Correlation functions of scalar field theories from homotopy algebras

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arXiv:2203.05366 and work in progress

October 21, 2022

HirosiFest @ Kavli IPMU

1. Prelude

My postdoc days at Caltech (September of 2000 to August of 2003)

Seiberg and Witten found that noncommutative gauge theories (gauge theories based on noncommutative geometry) can be realized on D-branes in string theory.

Seiberg and Witten, hep-th/9908142

We studied the coupling of noncommutative gauge theories to gravity by calculating disk amplitudes with one closed string and an arbitrary number of open strings.

Okawa and Ooguri, hep-th/0012218

We then realized that the energy-momentum tensor of the gauge theory on D-branes in the decoupling limit (which was also called the DKPS limit or the Sen-Seiberg limit) can be derived using the same method.

Okawa and Ooguri, hep-th/0103124

Futhermore, we derived an exact expression for the Seiberg-Witten map by studying the coupling of non-commutative gauge theories to Ramond-Ramond gauge fields in string theory.

Okawa and Ooguri, hep-th/0104036

On Wednesdays, the Caltech people make a trip to USC to participate in the seminar.

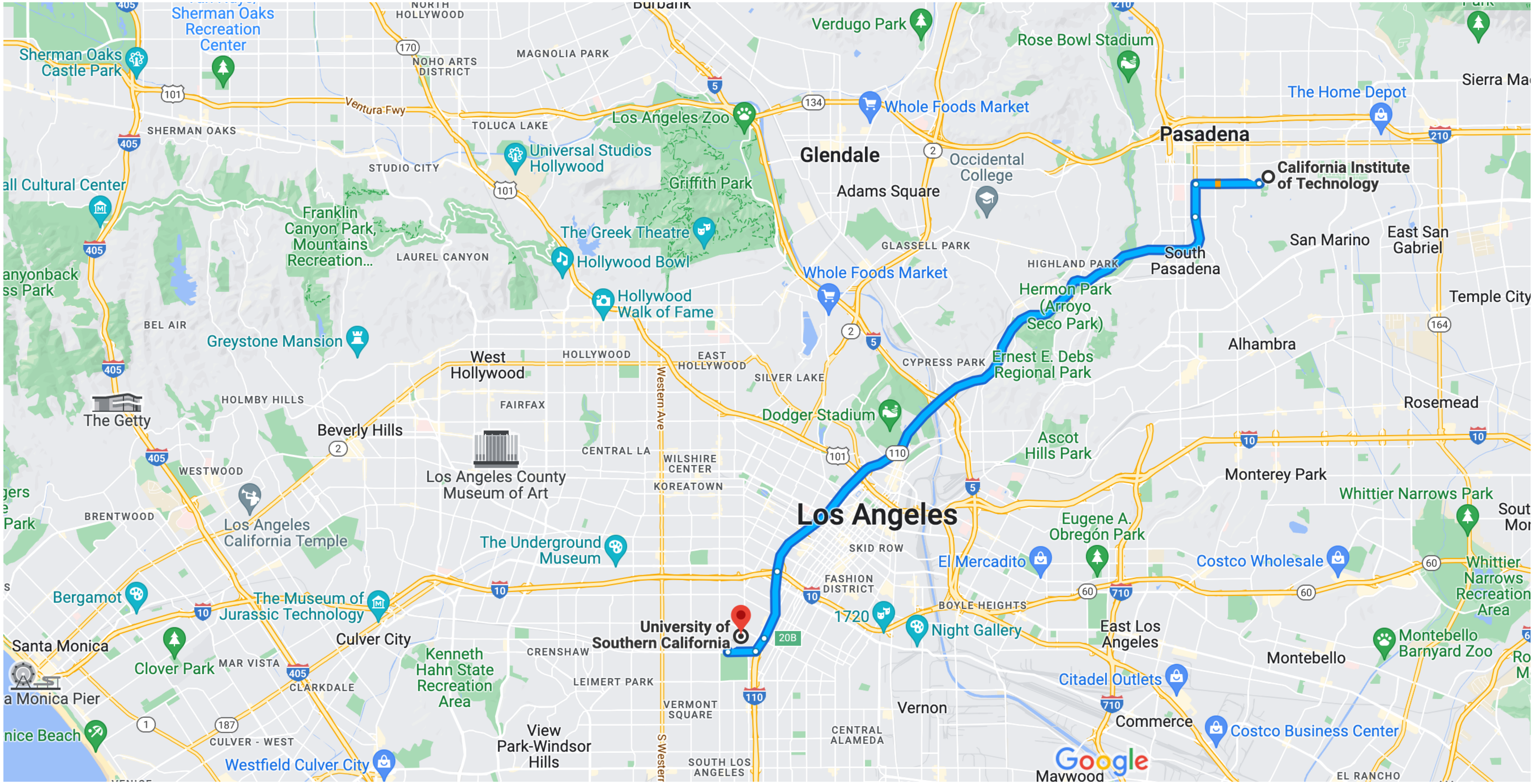
On Fridays, the USC people come to Caltech to participate in the seminar.

Sometimes we have two seminars a day.

Sometimes we have four seminars a week!

We had a gorgeous selection of seminars, which was an ideal environment for postdocs like me.

