component $\mathrm{Op}^{\gamma}_{SL_2}(X)$ is a holomorphic connection on the rank 2 vector bundle $\mathcal{V}_{\omega_1}$

$$0 \to K_X^{1/2} \to \mathcal{V}_{\omega_1} \longrightarrow K_X^{-1/2} \to 0$$

satisfying a *transversality condition*.

Here's an alternative description of this component.

A *projective connection* associated to $K_X^{1/2}$ is a second-order differential operator $P : K_X^{-1/2} \to K_X^{3/2}$ such that

1. $\mathrm{symb}(P) = 1 \in \mathcal{O}_X$, and
2. P is algebraically self-adjoint.

They form an affine space $\mathcal{P}roj_\gamma(X)$. Locally, $P = \partial_z^2 - v(z)$.

Lemma 9

There is a bijection $\mathrm{Op}^\gamma_{SL_2}(X) \simeq \mathcal{P}roj_\gamma(X)$

$$\chi \in \mathrm{Op}^\gamma_{SL_2}(X) \quad \mapsto \quad P_\chi \in \mathcal{P}roj_\gamma(X)$$

such that the section $s_{\omega_1} \in \Gamma(X, K_X^{-1/2} \otimes \mathcal{V}_{\omega_1})$ *corresponding to the embedding* $K_X^{1/2} \hookrightarrow \mathcal{V}_{\omega_1}$ *satisfies* $P_\chi \cdot s_{\omega_1} = 0$
(here we use the $\mathcal{D}_X$*-module structure on* $\mathcal{V}_{\omega_1}$ *corresponding to* ∇_χ*).*

Let $\mathcal{V}_{\omega_1}^{\text{univ}}$ be the universal vector bundle over $\text{Op}_{SL_2}^{\gamma}(X) \times X$ with a partial connection ∇^{univ} along X, such that

$$(\mathcal{V}_{\omega_1}^{\text{univ}}, \nabla^{\text{univ}})|_{\chi \times X} = (\mathcal{V}_{\omega_1}, \nabla_{\chi})$$

Let $\mathcal{V}_{\omega_1,X}^{\text{univ}} := \pi_*(\mathcal{V}_{\omega_1}^{\text{univ}})$, where $\pi : \text{Op}_{SL_2}^{\gamma}(X) \times X \to X$. The connection ∇^{univ} makes $\mathcal{V}_{\omega_1,X}^{\text{univ}}$ into a left $\mathcal{D}_X$-module.

The algebra $D_{PGL_2} \simeq \text{Fun}\,\text{Op}_{SL_2}^{\gamma}(X)$ acts on $\mathcal{V}_{\omega_1,X}^{\text{univ}}$ and commutes with the action of $\mathcal{D}_X$.

Lemma 10

The second-order differential operator we constructed earlier

$$\sigma : K_X^{-1/2} \to D_{PGL_2} \otimes K_X^{3/2}$$

is the unique operator such that for any $\chi \in \text{Op}_{SL_2}^{\gamma}(X)$, applying the corresponding homomorphism $D_{PGL_2} \to \mathbb{C}$ we obtain P_{χ}.

Differential equation on Hecke operators

As x varies along X, the Hecke operators $\widehat{H}_{\omega_1}(x)$ combine into a section of the C^∞ line bundle $|K_X|^{-1/2}$ on X with values in operators $\mathcal{H}_{PGL_2} \to \mathcal{H}_{PGL_2}$. We denote it by $\widehat{H}_{\omega_1}$.

Theorem 11

The Hecke operator $\widehat{H}_{\omega_1}$, viewed as an operator-valued section of $|K_X|^{-1/2}$, satisfies the system of differential equations

$$\sigma \cdot \widehat{H}_{\omega_1} = 0, \qquad \overline{\sigma} \cdot \widehat{H}_{\omega_1} = 0$$

This is the system of second-order differential equations (one holomorphic and one anti-holomorphic) we discussed earlier.
In our second paper, we derived it from a theorem of Beilinson and Drinfeld about Hecke eigensheaves. It can also be proved by a local-to-global argument we discussed earlier.

