# Quantum mechanics of bi-partite ribbon graphs and Kronecker coefficients.

Sanjaye Ramgoolam

Queen Mary, University of London

Kavli IPMU workshop on Number theory, Strings and Quantum Physics.

Based on

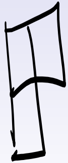
J. Ben Geloun and S. Ramgoolam, "Quantum mechanics of bipartite ribbon graphs: Integrality, Lattices and Kronecker coefficients" arXiv:2010.04054v1 [hep-th]

## Intro: Kronecker coefficients

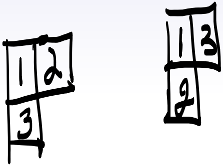
Many quantities in symmetric group representation theory are combinatorially constructible.

E.g. dimensions of irreps (standard tableaux) ;

Little-wood Richardson coefficients : counting labelled skew Young tableaux according to some labelling rules



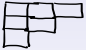
→ · Dim of $S_3$ irrep $= 2$

· 

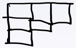
2 standard Young tableaux

27 May 2021 13:28

 $\rightarrow$ $S_3$ irrep. : Dim = 2

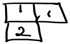
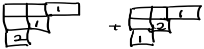
 $\rightarrow$ $S_6$ irrep. : Dim. = 16

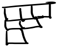
$S_3 \times S_3 \subset S_6$

 $\rightarrow$ $\underbrace{2}_{}$ (  ) ,

Decompose this irrep of $S_6$ $\longrightarrow$ into (irreps of $S_3 \times S_3$)

There is a combinatoric count. of the multiplicities (Littlewood Richard. coeff.)

 ·  $\rightarrow$ 

Construct the "skew Young tableaux" (labelled according to Littlewood Richardson rule) — count how many times the shape  appears :

. $\underline{\underline{2}}$

The multiplicities are properties of representations defined over $\mathbb{C}$, but they can be calculated by the LR rule – just using

- **integers**
- **discrete objects** (partially labelled Young diagrams)
- **and counting.**

The Littlewood-Richardson coefficients are multiplicities of reduction of irreps of $S_{m+n}$ into irreps of $S_m \times S_n$.

They also give the multiplicities of tensor product decompositions of $U(N)$ tensor products $R \otimes S \rightarrow T$. The relation between these two interpretations is given by Schur-Weyl duality.

Characters of $S_n$ also have a combinatoric construction by the Murnaghan-Nakayama Lemma

$$\chi_\rho^R = \sum_{T \in BST(R;\rho)} (-1)^{ht(T)}$$

Count labellings of young diagram $R$; labelling rules depend on $\rho$ ; height function for each $T$.

From this construction, it is clear that these characters are integers.

The Kronecker coefficient for a triple of Young diagrams with $n$ boxes is the number of times the one-dimensional (trivial) irrep of $S_n$ appears in

$(6 \otimes 6 \otimes 6)$

$$V_{R_1}^{(S_n)} \otimes V_{R_2}^{(S_n)} \otimes V_{R_3}^{(S_n)}$$

Equivalently the number of times $V_{R_3}^{(S_n)}$ appears in the decomposition of $V_{R_1}^{(S_n)} \otimes V_{R_2}^{(S_n)}$.

For Kronecker coefficients, we know from rep theory interpretation that they are non-negative. But there is no manifestly positive construction ( such as LR) or formula :

$$C(R_1, R_2, R_3) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{R_1}(\sigma)\chi^{R_2}(\sigma)\chi^{R_3}(\sigma)$$

Is there a manifestly positive construction ?

Discussed in Stanley (1999) - positivity problems and conjectures , also recent papers in connection with computational complexity, e.g. Pak and Panova, “On the complexity of computing Kronecker coefficients,” Comp. Complexity 2017.

## Intro: Algebras $\mathcal{K}(n)$

We will be approaching this problem using a family of algebras $\mathcal{K}(n)$, one for every positive integer $n$

These algebras $\mathcal{K}(n)$ have a lot of information about tensor model observables, their counting, associated large $N$ phase transitions, and correlators of tensor model observables – which are of interest in the context of holography for tensor models.

The rapid growth of the dimension of $\mathcal{K}(n)$ as a function of $n$ implies a vanishing large $N$ Hagedorn temperature ( Beccaria, Tseytlin, 2017).

Here we focus on a mathematical application of $\mathcal{K}(n)$.

$\mathcal{K}(n)$ is a sub-algebra of $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$.

$\mathbb{C}(S_n)$ is a vector space of dimension $n!$ spanned by permutations $\sigma \in S_n$. For two elements

$$a = \sum_{\sigma \in S_n} a_\sigma \sigma$$
$$b = \sum_{\tau \in S_n} b_\tau \tau$$
$$ab = \sum_{\sigma\tau} a_\sigma b_\tau (\sigma\tau)$$

$\mathcal{K}(n)$ has a nice combinatoric basis related to tensor invariants of $U(N)$ in the large $N$ limit

## Intro: A basis for tensor invariants

Consider $\Phi_{ijk}$ is a 3-index tensor variable. $\bar{\Phi}^{ijk}$ is the conjugate

Transform as $V_N \otimes V_N \otimes V_N$ of $U(N) \times U(N) \times U(N)$. And $\bar{V}_N \otimes \bar{V}_N \otimes \bar{V}_N$.

In "tensor models" (generalizations of random matrix theories) we are interested in polynomial functions $\Phi, \bar{\Phi}$ which are invariant under $U(N) \times U(N) \times U(N)$.

Ben Geloun, Ramgoolam, "Counting Tensor Model Observables and Branched Covers of the 2-Sphere," AIHPD 2014

Also from quantum information theory motivations
"Stable Hilbert series as related to the measurement of quantum entanglement," MW Hero, JF Willenbring, Discrete Maths 2009.

