Interpolation, integrals, and indices

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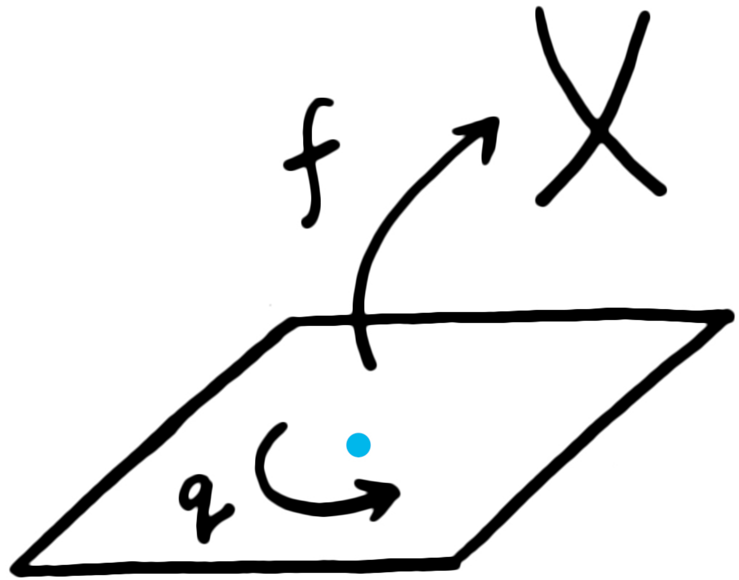
Summary:

For a (2+1)d very susy gauge theory with Higgs branch X, we consider the analog of (4+1)d Nekrasov function with an insertion α at the origin and β at infinity of the flat space $C = \mathbb{C}$. Work equivariantly with respect to $GL(1) \ni q$ acting in the source.



Here α is a K-theory class on X and β is best treated as an elliptic cohomology class on X, so this count is like an operator Ψ from one theory to the other.

Main formula:

$$(\alpha, \Psi\beta) = \int_{|x_i|=1} \mathsf{Stab}(\alpha)\Big|_{q=0} \mathsf{Stab}(\beta) \frac{\prod \Gamma_q(\dots)}{\prod \Gamma_q(\dots)} \, d_{\mathsf{Haar}}(x) \,, \qquad (\text{☆})$$

where Γ_q is the q-analog of the Γ-functions, and $\mathsf{Stab}(\beta)$ is a certain canonical extention (interpolation) of β from X to the ambient stack.

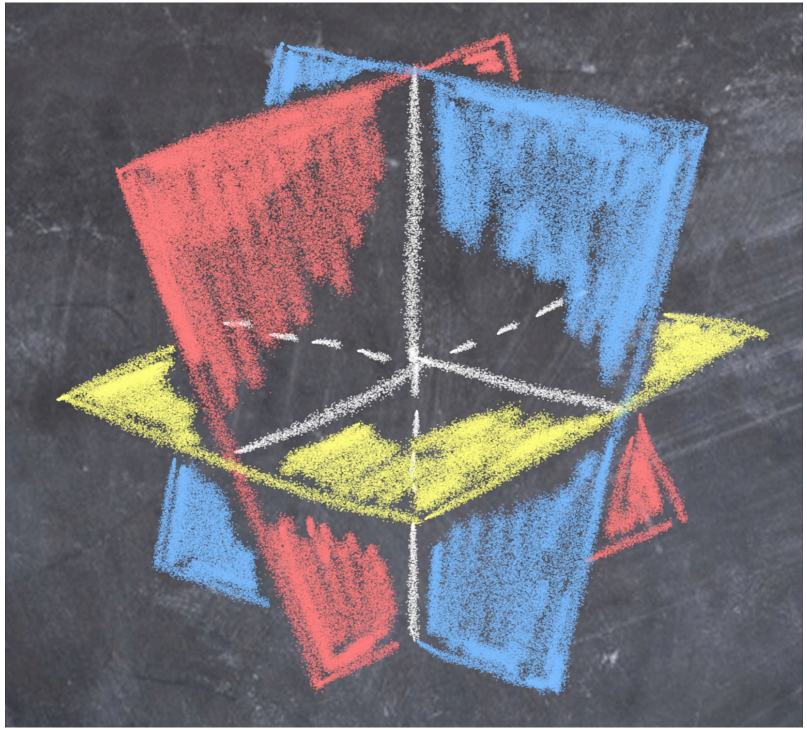
Such integrals solve interesting q-difference equations and contain lots of other interesting (including, number-theoretic) information.