We can use this result to describe the eigenvalues of the Hecke operators in terms of the eigenvalues of the global differential operators.

We have $\mathrm{Bun}_{PGL_2} = \mathrm{Bun}^0_{PGL_2} \bigsqcup \mathrm{Bun}^1_{PGL_2}$

Hence $\mathcal{H}_{PGL_2} = \mathcal{H}^0_{PGL_2} \oplus \mathcal{H}^1_{PGL_2}$

Let $\chi \in \mathrm{Op}^{\gamma}_{SL_2}(X)_{\mathbb{R}}$. Here's a more precise version of Corollary 8:

Conjecture 12

There is a one-dimensional eigenspace $\mathcal{E}^i_\chi$ of $\mathcal{A}_{PGL_2}$ corresponding to χ in $\mathcal{H}^i_{PGL_2}, i = 0, 1$, and their direct sum $\mathcal{E}^0_\chi \oplus \mathcal{E}^1_\chi$ is stable under the action of the Hecke operators.

Corollary 13

We can choose generators $\psi^i_\chi \in \mathcal{E}^i_\chi$ in such a way that $\Psi^{\pm}_\chi = \psi^0_\chi \pm \psi^1_\chi$ are eigenvectors of H_{ω_1}. The corresponding eigenvalues of H_{ω_1} differ by a sign.

Denote these eigenvalues of H_{ω_1} by $\pm\Phi_{\omega_1}(\chi)$. Recall that these are sections of the line bundle $|K_X|^{-1/2}$ on X. We are going to write an explicit formula for $\pm\Phi_{\omega_1}(\chi)$ in terms of χ.

Recall $\qquad 0 \to K_X^{1/2} \to \mathcal{V}_{\omega_1} \longrightarrow K_X^{1/2} \to 0$

and $s_{\omega_1} \in \Gamma(X, K_X^{-1/2} \otimes \mathcal{V}_{\omega_1})$ corresponding to $K_X^{-1/2} \hookrightarrow \mathcal{V}_{\omega_1}$.

By definition of $\mathrm{Op}^{\gamma}_{SL_2}(X)_{\mathbb{R}}$,

$$(\mathcal{V}_{\omega_1}, \nabla_{\chi,\omega_1}) \simeq (\overline{\mathcal{V}}_{\omega_1}, \overline{\nabla}_{\chi,\omega_1})$$

as C^∞ flat bundles. Since $\mathcal{V}_{\omega_1} \simeq \mathcal{V}^*_{\omega_1}$, we get an Hermitian form

$$h_{\chi,\omega_1}(\cdot,\cdot) : (\mathcal{V}_{\omega_1}, \nabla_{\chi,\omega_1}) \otimes (\overline{\mathcal{V}}_{\omega_1}, \overline{\nabla}_{\chi,\omega_1}) \to (\mathcal{C}^\infty_X, d)$$

normalized (up to a sign) by a natural condition on the determinants.

Conjecture 14

$$\pm\Phi_{\omega_1}(\chi) = \pm h_{\chi,\omega_1}(s_{\omega_1}, \overline{s_{\omega_1}})$$

We prove this (up to a scalar) by showing that both sides satisfy the same system of second-order differential equations.

Recall that $\chi \mapsto P_\chi : K_X^{-1/2} \to K_X^{3/2}$ and $P_\chi \cdot s_{\omega_1} = 0$

Lemma 15

$h_{\chi,\omega_1}(s_{\omega_1}, \overline{s_{\omega_1}})$ is the unique, up to a scalar, section Φ of $|K_X|^{-1/2}$ which is a solution of the system

$$P_\chi \cdot \Phi = 0, \qquad \overline{P}_\chi \cdot \Phi = 0$$

Now recall from Theorem 11 that the Hecke operator H_{ω_1} satisfies

$$\sigma \cdot H_{\omega_1} = 0, \qquad \overline{\sigma} \cdot H_{\omega_1} = 0$$

This implies that the eigenvalues of the Hecke operator H_{ω_1} satisfy the system of Lemma 15. Hence we prove Conjecture 14 up to a scalar. There are analogous statements for any G (in our second paper).