Google Maps California Institute of Technology to University of Southern California Drive 14.1 miles, 18 min



Map data ©2022 2 km

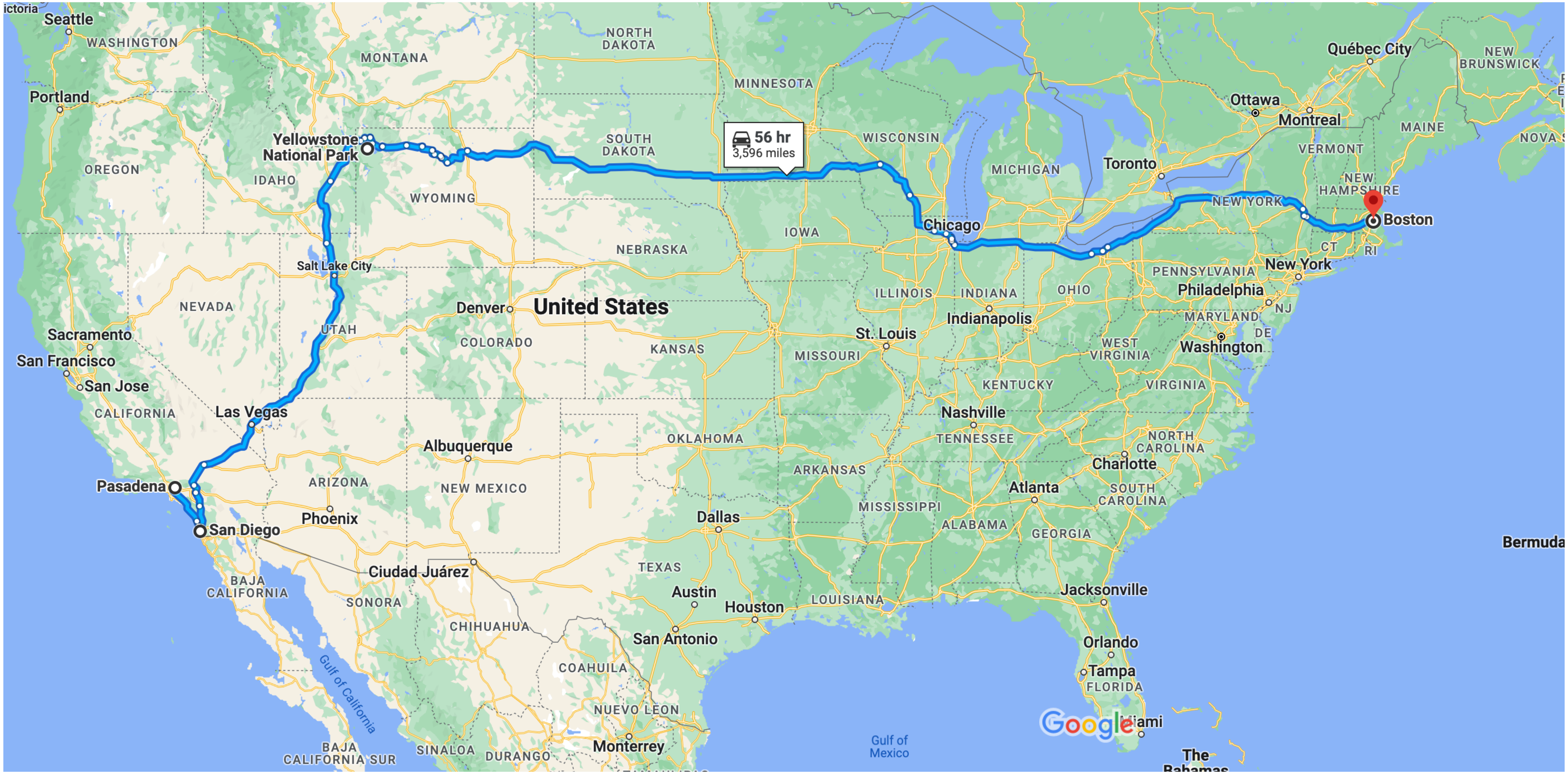
California Institute of Technology

1200 E California Blvd, Pasadena, CA 91125, United States

One of such seminars which influenced me a lot was the one by Ashoke Sen on open string field theory. I was impressed by their geometric method based on CFT to describe various string fields.

I got interested in this subject and wrote my first paper on string field theory, and got a postdoc ticket to MIT.

Google Maps Pasadena, California, USA to Boston, Massachusetts, USA Drive 3,596 miles, 56 hr



Map data ©2022 Google, INEGI 200 km

⚠ This route has tolls.

⚠ This route includes roads that are closed in winter.

⚠ Your destination is in a different time zone.

Pasadena

I realized that my current research is deeply rooted in my postdoc days at Caltech.

The motivation behind the work I will explain today is directly related to the calculation of the energy-momentum tensor of the gauge theory on D-branes in the decoupling limit by Hirosi and myself.

The aim of the work I will explain today is its application to open string field theory.

2. Introduction

Closed string theory is a consistent theory including quantum gravity, but it is defined only perturbatively.

While closed string field theory is useful, for example, in the discussion of mass renormalization or vacuum shift, it would not be promising to use closed string field theory for a nonperturbative definition of closed string theory.

This is because gauge invariance in the classical theory is anomalous and we need correction terms at every loop order to recover gauge invariance.