The tensor product $V_N \otimes \bar{V}_N$ decomposes as the adjoint of dimension $N^2 - 1$ and the invariant.
The invariant, where $\{e^i, \bar{e}_j\}$ are the standard bases for $U(N)$ fundamental/anti-fundamental reps

$$\sum_i e^i \otimes \bar{e}_i = \sum_{i,j} \delta^i_j e^j \otimes \bar{e}_i$$

States in $V^{\otimes n} \otimes \bar{V}^{\otimes n}$ (for $n \le N$ ), invariant under the diagonal $U(N)$ action, span a space of dimension $n!$ (follows from Schur-Weyl duality) and a basis of invariants is given by

$$\mathcal{O}_\sigma = \sum_{i_1, i_2, \cdots i_n} \sum_{j_1, \cdots, j_n} \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_n}_{j_{\sigma(n)}} e^{j_1} \otimes \cdots \otimes e^{j_n} \otimes \bar{e}_{i_1} \otimes \cdots \otimes \bar{e}_{i_n}$$
$$= \sum_{j_1, \cdots, j_n} e^{j_1} \otimes \cdots \otimes e^{j_n} \otimes \bar{e}_{j_{\sigma(1)}} \otimes \bar{e}_{j_{\sigma(2)}} \otimes \cdots \bar{e}_{j_{\sigma(n)}}$$

Schur-Weyl duality :

$$V_N^{\otimes n} = \bigoplus_{R\vdash n; l(R)\leq N} V_R^{U(N)} \otimes V_R^{S_n}$$

Considering $U(N)$ invariants in

$$V_N^{\otimes n} \otimes \bar{V}_N^{\otimes n} = \bigoplus_{R\vdash n; l(R)\leq N} V_R^{U(N)} \otimes V_R^{S_n} \otimes \bigoplus_{S\vdash n; l(S)\leq N} V_S^{U(N)} \otimes V_S^{S_n}$$

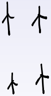
For large $N$

$$\sum_{R\vdash n} d_R^2 = n!$$

For finite $N$

$$\sum_{R\vdash n; l(R)\leq N} d_R^2$$

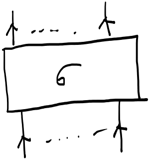
$\underline{n=1}$: $e^i \otimes \bar{e}_j \to$  ; $e^i \otimes e_i \to$ 

$\underline{n=2}$: $e^{i_1} \otimes e^{i_2} \otimes \bar{e}_{j_1} \otimes \bar{e}_{j_2} \to$ 

$\sigma = (1)(2)$ : $e^{i_1} \otimes e^{i_2} \otimes \bar{e}_{i_1} \otimes \bar{e}_{i_2} \longrightarrow$ 

$\sigma = (12)$ : $e^{i_1} \otimes e^{i_2} \otimes \bar{e}_{i_2} \otimes \bar{e}_{i_1} \longrightarrow$ 

$\underline{\text{general } n}$ :

$\theta_\sigma =$ 

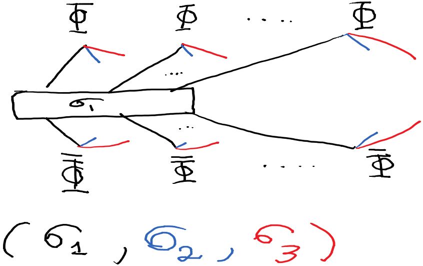
Invariant polynomials of degree $n$ in $\Phi_{ijk}, \bar{\Phi}_{ijk}$ correspond to $U(N)$ invariants in

$$Sym^n(V_N \otimes V_N \otimes V_N) \otimes Sym^n(\bar{V}_N \otimes \bar{V}_N \otimes \bar{V}_N)$$

Invariant polynomials can be labelled by a triple of permutations

$$\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3}(\Phi, \bar{\Phi}) = \Phi^{i_1 j_1 k_1} \cdots \Phi^{i_n j_n k_n} \bar{\Phi}_{i_{\sigma_1(1)} j_{\sigma_2(1)} k_{\sigma_3(1)}} \cdots \bar{\Phi}_{i_{\sigma_1(n)} j_{\sigma_2(n)} k_{\sigma_3(n)}}$$

01 November 2018 10:25



There are equivalences

$$\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} = \mathcal{O}_{\gamma_1\sigma_1\gamma_2,\gamma_1\sigma_2\gamma_2,\gamma_1\sigma_3\gamma_2}$$

for $\gamma_1, \gamma_2 \in S_n$.

These equivalence classes form a double coset.

$$Diag(S_n) \setminus (S_n \times S_n \times S_n)/Diag(S_n)$$

Points are in 1-1 correspondence with 3-edge-colored graphs with $n$ black and $n$ white vertices, with lines joining black to white. Each vertex is trivalent and has one incident edge of each color.

The space of functions on the double coset forms an algebra $\mathcal{K}(n)$ – we will give more concrete ways of thinking about this algebra shortly.

# Intro: Dimension of $\mathcal{K}(n)$

The number of these equivalence classes can be counted using Burnside's Lemma

$$\mathcal{N}(n) = \frac{1}{n!^2} \sum_{\gamma_1,\gamma_2} \text{Number of fixed points}$$
$$= \frac{1}{n!^2} \sum_{\gamma_1,\gamma_2} \sum_{\sigma_1,\sigma_2,\sigma_3} \delta(\gamma_1\sigma_1\gamma_2\sigma_1^{-1})\delta(\gamma_1\sigma_2\gamma_2\sigma_2^{-1})\delta(\gamma_1\sigma_3\gamma_2\sigma_3^{-1})$$

Simplifying this leads to

$$\mathcal{N}(n) = \sum_{p \vdash n} \mathrm{Sym}\ p$$

For a partition $p$ consisting of $p_i$ parts of length $i$

$$\mathrm{Sym} p = \prod_i i^{p_i} p_i!$$

## Intro: Dimension of $\mathcal{K}(n)$

Using the expansion of $\delta(\sigma)$ in terms of irreducible characters of $S_n$

$$\delta(\sigma) = \frac{1}{n!} \sum_{\sigma \in S_n} d_R \chi^R(\sigma)$$

we arrive at

$$\mathcal{N}(n) = \sum_{R_1, R_2, R_3 \vdash n} C(R_1, R_2, R_3)^2$$

There is another ribbon-graph-combinatoric interpretation, related by a "gauge-fixing" of the above permutation-triples/3-colored-graph description

# Intro: Colored graphs to bipartite ribbon graphs

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma_2, \gamma_1 \sigma_2 \gamma_2, \gamma_1 \sigma_3 \gamma_2)$$

All perms in $S_n$. These equivalence classes define the double coset

Take $\gamma_2 = \sigma_3^{-1}$ :

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\sigma_1 \sigma_3^{-1}, \sigma_2 \sigma_3^{-1}, 1) \equiv (\tau_1, \tau_2, 1)$$

Apply $\gamma_1$, then $\gamma_2 = \sigma_3^{-1} \gamma_1^{-1}$ to get

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1, \gamma_1 \sigma_2, \gamma_1 \sigma_3) \sim (\gamma_1 \sigma_1 \sigma_3^{-1} \gamma_1^{-1}, \gamma_1 \sigma_2 \sigma_3^{-1} \gamma_1^{-1}, 1)$$
$$\equiv (\gamma_1 \tau_1 \gamma_1^{-1}, \gamma_1 \tau_2 \gamma_1^{-1}, 1)$$

$\tau_1 = \sigma_1 \sigma_3^{-1}, \tau_2 = \sigma_2 \sigma_3^{-1}$

## Equivalence classes in $S_n \times S_n$ : Bipartite ribbon graphs

The counting of 3-colored graphs is equivalent to counting equivalence classes of pairs in $S_n$, generated by the diagonal conjugation:

$$(\tau_1, \tau_2) \sim (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1})$$

These describe bi-partite ribbon graphs embedded on a surface.

