On Symmetries

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Hirosifest, Kavli IPMU, October 2022

Happy birthday, Hirosi!

Strings 2002

Cambridge, July 15-20

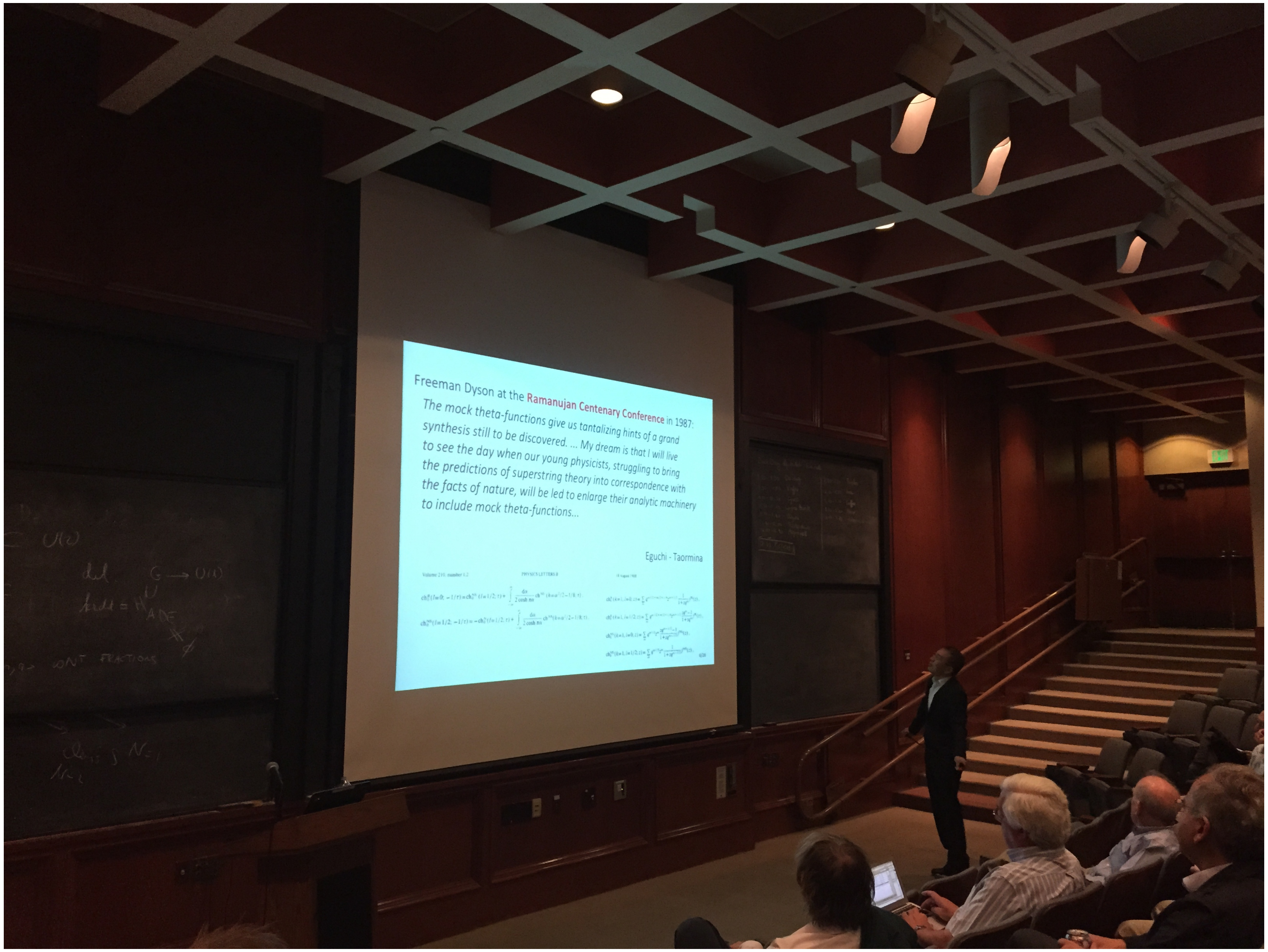
Caltech 2006: True California (Boulevard) View



... and where serious work was done, in a truly inspiring environment:

Pre-fab-era IPMU: First F-theory (annual) conference

... and at Caltech



2006: the solitary scooter

Some serious trendsetting: 2022, worldwide

The real question is: how does he do ALL these things?

We should all aspire to a fraction of what Hirosi has so far achieved in our field, and for our field and beyond! Apart from the obvious research leadership, he's an author, science educator, a movie advisor, etc, etc, and chairs these wonderful institutions:

Symmetries

One outstanding feature of Hirosi's research is its breadth in terms of relevance for different fields: CFT, string theory, quantum gravity, holography, mathematics, etc.