The most promising approach to the nonperturbative definition of closed string theory would be the AdS/CFT correspondence, but the world-sheet picture is gone in the strict low-energy limit of the gauge theory on D-branes.

It might be useful to consider the theory on D-branes before taking the low-energy limit.

We may think that such a theory would be open-closed string field theory, but my claim is that it can be described by open string field theory with the source term for gauge-invariant operators.

This seems to be the case at least for the bosonic string as a consequence of a few nontrivial facts.

Unlike closed string field theory, gauge invariance of open bosonic string field theory is not anomalous, and we do not need correction terms to the classical action.

It is in general difficult to construct gauge-invariant operators in string field theory, but a class of gauge-invariant operators are constructed in open bosonic string field theory.

hep-th/0111092, Hashimoto and Itzhaki
hep-th/0111129, Gaiotto, Rastelli, Sen and Zwiebach

We can construct a gauge-invariant operator for each on-shell closed string state, and peculiarly it is linear in the open string field.

Open string field theory with the source term for gauge-invariant operators can be obtained in a special limit of open-closed string field theory, and it generates all Feynman diagrams which contain at least one boundary.

hep-th/9202015, Zwiebach

Purely closed-string diagrams without boundaries are not generated, but their contributions vanish in the low-energy limit we are interested in.

It is crucially important whether or not this scenario can be extended to open superstring field theory.

The long-standing problem of constructing an action involving the Ramond sector has been overcome in superstring field theory.

Kunitomo and Okawa, arXiv:1508.00366
Sen, arXiv:1508.05387

While the formulations of open superstring field theory need to be developed further, we consider that we are in a position to discuss how we use open superstring field theory to understand the mechanism which realizes the AdS/CFT correspondence.

So what should we do?

Instead of scattering amplitudes, we should consider correlation functions of gauge-invariant operators in open string field theory.

We evaluate correlation functions in the $1/N$ expansion and turn it into the genus expansion of closed string theory.

This step would be the most difficult part and we need to generalize the world-sheet derivation of the large N duality of the topological string by Ooguri and Vafa to the superstring.

hep-th/0205297, Ooguri and Vafa

The quantum treatment of open string field theory must be crucial for the duality, but such calculations would be technically difficult for open superstring field theory.

We do not need to calculate correlation functions explicitly, but we want to understand the structure of the theory in the low-energy limit.

Homotopy algebras can be useful tools for this purpose.

We have used homotopy algebras such as A_∞ algebras and L_∞ algebras in the construction of actions of string field theory.

However, we might not have fully appreciated the power of homotopy algebras, and they can be also useful in solving the theory.

The plan of the talk

1. Prelude
2. Introduction
3. A_∞ algebra
4. Formula for correlation functions
5. Renormalization group
6. Summary

3. A_∞ algebra

Open bosonic string field theory is described in terms of string field, which is a state of the boundary conformal field theory.

The Hilbert space $\mathcal{H}$ can be decomposed based on the ghost number as

$$\mathcal{H} = \ldots \oplus \mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots\,,$$

and the classical action is written in terms of Ψ in $\mathcal{H}_1$.

Consider an action of the form:

$$S = -\frac{1}{2}\langle\, \Psi, V_1(\Psi)\,\rangle - \frac{g}{3}\langle\, \Psi, V_2(\Psi,\Psi)\,\rangle - \frac{g^2}{4}\langle\, \Psi, V_3(\Psi,\Psi,\Psi)\,\rangle + O(g^3)\,,$$

where $\langle\, A_1, A_2\,\rangle$ is the BPZ inner product of A_1 and A_2, V_n is an n-string product, and g is the string coupling constant.

This action is invariant up to $O(g^3)$ under the gauge transformation with the gauge parameter Λ in $\mathcal{H}_0$ given by

$$\begin{aligned}\delta_\Lambda \Psi = {} & V_1(\Lambda) + g\,(\,V_2(\Psi,\Lambda) - V_2(\Lambda,\Psi)\,) \\ & + g^2\,(\,V_3(\Psi,\Psi,\Lambda) - V_3(\Psi,\Lambda,\Psi) + V_3(\Lambda,\Psi,\Psi)\,) + O(g^3)\end{aligned}$$

if the multi-string products satisfy the following relations:

$$\begin{aligned}& V_1(V_1(A_1)) = 0\,, \\ & V_1(V_2(A_1,A_2)) - V_2(V_1(A_1),A_2) - (-1)^{A_1} V_2(A_1,V_1(A_2)) = 0\,, \\ & V_1(V_3(A_1,A_2,A_3)) + V_3(V_1(A_1),A_2,A_3) \\ & \; + (-1)^{A_1} V_3(A_1,V_1(A_2),A_3) + (-1)^{A_1+A_2} V_3(A_1,A_2,V_1(A_3)) \\ & \; - V_2(V_2(A_1,A_2),A_3) + V_2(A_1,V_2(A_2,A_3)) = 0\,.\end{aligned}$$

These relations can be extended to higher orders and called A_∞ relations.
(In this talk all the discussions on cyclic properties are omitted.)

Let us simplify the description of A_∞ relations in three steps.