A bipartite ribbon graph, also called a hypermap, is a graph embedded on a two-dimensional surface with black and white vertices, such that edges connect black to white vertices and cutting the surface along the edges leaves a disjoint union of regions homeomorphic to open discs. Bipartite ribbon graphs, denoted ribbon graphs for short in this paper, with $n$ edges can be described using permutations of $\{1, 2, \cdots, n\}$ forming the symmetric group $S_n$.

See for example : Graphs on surfaces and their applications, Lando and Zvonkin.

$\mathcal{N}(n)$ is the number of bipartite graphs with $n$ edges.

# Permutation pairs and bi-partite ribbon graphs

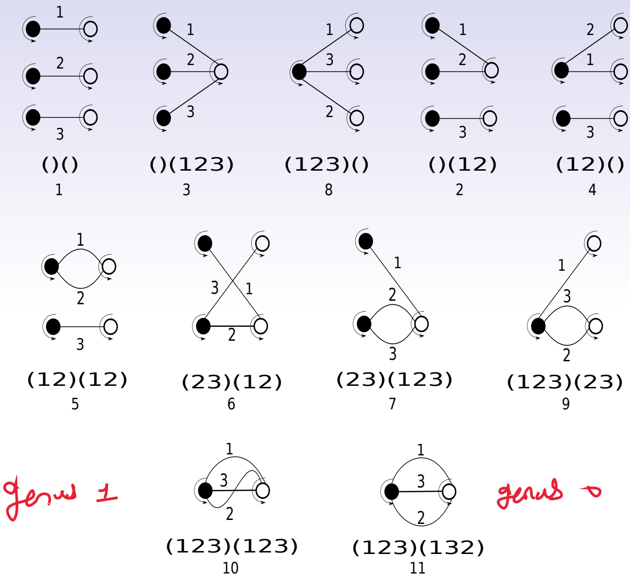



**Figure:** Bipartite ribbon graphs with $n = 3$ edges

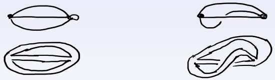
# Tensor observables and Matrix Feynman graphs

29 May 2021 11:30

→ These bipartite ribbon graphs can be thickened to the usual double line diagrams of large N matrix theory.

→ e.g.

3 faces
↓
3 index lines



$\langle \mathrm{tr} ZZZ \quad \mathrm{tr} Z^{+}Z^{+}Z^{+} \rangle$ $\qquad$ $\langle \mathrm{tr} ZZZ \quad \mathrm{tr} Z^{+}Z^{+}Z^{+} \rangle$

$\langle Z^{i}_{j} Z^{+k}_{l} \rangle = \delta^{i}_{l} \delta^{k}_{j}$

↓ $N^3$ $\qquad$ ↓ $N$

Observables of tensor models → Feynman graphs of Matrix Models.

= Double line diagrams

= Bipartite Ribbon graphs.

↓

$(\tau_1, \tau_2)$

$\sim (\gamma \tau_1 \gamma^{-1}, \gamma \tau_2 \gamma^{-1})$

$\tau_1, \tau_2, \gamma$ : here in $S_3$

## Intro : $\mathcal{K}(n)$

$\mathcal{K}(n)$ is the subspace of $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ which is invariant under

$$\gamma : \sum_{\tau_1,\tau_2} \lambda_{\tau_1,\tau_2} \tau_1 \otimes \tau_2 \to \sum_{\tau_1,\tau_2} \lambda_{\tau_1,\tau_2} \gamma\tau_1\gamma^{-1} \otimes \gamma\tau_2\gamma^{-1}$$

The dimension of this subspace is the same as the $\mathcal{N}(n)$, which is the number of equivalence classes of pairs in $(\tau_1, \tau_2) \in S_n \times S_n$ under the equivalence $(\tau_1, \tau_2) \sim (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1})$.

## Intro: $\mathcal{K}(n)$ and ribbon graph basis.

Let $r$ be an index rrunning over distinct equivalence classes or permutation pairs : $(\tau_1^{(r)}, \tau_2^{(r)})$ are representatives of the classes.

For each equivalence class

$$
\begin{aligned}
E_r &= \frac{1}{n!} \sum_{\gamma \in S_n} \gamma \tau_1^{(r)} \gamma^{-1} \otimes \gamma \tau_2^{(r)} \gamma^{-1} \\
&= \frac{1}{|\mathrm{Orb}(r)|} \sum_{a \in \mathrm{Orb}(r)} \tau_1^{(r)}(a) \otimes \tau_2^{(r)}(a)
\end{aligned}
$$

obeys $(\gamma \otimes \gamma) E_r (\gamma^{-1} \otimes \gamma^{-1}) = E_r$ and hence $E_r \in \mathcal{K}(n)$.

There is one ( and only one ) element in $\mathcal{K}(n)$ for every orbit, i.e. every ribbon graph.

Use the product in $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ to multiply these. The outcome is within the subspace.

$\mathcal{K}(n)$ is a sub-algebra.

The $E_r$ form the geometric basis of $\mathcal{K}(n)$.

## Intro: $\mathcal{K}(n)$

$$\mathrm{Dim}(\mathcal{K}(n)) = |\mathrm{Rib}(n)| = \sum_{p \vdash n} |\mathrm{Sym}(p)| = \sum_{R_1, R_2, R_3 \vdash n} C(R_1, R_2, R_3)^2$$

Bengeloun, Ramgoolam , "Tensor models, Kronecker coefficients and permutation centralizer algebras," JHEP2017

The dimension of $\mathcal{K}(n)$ is the number of ribbon graphs which is also the sum of squares of Kronecker coefficients. This sum of squares thus has a combinatoric construction.

J. Bengeloun and S. Ramgoolam, "Quantum mechanics of bipartite ribbon graphs: Integrality, Lattices and Kronecker coefficients" arXiv:2010.04054v1 [hep-th]

Can we use $\mathcal{K}(n)$ to give a combinatoric construction for $C(R_1, R_2, R_3)^2$ and $C(R_2, R_2, R_3)$ ?