It is an ancient problem to find a polynomial $f(x)$ that takes given values $f(a_i)$ at the poins $a_1, \ldots, a_n$:

$$f(x) = \sum_{i=1}^{n} f(a_i) \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \,.$$

This also works when some a_i collide and is really an iterpolation from the subset

$$\mathscr{S} = \bigcup_i \{x = a_i\} \subset \{(x, a_1, \ldots, a_n)\} \,.$$



Same logic works for Laurent polynomials, where there is more flexibilty:

$$f(x) = \sum_{i=1}^{n} f(a_i) \left(\frac{x}{a_i}\right)^{L} \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \,.$$

This interpolation polynomial satisfies

$$\text{deg}\, f = \text{Newton polygon}(f) \subset [L, L + n - 1] \,.$$

Next comes intepolation for sections of a line bundle $\mathscr{L}$ on an elliptic curve E.

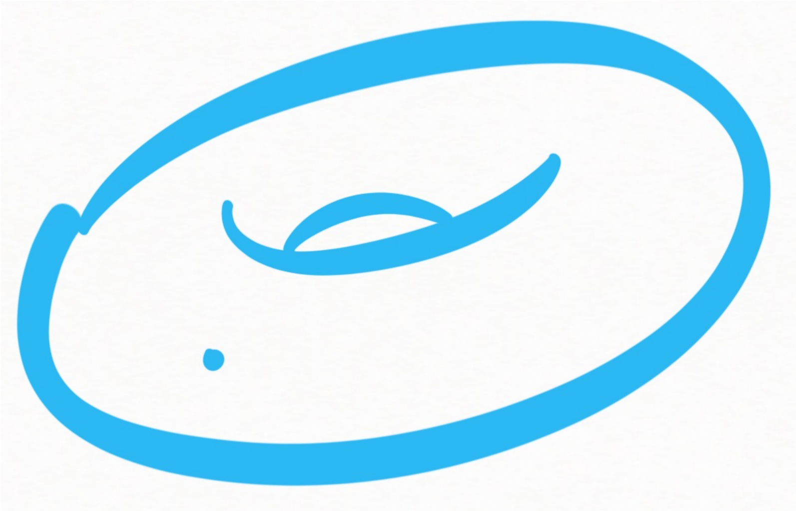
Line bundles $\mathscr{L}$ on

$$E = \mathbb{C}^\times / q^{\mathbb{Z}}\,, \quad |q| < 1\,,$$

may be described as solutions of

$$f(qx) = c x^{-d} f(x)\,, \quad d = \deg \mathscr{L}\,,$$

where d is the degree of a line bundle and $c \in E$ is a continuous parameter for $\mathscr{L}$.



By Riemann-Roch, $\mathscr{L}$ has d sections

$$\dim \Gamma(\mathscr{L}) = \deg \mathscr{L}$$

for $d > 0$.

Each section f has d zeros $\{x_1, \ldots, x_d\}$, such that

$$z = \prod x_i \mod q^{\mathbb{Z}} \in E$$

is fixed, and, basically, equals c above.

In practical terms,

$$f = \text{const} \prod \vartheta(x/x_i),$$

where ϑ is the unique section of the unique bundle with divisor $\{1\} \subset E$.

The interpolation problem is thus whether the restriction map

$$\Gamma(\mathscr{L}_z) \ni f(x) \mapsto (f(a_1), \ldots, f(a_d)) \tag{1}$$

is an isomorphism.