Step 1: Degree

We introduce *degree* defined by

$$\deg(A) = \epsilon(A) + 1 \mod 2\,,$$

where $\epsilon(A)$ is the Grassmann parity of A, and we define

$$\begin{aligned}
\omega(A_1, A_2) &= (-1)^{\deg(A_1)} \langle\, A_1, A_2 \,\rangle\,, \\
M_1(A_1) &= V_1(A_1)\,, \\
M_2(A_1, A_2) &= (-1)^{\deg(A_1)}\, V_2(A_1, A_2)\,, \\
M_3(A_1, A_2, A_3) &= (-1)^{\deg(A_2)}\, V_3(A_1, A_2, A_3)\,, \\
&\vdots
\end{aligned}$$

Step 2: Tensor products of $\mathcal{H}$

We denote the tensor product of n copies of $\mathcal{H}$ by $\mathcal{H}^{\otimes n}$. For an n-string product $c_n(A_1, A_2, \ldots, A_n)$ we define a corresponding operator c_n which maps $\mathcal{H}^{\otimes n}$ into $\mathcal{H}$ by

$$c_n\,(A_1 \otimes A_2 \otimes \ldots \otimes A_n) \equiv c_n(A_1, A_2, \ldots, A_n)\,.$$

We also introduce the vector space for the zero-string space denoted by $\mathcal{H}^{\otimes 0}$. It is a one-dimensional vector space given by multiplying a single basis vector $\mathbf{1}$ by complex numbers. The vector $\mathbf{1}$ satisfies

$$\mathbf{1} \otimes A = A\,, \qquad A \otimes \mathbf{1} = A$$

for any string field A.

The A_∞ relations are written as

$$
\begin{aligned}
& M_1\, M_1 = 0\,, \\
& M_1\, M_2 + M_2\, (\, M_1 \otimes \mathbb{I} + \mathbb{I} \otimes M_1\,) = 0\,, \\
& M_1\, M_3 + M_3\, (\, M_1 \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes M_1 \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes M_1\,) \\
& \; + M_2\, (\, M_2 \otimes \mathbb{I} + \mathbb{I} \otimes M_2\,) = 0\,, \\
& \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

where we denoted the identity map from $\mathcal{H}$ to $\mathcal{H}$ by $\mathbb{I}$.

Step 3: <u>Coderivations</u>

It is convenient to consider linear operators acting on the vector space $T\mathcal{H}$ defined by

$$T\mathcal{H} = \mathcal{H}^{\otimes 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \ldots .$$

We denote the projection operator onto $\mathcal{H}^{\otimes n}$ by π_n.

For a map c_n from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}$, we define an associated operator $\boldsymbol{c}_n$ acting on $T\mathcal{H}$ as follows.

$$\begin{aligned}
\boldsymbol{c}_n \, \pi_m &= 0 \quad \text{for} \quad m < n \,, \\
\boldsymbol{c}_n \, \pi_n &= c_n \, \pi_n \,, \\
\boldsymbol{c}_n \, \pi_{n+1} &= (\, c_n \otimes \mathbb{I} + \mathbb{I} \otimes c_n \,) \, \pi_{n+1} \,, \\
\boldsymbol{c}_n \, \pi_{n+2} &= (\, c_n \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes c_n \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes c_n \,) \, \pi_{n+2} \,, \\
&\vdots
\end{aligned}$$

An operator acting on $T\mathcal{H}$ of this form is called a *coderivation*.

We define $\mathbf{M}$ by

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \ldots$$

for coderivations $\mathbf{M}_n$ associated with M_n. Then the A_∞ relations can be compactly expressed as

$$\mathbf{M}^2 = 0\,.$$

When we consider projections onto subspaces of $\mathcal{H}$, homotopy algebras have turned out to provide useful tools.

- Projection onto on-shell states $\rightarrow$ on-shell scattering amplitudes
  Kajiura, math/0306332

- Projection onto the physical sector
  $\rightarrow$ mapping between covariant and light-cone string field theories
  Erler and Matsunaga, arXiv:2012.09521

- Projection onto the massless sector $\rightarrow$ the low-energy effective action
  Sen, arXiv:1609.00459
  Erbin, Maccaferri, Schnabl and Vošmera, arXiv:2006.16270
  Koyama, Okawa and Suzuki, arXiv:2006.16710

Let us decompose $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{Q} + \boldsymbol{m}\,,$$

where $\mathbf{Q}$ describes the free theory and $\boldsymbol{m}$ is for interactions. We consider projections which commute with Q.

We denote the projection operator by P:

$$P^2 = P\,, \qquad P\,Q = Q\,P\,.$$

We then promote P on $\mathcal{H}$ to $\mathbf{P}$ on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\mathbf{P}\,\pi_0 &= \pi_0\,,\\
\mathbf{P}\,\pi_1 &= P\,\pi_1\,,\\
\mathbf{P}\,\pi_2 &= (\,P \otimes P\,)\,\pi_2\,,\\
\mathbf{P}\,\pi_3 &= (\,P \otimes P \otimes P)\,\pi_3\,,\\
&\,\,\vdots
\end{aligned}$$

The operators $\mathbf{Q}$ and $\mathbf{P}$ satisfy

$$\mathbf{P}^2 = \mathbf{P}\,, \qquad \mathbf{Q}\,\mathbf{P} = \mathbf{P}\,\mathbf{Q}\,.$$