Symmetries are central to a multitude of questions

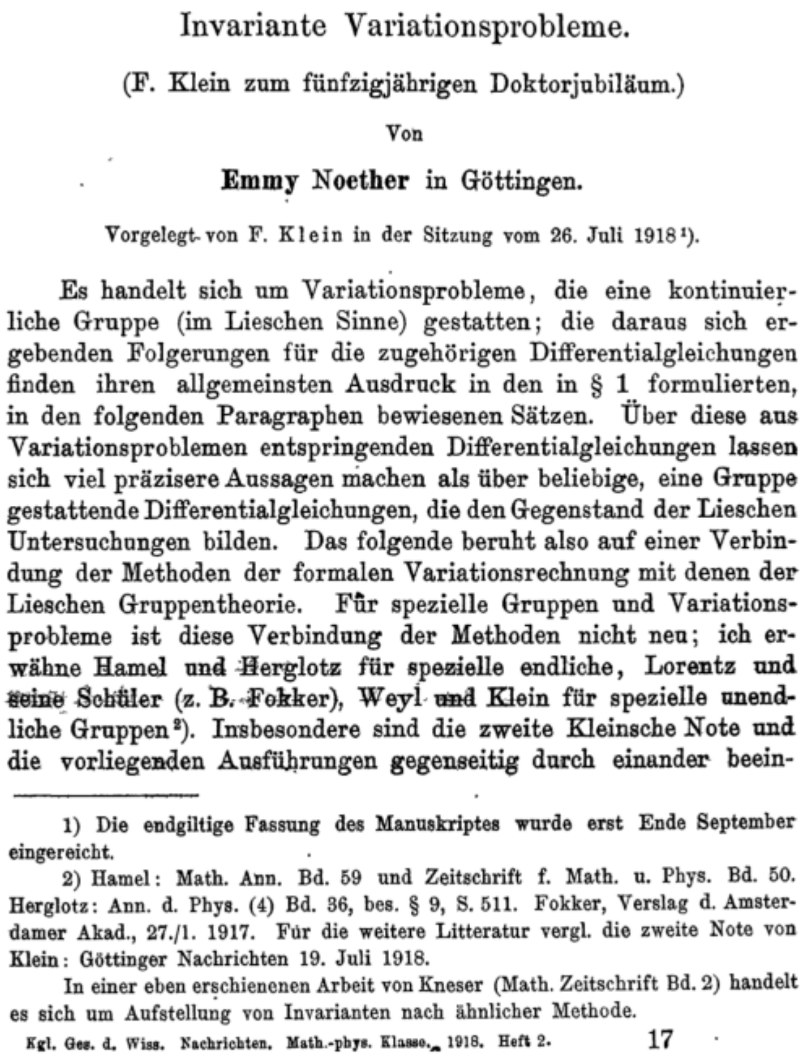
- QFT: $d = 2, 3, 4, \cdots$
- Holography
- Quantum Gravity, string theory, Swampland, Weak Gravity
- Math: topology, category theory

Goal of this talk is to summarize recent progress on constructions of somewhat unexpected symmetries, aspiring to touch upon almost all of the above topics.

Symmetries from Topological Operators

2022: Topological defects correspond to symmetries.

This is a long way from Emmy Noether*'s 1918 continuous "Lieschen" type symmetries, though the core idea is the same:



Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

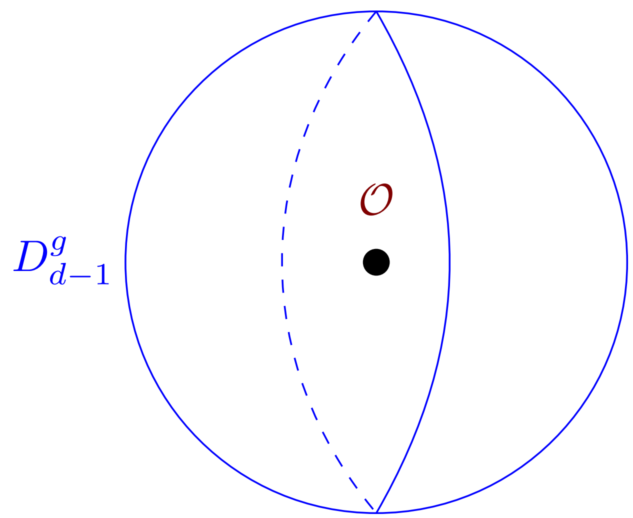
Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17



* Whose 140th birthday was also this year in March.

Higher Form, Higher Group, Higher Cat(egorie)s

Recent explosion of types of symmetries:

1. Higher-form symmetries $\Gamma^{(p)}$:
   p-dimensional charged defects, whose charge is measured by topological operators $D^g_{q=d-(p+1)}$, $g \in \Gamma^{(p)}$ [Gaiotto, Kapustin, Seiberg, Willett, 2014]

   $$D_q^g \otimes D_q^h = D_q^{gh}\,, \qquad g, h \in \Gamma^{(p)}$$

   Background fields are $B_{p+1} \in H^{p+1}(M, \Gamma^{(0)})$.