## Intro: A Fourier basis of $\mathcal{K}(n)$

We used the Wedderburn-Artin decomposition of $\mathcal{K}(n)$ into matrix blocks, which can be viewed as a Fourier decomposition of the algebra.

$\mathcal{K}(n)$ has a combinatoric basis set $\{E_r\}$. One basis vector $E_r$ for each 3-colored graph ( or each bi-partite graph )

And a Fourier basis given in terms of triples of YDs $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$. $1 \leq \tau_1, \tau_2 \leq C(R_1, R_2, R_3)$.

## Intro: Fourier subspace of a triple $R_1, R_2, R_3$

We define the Fourier subspace, of dimension $C^2$, of $\mathcal{K}(n)$ as

$$V^{R_1,R_2,R_3} = \bigoplus_{\tau_1,\tau_2} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$

There is an explicit formula for $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$ in terms of Clebsch-Gordan coefficients (expansion coefficients of $S_n$ invariant vectors in $R_1 \otimes R_2 \otimes R_3$)

Useful in obtaining a characterisation of $V^{R_1,R_2,R_3}$ as the simultaneous eigenspace of a set of operators

$$T^{(i)}_k$$

acting on the algebra $\mathcal{K}(n)$.

$$1 \leq i \leq 3$$
$$1 \leq k \leq \tilde{k}_*$$

## Intro : Lattices of ribbon graphs

Consider the space of real linear combinations $\sum_r a_r E_r$.

These $a_r$ define vectors in

$$\mathbb{R}^{\mathcal{N}(n)}$$

Insider this Euclidean space is a lattice formed by the integer $a_r$

$$\mathbb{Z}^{\mathcal{N}(n)}$$

## Intro : Kronecker coefficients and lattices of ribbon graphs

We showed, using the $T_k^{(i)}$, that this integer lattice contains sub-lattices of dimension $C(R_1, R_2, R_3)^2$ ( and $C(R_1, R_2, R_3)$ ) for each triple $(R_1, R_2, R_3)$.

For each triple, a basis for the sub-lattice can be identified by constructing the null vectors of an integer matrix

$$X^{R_1, R_2, R_3} v = 0$$

There are combinatoric algorithms for calculating such integer null spaces of integer matrices, e.g. techniques for Hermite normal forms.

## Intro : $\mathcal{K}(n)$ as a Hilbert space

$\mathcal{K}(n)$ is a Hilbert space and the $T_k^{(i)}$ are hermitian operators.

They can be used to construct Hamiltonians whose null states span spaces of dimension $C^2$ (or $C$).

These quantum mechanical systems have an interpretation as a model of quantum mechanical membranes.

## OUTLINE

- Fourier transform on $\mathbb{C}(S_n)$ and $\mathcal{Z}(\mathbb{C}(S_n))$ : permutations and Young diagrams.
- Fourier transform on $\mathcal{K}(n)$ : ribbon graphs and Young diagram triples.
- $C^2$ as the number of integer vectors in a lattice of ribbon graphs: integer matrix algorithms.
- Quantum mechanics on $\mathcal{K}(n)$.
- Belyi maps and quantum membrane interpretation

## Part 1 : $\mathbb{C}(S_n)$ and $\mathcal{Z}(\mathbb{C}(S_n))$

A basis for $\mathcal{Z}(\mathbb{C}(S_n))$ is given by sums of permutations in conjugacy classes. Let $\mu$ be a partition of $n$, which determines a cycle structure and a conjugacy class of $S_n$. Call this class $\mathcal{C}_\mu$.

$$T_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \sigma$$

$$T_\mu T_\nu = \sum_\lambda n_{\mu\nu}^\lambda T_\lambda$$

The coefficients $n_{\mu\nu}^\lambda$ are integers.

## Part 1 : $\mathbb{C}(S_n)$ and $\mathcal{Z}(\mathbb{C}(S_n))$

Another basis for the centre is given by the projectors, labelled by irreps or Young diagrams $R$,

$$P_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi^R(\sigma)\sigma$$

These obey

$$P_R P_S = \delta_{RS} P_R$$

We also have

$$T_\mu P_R = \frac{\chi^R(T_\mu)}{d_R} P_R$$
$$T_\mu = \sum_R \frac{\chi^R(T_\mu)}{d_R} P_R$$

$T_\mu$ acts on $\mathcal{Z}(\mathbb{C}(S_n))$ by left multiplication. The matrix $(N_\mu)^\lambda_\nu = n^\lambda_{\mu\nu}$ is a ( non-negative ) integer matrix.

The eigenvalues of the integer matrix are the normalized characters.

We also know that the normalized characters are rational - because of the Murnaghan-Nakayama construction.

Combining these two facts, we know that $\frac{\chi^R(T_\mu)}{d_R}$ are in fact integers.

see e.g. Simon, representation theory.

Consider the special cases where $\mu$ is of the form $[k, 1^{n-k}]$ ( $k \geq 2$ ). Permutations in this conjugacy class have one non-trivial cycle and remaining cycles of length 1.

**Lemma 1:** $T_2, T_3, \cdots, T_n$ generate the centre of $\mathcal{Z}(\mathbb{C}(S_n))$.

**Lemma 2:** The normalized characters of $\{\frac{\chi^R(T_2)}{d_R}, \frac{\chi^R(T_3)}{d_R}, \cdots, \frac{\chi^R(T_n)}{d_R}\}$ distinguish Young diagrams.

Kemp-Ramgoolam “BPS states and centres of symmetric group algebras.” JHEP2020. (arXiv:1911.11649 [hep-th])

The first one uses results in MacDonald (symmetric functions and Hall Polynomials).

The connection between the two uses the fact that a linear combination of the form $T = \sum_R a_R P_R$ with all distinct $a_R$ generates the centre.

Canonical idempotents of multiplicity-free families of algebras Stephen Doty, Aaron Lauve, George H.

Seelinger,arXiv:1606.08900v5 [math.RT]

In fact (using GAP), we only need a small subset of
$\{\frac{\chi^R(T_2)}{d_R}, \frac{\chi^R(T_3)}{d_R}, \cdots, \frac{\chi^R(T_n)}{d_R}\}$

Using $\frac{\chi^R(T_2)}{d_R}$ works for $n = 2, 3, 4, 5, 7$.

Using $\{\frac{\chi^R(T_2)}{d_R}, \frac{\chi^R(T_3)}{d_R}\}$ works up to $n = 14$.

Using $\{\frac{\chi^R(T_2)}{d_R}, \frac{\chi^R(T_3)}{d_R}, \cdots, \frac{\chi^R(T_6)}{d_R}\}$ works up to $n = 79$.