In the context of the projection onto the massless sector, the propagator h for massive fields is given by

$$h = \frac{b_0}{L_0}\,(\,\mathbb{I} - P\,)\,.$$

In general we consider h satisfying the following relations:

$$Q\,h + h\,Q = \mathbb{I} - P\,, \qquad h\,P = 0\,, \qquad P\,h = 0\,, \qquad h^2 = 0\,.$$

We then promote h on $\mathcal{H}$ to $\boldsymbol{h}$ on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\boldsymbol{h}\,\pi_0 &= 0\,,\\
\boldsymbol{h}\,\pi_1 &= h\,\pi_1\,,\\
\boldsymbol{h}\,\pi_2 &= (\,h\otimes P + \mathbb{I}\otimes h\,)\,\pi_2\,,\\
\boldsymbol{h}\,\pi_3 &= (\,h\otimes P\otimes P + \mathbb{I}\otimes h\otimes P + \mathbb{I}\otimes\mathbb{I}\otimes h\,)\,\pi_3\,,\\
&\ \ \vdots
\end{aligned}$$

The relations involving Q, P, and h are promoted to the following relations

$$\mathbf{Q}\,\boldsymbol{h} + \boldsymbol{h}\,\mathbf{Q} = \mathbf{I} - \mathbf{P}\,, \qquad \boldsymbol{h}\,\mathbf{P} = 0\,, \qquad \mathbf{P}\,\boldsymbol{h} = 0\,, \qquad \boldsymbol{h}^2 = 0\,,$$

where $\mathbf{I}$ is the identity operator on $T\mathcal{H}$.

The important point is that the theory after the projection inherits the A_∞ structure from the theory before the projection as follows:

$$\mathbf{Q} + \boldsymbol{m} \quad \rightarrow \quad \mathbf{P}\,\mathbf{Q}\,\mathbf{P} + \mathbf{P}\,\boldsymbol{m}\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,\mathbf{P}\,,$$

which is known as *homological perturbation lemma*.

On-shell scattering amplitudes at the tree level can be calculated from this formula with the projection onto on-shell states.

On-shell scattering amplitudes at the loop level can also be calculated by extending A_∞ algebras to quantum A_∞ algebras (to be discussed later).

The formula from quantum A_∞ algebras has not been explored much.

In addition to scattering amplitudes we are also interested in correlation functions.

Actually, when actions are written in terms of homotopy algebras, expressions of on-shell scattering amplitudes are universal for both string field theories and ordinary field theories.

Let us study scalar field theories in terms of quantum A_∞ algebras to gain insights into quantum aspects of string field theories.

We also find that correlation functions of scalar field theories can also be described in terms of homotopy algebras.

Okawa, arXiv:2203.05366

4. Formula for correlation functions

Let us consider φ^3 theory in d dimensions:

$$S = \int d^d x \left[-\frac{1}{2}\, \partial_\mu \varphi(x)\, \partial^\mu \varphi(x) - \frac{1}{2}\, m^2\, \varphi(x)^2 + \frac{1}{6}\, g\, \varphi(x)^3 \right].$$

To describe this action in terms of an A_∞ algebra, we introduce two copies of the vector space of functions of x. We denote them by $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\mathcal{H}$ is given by

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\,.$$

We define ω, Q, and m by

$$\begin{aligned}
\omega\,(\,\varphi_1(x), \varphi_2(x)\,) &= \int d^d x\, \varphi_1(x)\, \varphi_2(x) \quad \text{for} \quad \varphi_1(x) \in \mathcal{H}_1\,,\ \varphi_2(x) \in \mathcal{H}_2\,, \\
Q\, \varphi(x) &= (\, -\partial^2 + m^2\,)\, \varphi(x) \in \mathcal{H}_2 \quad \text{for} \quad \varphi(x) \in \mathcal{H}_1\,, \\
m\,(\,\varphi_1(x) \otimes \varphi_2(x)\,) &= -\frac{g}{2}\, \varphi_1(x)\, \varphi_2(x) \in \mathcal{H}_2 \quad \text{for} \quad \varphi_1(x)\,, \varphi_2(x) \in \mathcal{H}_1\,.
\end{aligned}$$

The A_∞ structure of the classical action is described by $\mathbf{Q} + \boldsymbol{m}$. The A_∞ relations are trivially satisfied for this theory without gauge symmetries.

When we consider on-shell scattering amplitudes, we use the projection onto on-shell states. The action of h on $\varphi(x)$ in $\mathcal{H}_2$ is given by

$$h\,\varphi(x) = \int d^d y \int \frac{d^d p}{(2\pi)^d} \, \frac{e^{-ip\,(x-y)}}{p^2 + m^2 - i\epsilon} \, \varphi(y) \,.$$

In the case of the projection onto on-shell states, $\mathbf{P}\,\mathbf{Q}\,\mathbf{P}$ vanishes and on-shell scattering amplitudes at the tree level can be calculated from

$$\mathbf{P}\,\boldsymbol{m}\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,\mathbf{P} \,.$$

When we discuss the quantum theory, we need to include conterterms, and the counterterms are included in $\boldsymbol{m}$. On-shell scattering amplitudes including loop diagrams can be calculated from

$$\mathbf{P}\,\boldsymbol{m}\,\frac{1}{\mathbf{I}+\boldsymbol{h}\,\boldsymbol{m}+i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P}\,.$$