Clearly it has a kernel if and only if

$$\prod \vartheta(x/a_i) \in \Gamma(\mathscr{L}_z) \quad \Leftrightarrow \quad z = \prod a_i \in E\,.$$

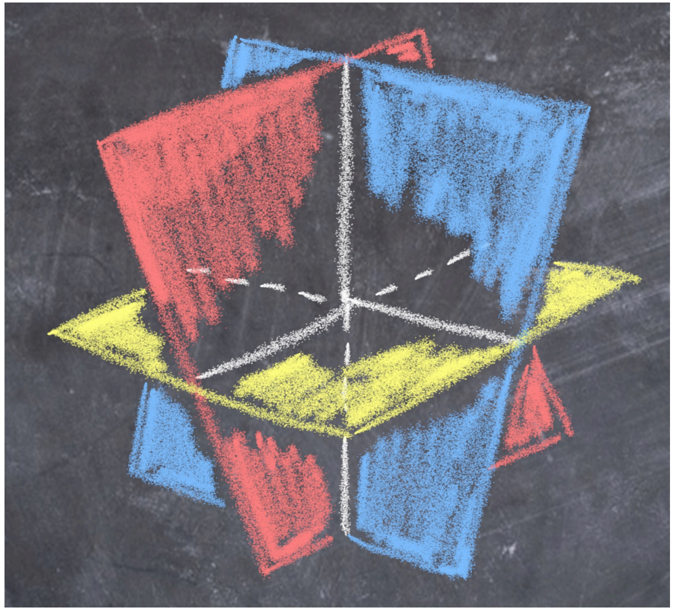
Thus (1) has an inverse with a pole in z of the form $\vartheta(z/\prod a_i)^{-1}$.

These basic facts may be revisited via the interpretation of the set

$$\mathscr{S} = \bigcup_i \{x = a_i\} \subset \{(x, a_1, \ldots, a_n)\}$$

as

$\mathscr{S} =$ spectrum of the equivariant cohomology/K-theory/elliptic cohomology of the projective space $\mathbb{P}^{n-1}$



Recall the role that characters play in representation theory.

The map

$$\begin{array}{c}\text{representation } V \\ \text{of a group } G\end{array} \xrightarrow{\text{character}} \begin{array}{c}\text{function } \chi_V(g) = \operatorname{tr}_V g \\ \text{on the group } G\end{array}$$

is a ring homomoprhism with respect to the operations $\oplus$, $\otimes$, and $\ominus$, where the latter is *added* formally.

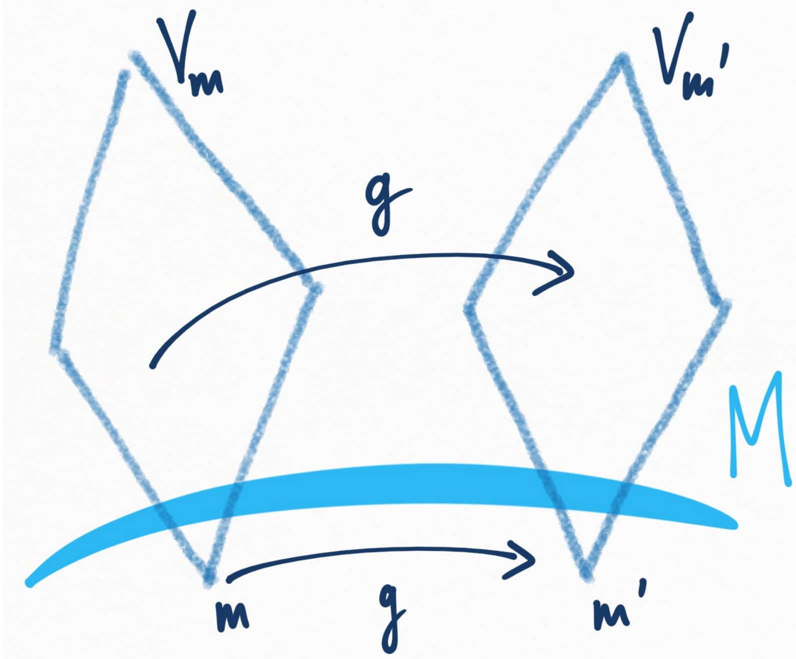
For instance, if $G = (\mathbb{C}^\times)^n$ with coordinates a_i then

$$\begin{array}{c}\text{representation} \\ \text{ring of } (\mathbb{C}^\times)^n\end{array} = \mathbb{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}] \,.$$

If G acts on a topological space M then $K_G(M)$ is the ring of vector bundles with linear action of G, under the operations $\oplus$, $\otimes$, and $\ominus$.