For our construction we can pick any $\tilde{k}_*$ between $k_*(n)$ and $n$.
Would be interesting to know more about $k_*(n)$.

## The study of $k_*(n)$ in

Kemp-Ramgoolam "BPS states and centres of symmetric group algebras." JHEP2020. (arXiv:1911.11649 [hep-th])

was motivated by

Balasubramanian, Czech, Larjo, Simon, "Integrability and information loss: a simple example," (2006)

which was studying the problem of distinguishing LLM geometries using a limited number of the multipole moments of the bulk AdS fields.

Lin, Lunin, Maldacena, "Bubbling AdS space and 1/2 BPS geometries," 2004

The CFT duals of these are constructed using projectors $P_R$ associated with Young diagrams.

Corley, Jevicki, Ramgoolam, "exact correlators of giant gravitons from dual N=4 SYM" 2001

A standard basis for $\mathbb{C}(S_n)$ is $\sigma \in S_n$.

Another useful basis for $\mathbb{C}(S_n)$ is

$$Q^R_{ij} = \frac{d_R}{n!} \sum_{\sigma} D^R_{ij}(\sigma)\sigma$$

The use of a group-invariant inner product on $V^R$, choice of an orthonormal basis, and reality of reps of $S_n$ ensures nice identities ( see math phys textbooks e.g. Hamermesh )

$$D^R_{ij}(\sigma^{-1}) = D^R_{ji}(\sigma)$$
$$\sum_{\sigma} D^R_{ij}(\sigma) D^S_{kl}(\sigma) = \frac{n!}{d_R}\delta^{RS}\delta_{ik}\delta_{jl}$$

These imply

$$Q^R_{ij} Q^S_{kl} = \delta_{RS}\delta_{jk} Q^R_{il}$$

The number of these matrix-basis elements is

$$\sum_R d_R^2 = n!$$

These $Q$'s give an explicit Wedderburn-Artin decomposition of $\mathbb{C}(S_n)$ into subspaces of dimension $d_R^2$ spanned by basis elements which multiply as blocks of elementary matrices labelled by $R$.

$$T_k Q_{ij}^R = \frac{\chi^R(T_k)}{d_R} Q_{ij}^R$$

We will see analogous matrix blocks for $\mathcal{K}(n)$ and analogous eigenvalue equations in terms of $T_k$ shortly.

## Part 2: Fourier transform on $\mathcal{K}(n)$

In $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ there is a subspace $\mathcal{K}(n)$ which is invariant under conjugation by $\gamma \otimes \gamma$. A basis in this subspace consists of averages over the equivalence classes $\mathrm{Rib}(n)$.

Let $r$ be an index for the equivalence classes in $\mathrm{Rib}(n)$. Recall

$$|\mathrm{Rib}(n)| = \sum_{p} |\mathrm{Sym}(p)| = \sum_{R_1, R_2, R_3} C(R_1, R_2, R_3)^2$$

For each equivalence class

$$E_r = \frac{1}{n!} \sum_{\gamma \in S_n} \gamma \tau_1^{(r)} \gamma^{-1} \otimes \gamma \tau_2^{(r)} \gamma^{-1}$$

## Part 2: Fourier transform on $\mathcal{K}(n)$

Use the product in $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ to multiply these. The outcome is within the subspace. $\mathcal{K}(n)$ is a sub-algebra.

$$E_r = \frac{1}{|\mathrm{Orb}(r)|} \sum_{a \in \mathrm{Orb}(r)} \tau_1^{(r)}(a) \otimes \tau_2^{(r)}(a)$$

The $E_r$ form the geometric basis of $\mathcal{K}(n)$.

# Part 2: Fourier transform on $\mathcal{K}(n)$

There is also a Fourier basis labelled by triples of Young diagrams.

$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \frac{d_{R_1} d_{R_2}}{n!^2} \sum_{\sigma_1,\sigma_2 \in S_n} \sum_{i_1,i_2,i_3,j_1,j_2} C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau_2}_{j_1,j_2;i_3} D^{R_1}_{i_1 j_1}(\sigma_1) D^{R_2}_{i_2 j_2}(\sigma_2)\, \sigma_1 \otimes \sigma_2$$
$$= \sum_{i_1,i_2,i_3,j_1,j_2} C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau_2}_{j_1,j_2;i_3} Q^{R_1}_{i_1,j_1} \otimes Q^{R_2}_{i_2 j_2}$$

Irreps $R$, orthonormal basis :

Young diagram $R, V^R, D^R(\sigma) : V_R \rightarrow V_R; D^R_{ij}(\sigma)$

$V^{R_1} \otimes V^{R_2}$ ; $D^{R_1}(\sigma) \otimes D^{R_2}(\sigma)$ ;

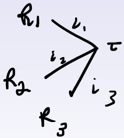
$$V^{R_1} \otimes V^{R_2} = \bigoplus_{R_3} V_{R_3} \otimes V^{R_3}_{R_1,R_2}$$

Clebsch-Gordan coefficients $C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3}$ are inner products $< R_1, i_1, R_2, i_2 | R_3, i_3, \tau >$ where $|R_3, i_3, \tau >$ chosen to be orthonormal basis for $V_{R_3} \otimes V^{R_3}_{R_1,R_2}$

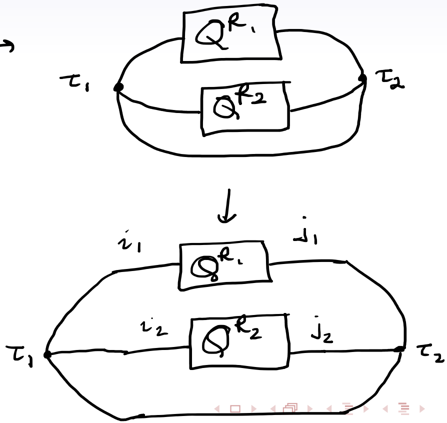
$$1 \leq \tau_1 \leq Dim(V^{R_1,R_2}_{R_3}) = C(R_1, R_2, R_3)$$

$Q^{R}_{ij} \rightarrow$


$C^{R_1 R_2 R_3 ; \tau_1}_{i_1 i_2 i_3} \rightarrow$




$Q^{R_1 R_2 R_3 ;}_{\tau_1 \tau_2} \rightarrow$



With $i, j$'s running over an orthonormal basis, using properties of the $D$'s and Clebsch's, we can show that

$$(\gamma \otimes \gamma) Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} (\gamma^{-1} \otimes \gamma^{-1}) = Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$

$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} Q^{R_1',R_2',R_3'}_{\tau_2',\tau_3} = \delta_{R_1 R_1'} \delta_{R_2 R_2'} \delta_{R_3 R_3'} \delta_{\tau_2 \tau_2'} Q^{R_1,R_2,R_3}_{\tau_1,\tau_3}$$

This gives the explicit decomposition into simple matrix algebras ( as expected according to Wedderburn-Artin theorem). Blocks labelled by triples $(R_1, R_2, R_3)$.