The operator $\mathbf{U}$ consists of maps from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}^{\otimes (n+2)}$. When the vector space $\mathcal{H}$ is given by $\mathcal{H}_1 \oplus \mathcal{H}_2$, the operator $\mathbf{U}$ incorporates a pair of basis vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$. We denote the basis vector of $\mathcal{H}_1$ by e^α, where α is the label of the basis vectors. For $\mathcal{H}_2$ we denote the basis vector by e_α, and repeated indices are implicitly summed over. We use the following choice for e^α and e_α:

$$\ldots \otimes e^\alpha \otimes \ldots \otimes e_\alpha \otimes \ldots = \int \frac{d^d p}{(2\pi)^d}\, \ldots \otimes e^{-ipx} \otimes \ldots \otimes e^{ipx} \otimes \ldots\,.$$

The action of $\mathbf{U}$ on $\mathcal{H}^{\otimes 0}$ is given by

$$\mathbf{U}\,\mathbf{1} = e^\alpha \otimes e_\alpha + e_\alpha \otimes e^\alpha\,,$$

and the action of $\mathbf{U}$ on $\mathcal{H}$ is given by

$$\begin{aligned}\mathbf{U}\,\varphi(x) = {} & e^\alpha \otimes e_\alpha \otimes \varphi(x) + (-1)^{\deg(\varphi)} e^\alpha \otimes \varphi(x) \otimes e_\alpha \\ & + (-1)^{\deg(\varphi)} \varphi(x) \otimes e^\alpha \otimes e_\alpha + e_\alpha \otimes e^\alpha \otimes \varphi(x) \\ & + e_\alpha \otimes \varphi(x) \otimes e^\alpha + (-1)^{\deg(\varphi)} \varphi(x) \otimes e_\alpha \otimes e^\alpha\,.\end{aligned}$$

A_∞ algebras are extended to quantum A_∞ algebras in the quantum theory. The quantum A_∞ relations are again trivially satisfied for this theory without gauge symmetries.

If we recall that the projection onto the massless sector corresponds to integrating out massive fields, carrying out the path integral *completely* should correspond to the projection with

$$P = 0\,.$$

The associated operator $\mathbf{P}$ corresponds to the projection onto $\mathcal{H}^{\otimes 0}$:

$$\mathbf{P} = \pi_0\,.$$

This may result in a trivial theory at the classical case, but it can be nontrivial for the quantum case and in fact it is exactly what we do when we calculate correlation functions.

Let us consider scalar field theories in Euclidean space. We define $\boldsymbol{f}$ by

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \boldsymbol{h}\,\mathbf{U}}\,,$$

which corresponds to

$$\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}$$

in Minkowski space.

While $\mathbf{P}\,\boldsymbol{m}\,\boldsymbol{f}\,\mathbf{P}$ vanishes, $\boldsymbol{f}$ is nonvanishing and this operator plays a central role in generating Feynman diagrams.

We claim that information on correlation functions is encoded in $\boldsymbol{f}\,\mathbf{1}$ associated with the case where $P = 0$.

More explicitly, correlation functions are given by

$$\begin{aligned}
&\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n)\, \rangle \\
&= \omega_n\, (\, \pi_n\, \boldsymbol{f}\, \mathbf{1}\, , \delta^d(x - x_1) \otimes \delta^d(x - x_2) \otimes \ldots \otimes \delta^d(x - x_n)\,)\, ,
\end{aligned}$$

where

$$\begin{aligned}
&\omega_n\, (\, \varphi_1(x) \otimes \varphi_2(x) \otimes \ldots \otimes \varphi_n(x)\, , \varphi'_1(x) \otimes \varphi'_2(x) \otimes \ldots \otimes \varphi'_n(x)\,) \\
&= \prod_{i=1}^{n} \omega\, (\, \varphi_i(x)\, , \varphi'_i(x)\,)\, .
\end{aligned}$$

The formula may look complicated, but it states that $\pi_n\, \boldsymbol{f}\, \mathbf{1}$ gives the n-point function by simply replacing x with x_i in the i-th sector in $\mathcal{H}^{\otimes n}$.

For example, when $\pi_3\, \boldsymbol{f}\, \mathbf{1}$ takes the form

$$\pi_3\, \boldsymbol{f}\, \mathbf{1} = \sum_a f_a(x) \otimes g_a(x) \otimes h_a(x)\,,$$

the three-point function is given by

$$\begin{aligned}
&\langle\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \rangle \\
&= \omega_3\, (\, \pi_3\, \boldsymbol{f}\, \mathbf{1}\,, \delta^d(x-x_1) \otimes \delta^d(x-x_2) \otimes \delta^d(x-x_3)\,) \\
&= \sum_a f_a(x_1)\, g_a(x_2)\, h_a(x_3)\,.
\end{aligned}$$

This can be summarized as the following replacement rule:

$$\pi_3\, \boldsymbol{f}\, \mathbf{1} = \sum_a f_a(x) \otimes g_a(x) \otimes h_a(x)$$

$$\downarrow$$

$$\langle\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \rangle = \sum_a f_a(x_1)\, g_a(x_2)\, h_a(x_3)\,.$$

Let us first demonstrate that correlation functions of the free theory are correctly reproduced. We denote correlation functions of the free theory by $\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n)\, \rangle^{(0)}$. We find

$$\pi_2\, \boldsymbol{f}\, \mathbf{1} = \pi_2\, \boldsymbol{h}\, \mathbf{U}\, \mathbf{1} = e^\alpha \otimes h\, e_\alpha = \int \frac{d^d p}{(2\pi)^d}\, e^{-ipx} \otimes \frac{1}{p^2+m^2}\, e^{ipx}\,.$$