2. Higher-group symmetries: [Sharpe][Tachikawa][Benini, Cordova, Hsin]...
   Higher-form symmetries might not form product groups, but a type of group extension. E.g. 0-form $\mathcal{F}^{(0)}$ and $\Gamma^{(1)}$ form a 2-group

   $$\delta B_2 = B_1^* \Theta$$

   where $\Theta \in H^2(B\mathcal{F}^{(0)}, \Gamma^{(1)})$, and $B_1 : M_d \to B\mathcal{F}^{(0)}$.

3. Non-invertible symmetries:
relax group law $\Rightarrow$ fusion algebra

$$D_p^i \otimes D_p^j = \bigoplus_k N_k^{ij} D_p^k$$

This is very well developed in 2d and to some extent 3d, but unchartered until recently in $d > 3$.

4. Higher-categorical symmetries:
topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion.

$\Rightarrow$ Formulation in terms of objects and higher-morphisms and mutual consistency conditions.

The main (surprising?) point to remember is:
these are symmetries that occur in vanilla 4d Yang-Mills theories (no susy, no matter).

Non-invertible Symmetries in $d > 3$:

In the context of QFTs in $d > 3$ within the last year

[Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela]

[Koide, Nagoya, Yamaguchi]

[Kaidi, Ohmori, Zheng]

[Choi, Cordova, Hsin, Lam, Shao]

[Roumpedakis, Seifnashri, Shao]

[Bhardwaj, Bottini, SSN, Tiwari]

[Antinucci, Galati, Rizi]

[Choi, Cordova, Hsin, Lam, Shao]

[Kaidi, Zafrir, Zheng]

[Choi, Lam, Shao]

[Cordova, Ohmori]

[Bhardwaj, SSN, Wu]

[Bartsch, Bullimore, Ferrari, Pearson]

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Plan

1. Generalized Gauging and Non-invertible Symmetries
2. Symmetry Categories and their Webs
3. Symmetry TFT and Non-invertible Symmetries in holography/string theory

This work was done together with phantastic collaborators:

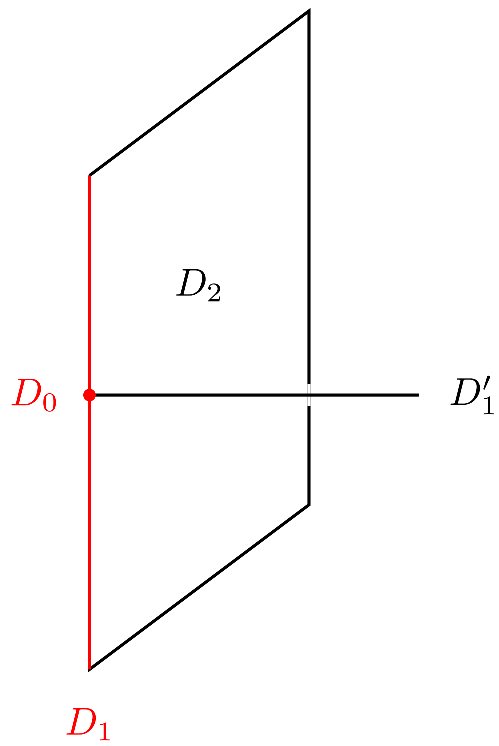
Lakshya Bhwardaj (Oxford), Lea Bottini (Oxford), Apoorv Tiwari (Stockholm)
Lakshya Bhwardaj (Oxford), Jingxiang Wu (Oxford)
Fabio Apruzzi (Bern/Padova), Ibou Bah (JHU), Federico Bonetti (Oxford)

Symmetry Categories

Consider a d-dimensional QFT $\mathfrak{T}$. Then the set of all topological defects

$$D_q^i\,, \qquad q = 0, \cdots, d-1$$

will form a $(d-1)$-category.



- Objects D_{d-1}
- 1-morphisms D_{d-2} between objects
- 2-morphism D_{d-3} between 1-morphisms
- $\cdots$
- $(d-2)$-morphisms: local operators

Topological operators can be **genuine** or non-genuine (ends of other topological operators)

The symmetry category $\mathcal{C}_{\mathfrak{T}}$ encodes the fusion of these topological defects. "Higher fusion category". Higher =2: [Douglas, Reutter].

Non-invertibles from Gauging

2d: gauging of fusion 1-categories [Runkel, Schweigert, Carqueville, ...][Bhardwaj, Tachikawa]

3d: [Barkeshli, Bonderson, M. Cheng, Z. Wang][Teo, Hughes, Fradkin].