## Part 2 : Fourier transform for $\mathcal{K}(n)$

For each Young diagram triple $(R_1, R_2, R_3)$ we define a Fourier subspace

$$V^{R_1,R_2,R_3} = \bigoplus_{\tau_1,\tau_2} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$$

These subspaces can be characterised as eigenspaces of operators $T_k^{(i)} \subset \mathcal{K}(n)$.

We define

$$T_k = \sum_{\sigma\in\mathcal{C}_k} \sigma$$

These are sums of permutations with the cycle structure $[k, 1^{n-k}]$.

From each $T_k \in \mathcal{Z}(\mathbb{C}(S_n))$ we define three linear operators on $\mathcal{K}(n)$, acting by left multiplication :

$$T_k^{(1)} = T_k \otimes 1 = \sum_{\sigma \in \mathcal{C}_k} \sigma \otimes 1$$

$$T_k^{(2)} = 1 \otimes T_k = \sum_{\sigma \in \mathcal{C}_k} 1 \otimes \sigma$$

$$T_k^{(3)} = \Delta(T_k) = \sum_{\sigma \in \mathcal{C}_k} \sigma \otimes \sigma$$

We find that

$$
\begin{aligned}
T_k^{(1)} Q_{\tau_1,\tau_2}^{R_1,R_2,R_3} &= \frac{\chi_{R_1}(T_k)}{d_{R_1}} Q_{\tau_1,\tau_2}^{R_1,R_2,R_3} \\
T_k^{(2)} Q_{\tau_1,\tau_2}^{R_1,R_2,R_3} &= \frac{\chi_{R_2}(T_k)}{d_{R_2}} Q_{\tau_1,\tau_2}^{R_1,R_2,R_3} , \\
T_k^{(3)} Q_{\tau_1,\tau_2}^{R_1,R_2,R_3} &= \frac{\chi_{R_3}(T_k)}{d_{R_3}} Q_{\tau_1,\tau_2}^{R_1,R_2,R_3}
\end{aligned}
$$

$T_k^{(i)}$ are central operators in $\mathcal{K}(n)$, and their eigenvalues only depend on the $R_i$ labels of the Fourier subspace $V^{R_1,R_2,R_3}$.

The Fourier subspace $V^{R_1,R_2,R_3}$ is uniquely characterised by using the eigenvalues of

$$\{T_2^{(1)},\cdots,T_{\widetilde{k}_*}^{(1)};T_2^{(2)},\cdots,T_{\widetilde{k}_*}^{(2)};T_2^{(3)},\cdots,T_{\widetilde{k}_*}^{(3)}\}$$

which are the normalized characters

$$\{\frac{\chi^{R_1}(T_2)}{d_{R_1}},\cdots,\frac{\chi^{R_1}(T_{\widetilde{k}_*})}{d_{R_k}};\frac{\chi^{R_2}(T_2)}{d_{R_2}},\cdots,\frac{\chi^{R_2}(T_{\widetilde{k}_*})}{d_{R_2}};\frac{\chi^{R_3}(T_2)}{d_{R_3}},\cdots,\frac{\chi^{R_3}(T_{\widetilde{k}_*})}{d_{R_3}}\}$$

Explicitly constructing the $D^R_{ij}$ and the Clebsch's is hard - and not obviously a combinatoric operation.

But we can construct the subspace $V^{R_1,R_2,R_3}$ using the geometric basis.

$$T^{(i)}_k E_r = \sum_s (\mathcal{M}^{(i)}_k)^s_r E_s$$

with

$(\mathcal{M}^{(i)}_k)^s_r =$ Number of times the multiplication of elements in the sum $T^{(i)}_k$ with a fixed element in orbit $r$ to the right produces an element in orbit $s$.

The vectors in the Fourier subspace for a triple $(R_1, R_2, R_3)$ solve the following matrix equation

$$\begin{bmatrix} \mathcal{M}_2^{(1)} - \frac{\chi_{R_1}(T_2)}{d(R_1)} \\ \vdots \\ \mathcal{M}_{\widetilde{k}_*}^{(1)} - \frac{\chi_{R_1}(T_{\widetilde{k}})}{d(R_1)} \\ \mathcal{M}_2^{(2)} - \frac{\chi_{R_2}(T_2)}{d(R_2)} \\ \vdots \\ \mathcal{M}_{\widetilde{k}_*}^{(2)} - \frac{\chi_{R_2}(T_{\widetilde{k}_*})}{d(R_2)} \\ \mathcal{M}_2^{(3)} - \frac{\chi_{R_3}(T_2)}{d(R_3)} \\ \vdots \\ \mathcal{M}_{\widetilde{k}_*}^{(3)} - \frac{\chi_{R_3}(T_{\widetilde{k}_*})}{d(R_3)} \end{bmatrix} \cdot v = \mathbf{0}$$

This rectangular array gives the matrix elements of a linear operator mapping $\mathcal{K}(n)$ to $3(\widetilde{k}_* - 1)$ copies of $\mathcal{K}(n)$, using the geometric basis of ribbon graph vectors for $\mathcal{K}(n)$. The normalized characters are integers.

Renaming as $X_{R_1,R_2,R_3}$ the integer matrix in the above equation we have

$$X_{R_1,R_2,R_3} \cdot v = 0$$

## Part 3: Integer matrix algorithms for null spaces

The null space of the integer matrix $X_{R_1,R_2,R_3}$ has a basis given by integer null vectors.

This can be found by taking $(X^T_{R_1,R_2,R_3})$ and finding its hermite normal form.

This amount to finding a unimodular matrix $U$ (an integer matrix with determinant$\pm 1$) and a matrix $h$ with special triangular form.

$$H = UX^T$$

- $H$ is upper triangular (that is, $H_{ij} = 0$ for $i > j$), and any rows of zeros are located below any other row.
- The leading coefficient (the first non-zero entry from the left, also called the pivot) of a non-zero row is always strictly to the right of the leading coefficient of the row above it; moreover, it is positive.
- The elements below pivots are zero and elements above pivots are non-negative and strictly smaller than the pivot.