If M is itself a vector space then

$$K_G(M) = K_G(\text{point})$$
$$= \text{representation ring of } G\,.$$



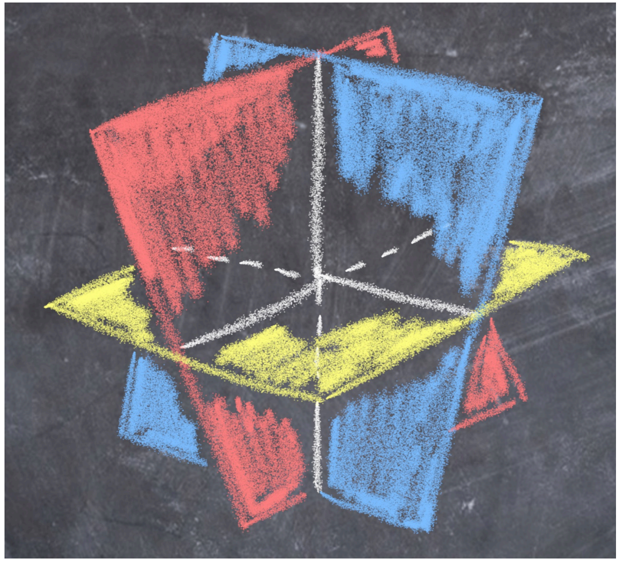
Characters are replaced by functions $\chi_V(g) = \sum_i (-1)^i \operatorname{tr}_{H^i(V)} g$

Let the tori

$$G_1 = \mathsf{Spec}\,\mathbb{C}[x^{\pm 1}]\,, \quad G_2 = \mathsf{Spec}\,\mathbb{C}[a_1^{\pm 1}, \dots, a_n^{\pm 1}]$$

act on $\mathbb{C}^n$ by $(x, a) \mapsto \mathsf{diag}(a_1/x, \dots, a_n/x)$. Then

$$\begin{aligned} \mathbb{Z}[x^{\pm 1}, a_i^{\pm 1}] &= K_{G_1 \times G_2}(\mathbb{C}^n) \\ &\to K_{G_1 \times G_2}(\mathbb{C}^n \setminus 0) \\ &= K_{G_2}((\mathbb{C}^n \setminus 0)/G_1) \\ &= K_{G_2}(\mathbb{P}^{n-1}) \\ &= \mathbb{Z}[\mathscr{S}] \end{aligned}$$



is the restriction map that the interpolation aims to invert.

Indeed, $\prod(1 - x/a_i) =$ the equation of $\mathscr{S}$ is the K-class of the skyscraper sheaf at $0 \in \mathbb{C}^n$.

In this example,

$$\mathbb{C}^n \setminus 0 = \text{(semi)stable locus for the action of } G_1 \text{ on } \mathbb{C}^n$$

meaning that

$$\mathbb{P}^{n-1} = \text{GIT quotient } \mathbb{C}^n /\!\!/ G_1\,.$$

Thus, a fancy way to phrase the interpolation problem is to ask for a section of the restriction map

$$K_{G_2}(\text{quotient stack}\,[\mathbb{C}^n/G_1]) \to K_{G_2}(\text{GIT quotient}\,\mathbb{C}^n /\!\!/ G_1),$$

and similarly for cohomology classes, and elliptic cohomology classes.

If we replace $\mathbb{P}^{n-1}$ by the Grassmannian

$$\mathrm{Gr}(k,n) = \mathrm{Mat}(n \times k) /\!\!/ GL(k)$$

then it is about interpolation of symmetric polynomials in $x_1, \ldots, x_k$ at points of the form

$$x = (a_{i_1}, \ldots a_{i_k}) \,, \quad 1 \leq a_{i_1} < \cdots < a_{i_k} \leq n \,.$$

Schubert calculus is closely related the corresponding Newton interpolation polynomials, as first noticed probably by Lascoux and Schützenberger a long time ago.

One can encounter the K-theory of the Grassmannian, and the interpolation questions, computing integrals of the form

$$\chi_f(a) = \int f(x) \, \frac{\prod_{i \neq j}(1 - x_i/x_j)}{\prod_{i,k}(1 - x_i/a_k)} \, d_{\mathsf{Haar}}(x) \in \mathbb{Z}[a_i^{\pm 1}]$$

by residues.