Following the replacement rule, the two-point function is given by

$$\langle\, \varphi(x_1)\, \varphi(x_2)\, \rangle^{(0)} = \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip\,(x_1-x_2)}}{p^2+m^2}\,.$$

The four-point function can be calculated from $\pi_4\, \boldsymbol{f}\, \mathbf{1}$. Since

$$\begin{aligned}\pi_4\, \boldsymbol{f}\, \mathbf{1} &= \pi_4\, \boldsymbol{h}\, \mathbf{U}\, \boldsymbol{h}\, \mathbf{U}\, \mathbf{1} \\ &= e^\beta \otimes e^\alpha \otimes h\, e_\alpha \otimes h\, e_\beta + e^\alpha \otimes e^\beta \otimes h\, e_\alpha \otimes h\, e_\beta \\ &\quad + e^\alpha \otimes h\, e_\alpha \otimes e^\beta \otimes h\, e_\beta\,,\end{aligned}$$

the four-point function is given by

$$\begin{aligned}&\langle\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \varphi(x_4)\, \rangle^{(0)} \\ &= \langle\, \varphi(x_2)\, \varphi(x_3)\, \rangle^{(0)} \langle\, \varphi(x_1)\, \varphi(x_4)\, \rangle^{(0)} + \langle\, \varphi(x_1)\, \varphi(x_3)\, \rangle^{(0)} \langle\, \varphi(x_2)\, \varphi(x_4)\, \rangle^{(0)} \\ &\quad + \langle\, \varphi(x_1)\, \varphi(x_2)\, \rangle^{(0)} \langle\, \varphi(x_3)\, \varphi(x_4)\, \rangle^{(0)}\,.\end{aligned}$$

We have thus reproduced Wick's theorem for four-point functions, and it is not difficult to extend the analysis to six-point functions and further.

Let us next consider φ^3 theory. The action including counterterms is given by

$$S = \int d^d x \left[\frac{1}{2} Z_\varphi \, \partial_\mu \varphi(x) \, \partial_\mu \varphi(x) + \frac{1}{2} Z_m \, m^2 \, \varphi(x)^2 - \frac{1}{6} Z_g \, g \, \varphi(x)^3 - Y \varphi(x) \right],$$

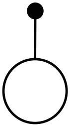
where Y, Z_φ, Z_m, and Z_g are constants. We expand Y, Z_φ, Z_m, and Z_g in g as follows:

$$\begin{aligned} Y &= g \, Y^{(1)} + O(g^3) \,, \\ Z_\varphi &= 1 + g^2 Z_\varphi^{(1)} + O(g^4) \,, \\ Z_m &= 1 + g^2 Z_m^{(1)} + O(g^4) \,, \\ Z_g &= 1 + g^2 Z_g^{(1)} + O(g^4) \,. \end{aligned}$$

The one-point function is given by

$$\langle \, \varphi(x_1) \, \rangle = \frac{g}{m^2} \left[\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{p^2 + m^2} + Y^{(1)} \right] + O(g^2) \,.$$

We have reproduced the contribution from the one-loop tadpole diagram:



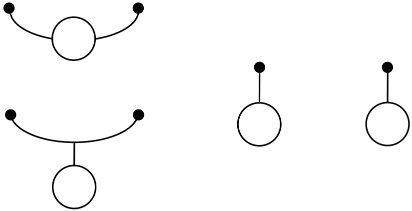
Note that the correct symmetry factor appeared.

The two-point function is given by

$$\begin{aligned}\langle\, \varphi(x_1)\, \varphi(x_2)\,\rangle &= \omega_2\,(\,\pi_2\, \boldsymbol{f}\, \mathbf{1}\,, \delta^d(x-x_1)\otimes\delta^d(x-x_2)\,)\\ &= \langle\, \varphi(x_1)\, \varphi(x_2)\,\rangle^{(0)} + \langle\, \varphi(x_1)\, \varphi(x_2)\,\rangle_C^{(1)}\\ &\qquad + \langle\, \varphi(x_1)\,\rangle^{(1)}\,\langle\, \varphi(x_2)\,\rangle^{(1)} + O(g^3)\,.\end{aligned}$$

The connected part is given by

$$
\begin{aligned}
&\langle\, \varphi(x_1)\, \varphi(x_2)\, \rangle_C^{(1)} \\
&\quad = \; -\, g^2 \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip\,(x_1 - x_2)}}{(\, p^2 + m^2\,)^2} \left[-\, \frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d}\, \frac{1}{(\ell + p)^2 + m^2}\, \frac{1}{\ell^2 + m^2} \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. +\, Z_\varphi^{(1)}\, p^2 + Z_m^{(1)}\, m^2 \right] \\
&\qquad +\, g^2 \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip\,(x_1 - x_2)}}{m^2\,(\, p^2 + m^2\,)^2} \left[\frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d}\, \frac{1}{\ell^2 + m^2} + Y^{(1)} \right].
\end{aligned}
$$



We can show that correlation functions from our formula satisfy the Schwinger-Dyson equations as an immediate consequence of the structure

$$(\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \boldsymbol{h}\,\mathbf{U})\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \boldsymbol{h}\,\mathbf{U}}\,\mathbf{1} = \mathbf{1}\,.$$

We can extend the formula for correlation functions to incorporate Dirac fermions.