Example :

$$H = \begin{pmatrix} 1 & 0 & 40 & -11 \\ 0 & 3 & 27 & -2 \\ 0 & 0 & 61 & -13 \end{pmatrix}$$

There are integer algorithms for doing producing $H$ and $U$. We implement a sequence of steps involving :

- Swop two rows.
- Multiply a row by $-1$.
- Add an integer multiple of a row to another row of $A$.

Explicit algorithms are described in textbooks such as

[H. Cohen, “A Course in Computational Algebraic Number Theory,” Springer Science & Business Media, Springer, 2000.]

And HNF algorithms for general rectangular matrices are available in GAP.

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 3 & 2 & 5 \end{pmatrix}$$

$Mv = 0$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 3 & 2 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(-1) \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + (1) \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The null vectors can be constructed by integer algorithms (e.g. in GAP) for finding Hermite Normal Forms $H$ of matrix $M^T$

$$H = UM^T$$

$U$ is unimodular - integer matrix with determinant $\pm 1$. In this case

$$H = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

To see that the connection between vanishing rows of $H$ and the corresponding rows of $U$ is general :

$$UX^T = H$$

can be written as :

$$\sum_j U_{ij} X^T_{jk} = H_{ik}$$
$$\sum_j U_{ij} X_{kj} = H_{ik}$$

A vanishing row of $H$ means that we have an $i$ for which $H_{ik} = 0$ for all $k$. Then fixing $i$, we have a vector $U_{ij}$ obeying

$$\sum_j X_{kj} U_{ij} = 0$$

The number of zero rows is equal to the dimension of the null space of $X$.

The rows of $U$ corresponding to the zero rows of $h$ give integer null vectors of $X^{R_1,R_2,R_3}$.

The number of these null vectors is equal to $C(R_1, R_2, R_3)^2$.

To summarize

- So we start with an integer matrix (constructed from integer structure constants of $\mathcal{K}(n)$ @ $T_k^{9i)}$ on $E_r$ basis and integer normalized characters ).
- Perform integer row operations and arrive at the null vectors.
- We count the null vectors. We obtain $C^2$.
- The null vectors are a set of vectors in

$$\mathbb{Z}^{|\mathrm{Rib}(n)|}$$

Taking integer linear combinations of these basis null vectors generates a sub-lattice.

## Part 4: Quantum mechanics - non-degenerate inner product

We mentioned the Wedderburn-Artin decomposition of the algebra $\mathcal{K}(n)$. This exists for algebras which are associative and have a non-degenerate bilinear form.
The non-degenerate bilinear form on $\mathcal{K}(n)$ is inherited from corresponding one on $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$.

$$\delta(\sigma_1 \otimes \sigma_2; \tau_1 \otimes \tau_2) = \delta(\sigma_1 \tau_1^{-1})\delta(\sigma_2 \tau_2^{-1})$$

On the geometric basis

$$\delta(E_r; E_s) = \frac{1}{|\mathrm{Orb}(r)|}\delta_{rs}$$

Can use this to define a non-degenerate inner product (sesquilinear non-degenerate pairing)

$$g(\sum_r \lambda_r E_r; \sum_s \mu_s E_s) = \sum_{r,s} \lambda^* \mu_s \delta(E_r, E_s)$$

## Part 4: Quantum mechanics - Hermitian $T_k^{(i)}$

The $T_k^{(i)}$ operators are hermitian

$$g(T_k^{(i)} v; w) = g(v; T_k^{(i)} w)$$

Linear combinations

$$\mathcal{H} = \sum_{k,i} a_{k,i} T_k^{(i)}$$

for real $a_{k,i}$ are hermitian operators.

## Part 4: Quantum mechanics - Hamiltonians

In addition to distinguishing the Fourier subspaces with these lists, we can also distinguish them using linear combinations

$$\mathcal{H} = \sum_{i=1}^{3} \sum_{k=2}^{k_*} a_{i,k} T_k^{(i)}$$

For appropriate choices of integers $a_{i,k}$.

The corresponding eigenvalues are :

$$\omega_{R_1,R_2,R_3} = \sum_{i=1}^{3} \sum_{k=2}^{k_*} a_{i,k} \widetilde{\chi}_{R_i}(T_k)$$

$\mathcal{H} v = \omega v$

$(\mathcal{H} - \omega 1) v$

$\mu$

(square mtx.)

$\mathcal{M}_k^{(i)}$ is an integer matrix ( entries are either zero or positive integer).

Finding the eigenvalues and eigenvectors of $\mathcal{H}$ amounts to finding the eigenvalues/eigenvectors of

$$X = \sum_{k,i} a_{i,k} \mathcal{M}_k^{(i)}$$

The eigenvalues are known combinatorially constructible (Murnaghan-Nakayama Lemma) quantities. The eigenvectors in $V^{R_1,R_2,R_3}$ for fixed triple $(R_1, R_2, R_3)$ obey the equation

$$Xv = \omega_{R_1,R_2,R_3} v$$

So the constructions which we described in terms of rectangular matrices can be done with square matrices since $\mathcal{H} : \mathcal{K}(n) \to \mathcal{K}(n)$.

## Part 5: Bi-partite graphs and Branched covers

We saw that bi-partite grphs with *n* edges are in 1-1 correspondence with equivalence classes of permutation pairs $(\tau_1, \tau_2)$ with the equivalence

$$(\tau_1, \tau_2) = (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1})$$

Given a pair $(\tau_1, \tau_2)$, define a third

$$\tau_3 = (\tau_1\tau_2)^{-1}$$

so that

$$\tau_1\tau_2\tau_3 = 1$$

# Part 5: Bi-partite graphs and Branched covers

Equivalently bi-partite graphs are in 1-1 correspondence with triples $(\tau_1, \tau_2, \tau_3)$ with

$$\begin{aligned} \tau_1\tau_2\tau_3 &= 1 \\ (\tau_1, \tau_2, \tau_3) &\sim (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1}, \gamma\tau_3\gamma^{-1}) \end{aligned}$$

These describe branched covers of the sphere with 3 branch points (called Belyi maps). Genus $h$ of the covering surface is given by the Riemann-Hurwitz formula

$$(2h-2) = n(-2) + B(\tau_1) + B(\tau_2) + B(\tau_3)$$

$B(\tau_i)$ is the branching number.

$$(2h-2) = n - C_{\tau_1} - C_{\tau_2} - C_{\tau_3}$$

For each of these 3-point branched covers with degree $n$ we have a basis vector $E_r \in \mathcal{K}(n)$.

The time-evolved state

$$E_r(t) = e^{-iHt} E_r$$

using one of the Hamiltonians described above is a superposition of curves (each equipped with a branched covering map to a sphere).