These already look like the integral solutions of certain q-difference equations below, with the following differences.

The first difference, is that instead of poles of the form $(1 - x_i/a_k)^{-1}$, we want $\Gamma_q(x_i/a_k)$, where

$$\Gamma_q(w) = \frac{1}{(1-w)(1-qw)(1-q^2w)\cdots}, \quad |q| < 1\,.$$

Poles form progressions with denominator q, whence elliptic interpolation will be important.

Γ_q-functions arise when we replace an algebraic variety $X = \mathsf{Gr}(k,n)$ by $\mathsf{Maps}(\mathbb{C}, X)$, where $q \in \mathbb{C}^\times$ acts naturally on the domain of the map.

The second difference is that prefer to deal with balanced expressions like

$$\frac{\Gamma_q(qw)}{\Gamma_q(\hbar w)} \to (1-w)^{1-c}\,, \quad q \to 1, \log_q \hbar \to c\,.$$

This means replacing $X = \mathsf{Gr}(k,n)$ by the symplectic manifold T^*X, where the variable $\hbar \in \mathbb{C}^\times$ scales the cotangent directions. It is analogous to the variable t in the Macdonald-Cherednik theory.

In terms of quotients, this means

$$\mathsf{Gr}(k,n) = \mathsf{Mat}(n \times k) /\!\!/ GL(k) \hookrightarrow T^*\mathsf{Mat}(n\times k) /\!\!/\!\!/\!\!/ GL(k) = T^*\mathsf{Gr}(k,n)\,,$$

where $/\!\!/\!\!/\!\!/$ denotes algebraic symplectic reduction.

We now consider the following general setup:

$$X = T^*M /\!\!/\!\!/\!\!/ G$$

where G is reductive, and M is a representation of G. It can be replaced by an affine symplectic algebraic G-variety, and has to satisfy certain assumptions like the absense of strictly semistable points.

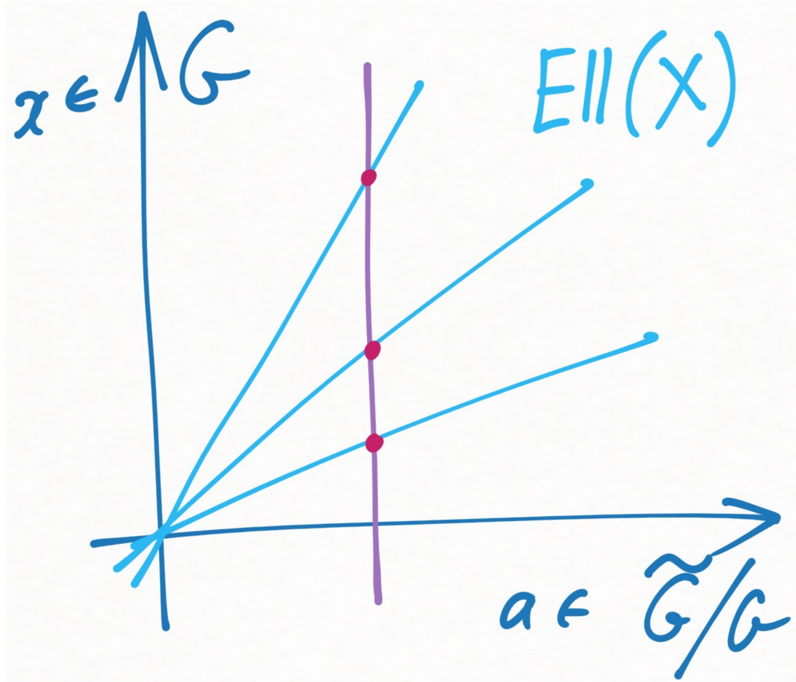
We choose $\widetilde{G}$ in

$$G \subset \widetilde{G} \subset \text{Normalizer}(G) \subset GL(T^*M)\,.$$

E.g. for $X = T^*\,\mathrm{Gr}(k,n)$ the variables a_i and $\hbar$ are in $\widetilde{G}$. In the gauge theory context, the group $\widetilde{G}$ combines gauge and flavor symmetries.