Konosu and Okawa, in progress

5. Renormalization group

The construction of $\boldsymbol{h}$ from h is not unique. In addition to $\mathbf{P}$ for $P = 0$, let us introduce $\mathbf{P}_\Lambda$ for the projection onto modes below the energy scale Λ, and use $\boldsymbol{h}$ given by

$$\boldsymbol{h} = \boldsymbol{h}_H + \boldsymbol{h}_L \,,$$

where the propagator $\boldsymbol{h}_H$ for high-energy modes satisfy

$$\mathbf{Q}\,\boldsymbol{h}_H + \boldsymbol{h}_H\,\mathbf{Q} = \mathbf{I} - \mathbf{P}_\Lambda \,, \qquad \boldsymbol{h}_H\,\mathbf{P}_\Lambda = 0 \,, \qquad \mathbf{P}_\Lambda\,\boldsymbol{h}_H = 0 \,, \qquad \boldsymbol{h}_H^2 = 0$$

and the propagator $\boldsymbol{h}_L$ for low-energy modes satisfy

$$\mathbf{Q}\,\boldsymbol{h}_L + \boldsymbol{h}_L\,\mathbf{Q} = \mathbf{P}_\Lambda - \mathbf{P} \,, \quad \boldsymbol{h}_L\,(\,\mathbf{I} - \mathbf{P}_\Lambda\,) = 0 \,, \quad (\,\mathbf{I} - \mathbf{P}_\Lambda\,)\,\boldsymbol{h}_L = 0 \,, \quad \boldsymbol{h}_L^2 = 0 \,.$$

Then we can write $\boldsymbol{f}\,\mathbf{P}$ as

$$
\begin{aligned}
&\frac{1}{\mathbf{I}+\boldsymbol{h}\,\boldsymbol{m}-\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P} \\
&=\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}-\boldsymbol{h}_H\,\mathbf{U}}\left(\mathbf{I}+\boldsymbol{h}_L\,(\,\boldsymbol{m}-\mathbf{U}\,)\,\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}-\boldsymbol{h}_H\,\mathbf{U}}\right)^{-1}\mathbf{P} \\
&=\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}-\boldsymbol{h}_H\,\mathbf{U}}\,\mathbf{P}_\Lambda\,\frac{1}{\mathbf{I}+\boldsymbol{h}_L\,\boldsymbol{m}_\Lambda-\boldsymbol{h}_L\,\mathbf{U}}\,\mathbf{P}\,,
\end{aligned}
$$

where

$$
\boldsymbol{m}_\Lambda=\mathbf{P}_\Lambda\left[\,(\,\boldsymbol{m}-\mathbf{U}\,)\,\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}-\boldsymbol{h}_H\,\mathbf{U}}+\mathbf{U}\,\right]\mathbf{P}_\Lambda\,.
$$

The operator $\boldsymbol{m}_\Lambda$ describes the Wilsonian effective action at the energy scale Λ, and correlation functions are calculated from a product of the operator for high-energy modes and the operator for low-energy modes.

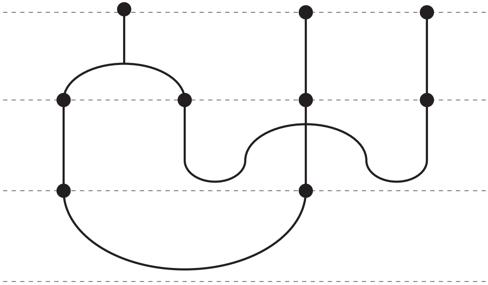
We can introduce a sequence of projections and write $\boldsymbol{f}\,\mathbf{P}$ as

$$\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \boldsymbol{h}\,\mathbf{U}}\,\mathbf{P} = \prod_i \frac{1}{\mathbf{I} + \boldsymbol{h}_i\,\boldsymbol{m}_i - \boldsymbol{h}_i\,\mathbf{U}}\,\mathbf{P}_i$$

with

$$\boldsymbol{h} = \sum_i \boldsymbol{h}_i\,.$$

While perturbative expressions for correlation functions with the previous $\boldsymbol{h}$ do not converge, this choice of $\boldsymbol{h}$ may lead to nonperturbative expressions for correlation functions.



6. Summary

We proposed the formula

$$\langle\, \varphi(x_1)\, \varphi(x_2)\; \ldots\; \varphi(x_n)\,\rangle = \omega_n\,(\,\pi_n\, \boldsymbol{f}\, \mathbf{1}\,, \delta^d(x{-}x_1){\otimes}\delta^d(x{-}x_2){\otimes}\ldots{\otimes}\delta^d(x{-}x_n)\,)$$

for correlation function of scalar field theories in perturbation theory using quantum A_∞ algebras.

Our ultimate goal is to provide a framework to prove the AdS/CFT correspondence using open string field theory with source terms for gauge-invariant operators. The quantum treatment of open string field theory must be crucial for this program, and we hope that quantum A_∞ algebras will provide us with powerful tools in this endeavor.

Happy 60th birthday, Ooguri san!

大栗さん、還暦おめでとうございます！