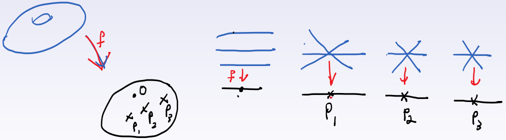
Can be viewed as a model describing a membrane theory with $S^2 \times \mathbb{R}$ target space. Initial state of a single cover can evolve into a superposition of curves.

**Recall:**


$\tau_1 = (123)$

$\tau_2 = (123)$

$\tau_3 = (123)$




$\rightarrow \; e^{-iAt} \; : \rightarrow \; \mathcal{H} = T_2^{(1)}$

$\rightarrow \; ((12)\otimes 1) \; (23)\otimes(123)$

$= (13) \otimes (123)$

Now $\tau_1 = (13)$ $\tau_2 = (123)$ $\tau_3 = (23)$

## Belyi and $\overline{\mathbb{Q}}$

**Belyi's theorem** It is known that curves defined over $\overline{\mathbb{Q}}$ admit 3-point branched covering maps to the sphere. These maps are also algebraic ( defined using numbers in $\mathbb{Q}$ ).

This has not played a big role in studies of $\mathcal{K}(n)$ so far – something for the future.

$(\tau_1, \tau_2)$

$G \subset S_n$

$\cdot\ Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longleftrightarrow$ "Dessins d'Enfant"

Grothendieck

Galois Inv?

cons.

combinator

## Remarks/open questions

- Estimate $k_*(n)$ as $n$ becomes large ....
- Verifying whether a vector $v$ in $\mathcal{K}(n)$ is annihilated by $X_{R_1,R_2,R_3}$ is computationally expensive - because of the $n!$ growth of the dimension of $\mathcal{K}(n)$.
- If there was a way to verify this in time which is polynomial in $n$ – the computational complexity theorists would be very interested.
- Minimal sets of generators for $\mathcal{K}(n)$ or for its centre would help improve the algorithms we have given so far.

Find continuum descriptions of these quantum membrane models ? ( e.g. along the lines of Horava (2008): Membranes at quantum criticality? )

In tensor models, we encounter

$$\mathcal{K}(\infty) = \bigoplus_{n=0}^{\infty} \mathcal{K}(n)$$

and Hamiltonians which mix different $n$. Tensor models could be an avenue towards continuum descriptions.

Similar to the relation between $\mathcal{K}(n)$, tensor model observables and Kronecker coefficients, there is a relation between a family $\mathcal{A}(m,n)$ of algebras, labelled by two integers $(m,n)$ ; LR coefficients ; and 2-matrix model invariants.

$$\mathrm{Dim}(\mathcal{A}(m,n)) = \sum_{\substack{R_1 \vdash m, R_2 \vdash n, \\ R_3 \vdash (m+n)}} g(R_1, R_2, R_3)^2$$

Fourier bases $Q^{R_1, R_2; R_3}_{\tau_1, \tau_2}$ have been constructed

P. Mattioli and S. Ramgoolam, " Permutation centralizer algebras and multi-matrix invariants ," Phys. Rev. D 93, 065040 (2016), arXiv:1601.06086v1 [hep-th]

originating from constructions in multi-matrix models

"Exact Multi-Matrix Correlators," Rajsekhar Bhattacharyya, Storm Collins(Witwatersrand U.), Robert de Mello Koch

and subsequent work reviewed in

"Permutations and the combinatorics of gauge invariants for general N" Sanjaye Ramgoolam; Proceeedings, Corfu 2016

Role of ribbon graphs in this case is played by 2-color necklaces. Permutation equivalence classes :

$$\sigma \in S_{m+n}$$
$$\sigma \sim \gamma\sigma\gamma^{-1} \text{ for } \gamma \in S_m \times S_n \subset S_{m+n}$$

LR coeffs and lattices of necklaces ..

## Appendix A : Constructing $C^2$ and constructing $C$.

We have given a sub-lattice construction of $C^2$. What about $C$?

There is an operation $S : \mathcal{K}(n) \rightarrow \mathcal{K}(n)$ which obeys $S^2 = 1$ ; acts by inverting the two permutations in $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$.

Acting on the geometric basis, a number of $E_r$ obey

$$S(E_r) = E_r$$

These are self-conjugate ribbons.

For a self-conjugate ribbon $(\tau_1, \tau_2)$, there exists a $\gamma$ such that $(\tau_1^{-1}, \tau_2^{-1}) = (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1})$ For non-self-conjugate $(\tau_1, \tau_2)$ and $(\tau_1^{-1}, \tau_2^{-1})$ belong to distinct orbits.

Remaining ribbons are paired up by $S$. We have corresponding vectors $\{E_n, S(E_n)\}$.

The $S = +1$ eigenspace of $\mathcal{K}(n)$ is spanned by the self-conjugate ribbons and the symmetric combinations $E_n + S(E_n)$.

On the Fourier basis $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$, the effect of $S$ is to keep $R_1, R_2, R_3$ unchanged and to swop the $\tau_1, \tau_2$. As a result $S = 1$ eigenspace in $V^{R_1,R_2,R_3}$ has dimension

$$C(C+1)/2$$

Integer matrix algorithms can be used to construct a sub-lattice of this dimension. Finding null vectors of

$$\begin{pmatrix} X_{R_1,R_2,R_3} \\ S-1 \end{pmatrix}$$

The dimension of $S=-1$ in $V^{R_1,R_2,R_3}$ is

$$C(C-1)/2$$

Find the sub-lattice basis vectors by finding null vectors of

$$\binom{X_{R_1,R_2,R_3}}{S+1}$$

Choose an injection between from the smaller set of sub-lattice generators to the bigger set. The complement of that will have exactly $C$ vectors.

This gives a construction of $C$.

An interesting corollary of the properties of $S$ is the identity

$$\text{Number of self-conjugate ribbons } = \sum_{R_1,R_2,R_3} C(R_1,R_2,R_3)$$

**Remarks**

This construction of $C^2$ ( subsequently $C$) uses rep theory input - action of $T_k$ in $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$ in terms of normalized characters (which are combinatorially constructible using MN-Lemma). Can this be made purely combinatoric ?

Requires proving directly that the matrices corresponding to $T_k^{(i)}$ on the geometric ribbon graph basis ( the $E_r$ ) have eigenvalues given by the MN result.

There are integer matrix algorithms for smith normal forms. $X = UDV$. $D$ is diagonal. Perhaps there is some connection between the entries of $D$ and the eigenvalues in this case. Also need to understand, without rep theory, but only integer matrices, why the multiplicities are $C^2$ (where these are expressed in terms of the sum of products of $\chi$ ). Does not look easy !