We have

$$\begin{array}{ccc} \mathsf{Ell}_{\widetilde{G}/G}(X) & \longrightarrow \mathsf{Ell}_{\widetilde{G}}(T^*M) \cong & E^{\mathsf{rk}\,\widetilde{G}}/\mathsf{Weyl} \\ \downarrow & & \downarrow \\ \mathsf{Ell}_{\widetilde{G}/G}(\mathsf{pt}) & = & E^{\mathsf{rk}(\widetilde{G}/G)}/\mathsf{Weyl} \end{array}$$



and we can ask about *extension/interpolation* of elliptic cohomology classes (that is, sections of certain line bundles).

Line bundles on an abelian variety have degree (which has to do with the codimension of an elliptic cohomology class) and also continous moduli — and we especially need those that come from **characters** of the group G like the variable z in the $\mathbb{P}^{n-1}$ example. We denote them collectively by

$$z \in \text{characters}(G) \otimes E\,.$$

For interpolation above we will choose the degree that correspond to **Lagrangian** cycles in X and we would further require the interpolants to be supported on $\mu^{-1}(0)$ where

$$\mu : T^*M \to \text{Lie}(G)^*$$

is the moment map in the symplectic reduction.

THEOREM 1

This interpolation problem has a unique solution in rational functions of z.

This is a nonabelian version of stable envelopes and we will denote it by **Stab**.

Its poles in z have a very interesting enumerative meaning — they are the so-called Kähler roots of X.

We will now turn to operator Ψ defined by

$$(\alpha, \Psi\beta) = \int_{|x_i|=1} \mathsf{Stab}(\alpha)\Big|_{q=0} \mathsf{Stab}(\beta) \frac{\prod \Gamma_q(\dots)}{\prod \Gamma_q(\dots)} \, d_{\mathsf{Haar}}(x) \,, \qquad (\text{☆})$$

where the argument of the Γ_q functions come from the weights in T^*M and Lie(G) like they did in the Grassmannian example.

THEOREM 2

The integral ($\star$) is the K-theoretic count of maps $f : \mathbb{C} \to X$ weighted by $z^{\deg f}$ and with certain boundary conditions at $0, \infty$ indexed by classes α and β.

thus one can apply an old theorem (from joint work with A. Smirnov) and deduce

COROLLARY

The integral ($\star$) satisfies q-difference equations in all variables that generalize the quantum Knizhnik-Zamolodchikov equations of I. Frenkel and N. Reshetikhin and the dynamical equations of A. Varchenko and his collaborators.

Why would one want to count curves in X and what is the meaning of these counts ?

X can be written as

$$X = \mathrm{Crit}(\mathscr{W})/G_c$$

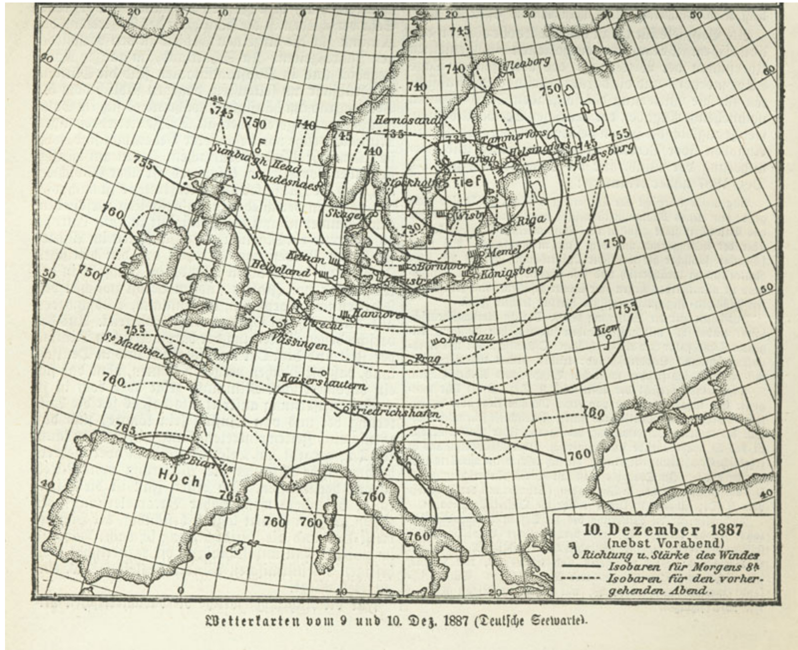
where the function $\mathscr{W}$ combines the complex and real maps for the action of

$$G_c = \text{maximal compact} \subset G\,.$$

X is the Higgs branch of the moduli space of vacua in a (2+1)-dim susy gauge theory with

$$G_c = \text{gauge symmetry}\,,$$
$$\widetilde{G}_c/G_c = \text{flavor symmetry}\,,$$
$$T^*M = \text{matter}\,.$$

Maps to X are modulated vacua and the holomorphic ones respect a certain supersymmetry.



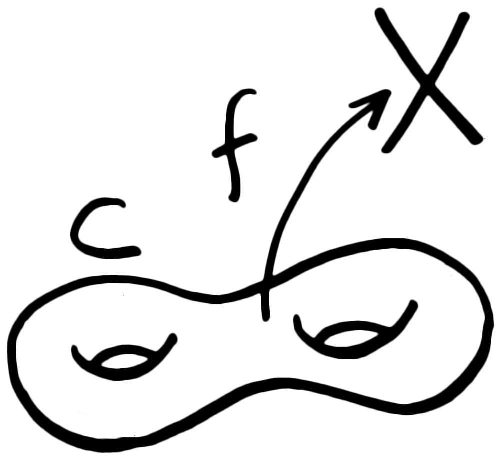
Wetterkarten vom 9 und 10. Dez. 1887 (Deutsche Seewarte).

In one language, we describe a physical theory as a susy σ-model with target X, the Higgs branch. In the dual Landau-Ginzburg-type description, we have the integrals and the Coulomb branch of the theory.

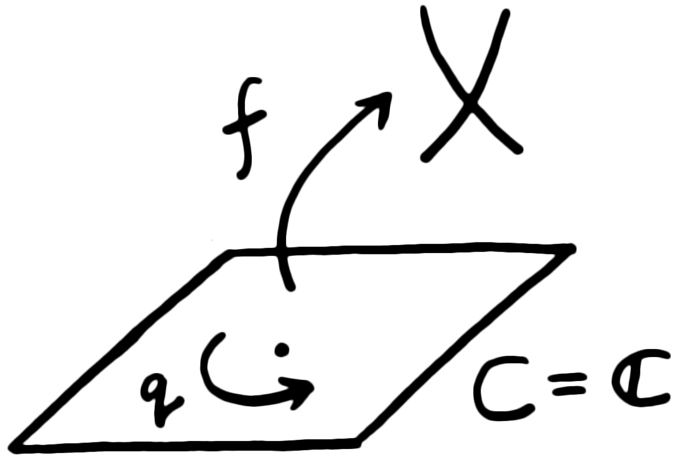
We compute the $\widetilde{G}$-equivariant index of a certain virtual Dirac operator on the moduli of holomorphic maps from a Riemann surface C to X.

Complex, but real computation in K-theory of certain well-defined moduli spaces.

The index with them is morally the same as the index of the full evolution operator in (2+1)-dimensional gauge theory, which is why many physicists think about it.



The most important count is in flat space $C = \mathbb{C}$ with boundary conditions at 0 and at infinity. Compare and contrast with Nekrasov functions in (4+1)-dimensional gauge theories. Have to be done equivariantly with respect to $GL(1) \ni q$ acting in the source.



Carries two indices in $K(X)$, so it is like a matrix which we denote by Ψ.

On formal geometric grounds, solve a q-difference equations in all variables. These include the difference analog of the Dubrovin connection from the quantum cohomology of X.

At $q = 1$, a flat q-difference connection gives a bunch of commuting matrices.

Around 2007, Nekrasov and Shatashvili computed their spectra and found it has a typical shape of Bethe equations. Whence the conjecture that the whole counting problem is solved by certain quantum groups.

The corresponding groups, including the ellliptic ones, can be constructed geometrically from the stable envelopes. The integrals (☆) summarize the theory of "Bethe Ansatz" for these groups, extend a great deal of prior research, and connect with a lot things that are being done at the IPMU !