$d > 2$: Distinct – and sometimes overlapping – approaches developed in the last year, motivated by QFT:

- [Kaidi, Ohmori, Zheng] Mixed anomalies to non-invertibles
- [Choi, Cordova, Hsin, Lam, Shao] Duality defects
- [Bhardwaj, Bottini, SSN, Tiwari] Gauging outer automorphisms

Examples:

- Spin$(4N)$ Yang-Mills in any dim has a $\mathbb{Z}_2^{(0)}$ outer automorphism, gauging results in Pin$^+(4N)$
- Gauging charge conjugation in Yang-Mills
- S_3-gauging of Spin(8) Yang-Mills $\Rightarrow$ allows non-abelian discrete gauging

Warmup: Gauging in 1-categories

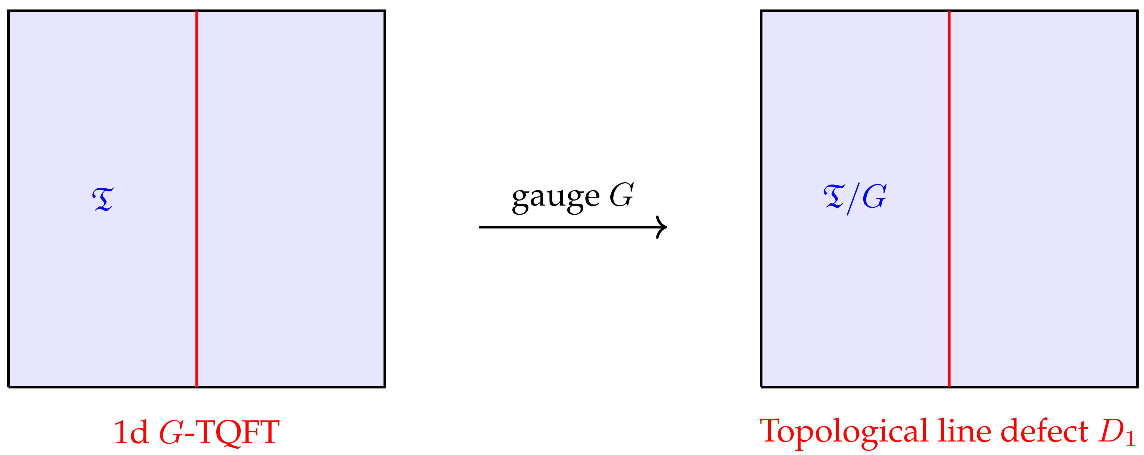
Consider a 3d theory $\mathfrak{T}$, finite 0-form symmetry G generated by D_2^g, $g \in G$. Let $\mathcal{C}$ and $\mathcal{C}_G$ be the cat of topological lines before and after gauging.

Objects of $\mathcal{C}_G$:

1. Gauge-invariants topological lines of $\mathcal{C}$, labeled by orbits O of G:

$$D_1^{(O)} = \bigoplus_{i \in O} D_1^{(i)}$$

2. Topological lines $D_1^{(R)}$, R= irrep of G.
   They form a subcategory $\mathrm{Rep}(G) \Rightarrow$ 1d TQFTs with G symmetry.



3. Mixture of both: G_O be the stabilizer group of object in $D_1^{(O)}$. We can dress $D_1^{(0)}$ by a rep R_O of G_O: $D_1^{(O,R_O)}$

Morphisms and Fusion

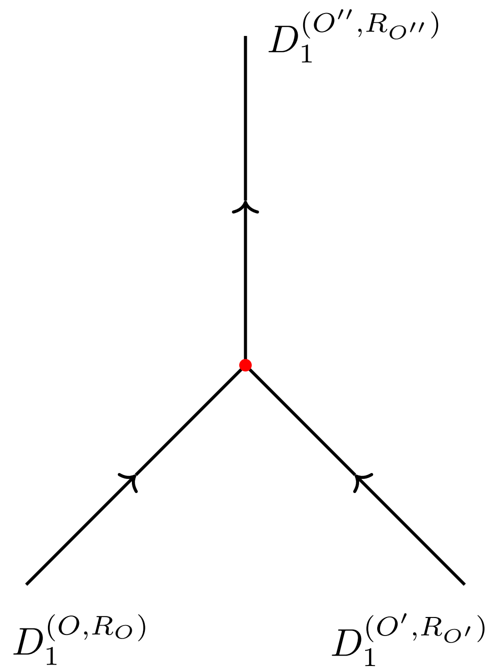
Morphisms in $\mathcal{C}_G$ are topological point-operators:

1. G-invariant combinations of morphisms of $\mathcal{C}$, i.e. operators on G-invariant topological lines
2. Morphisms in the category $\text{Rep}(G)$, i.e. 0d interfaces between 1d TQFTs

Fusion of Lines in $\mathcal{C}_G$:

1. Fusion in $\text{Rep}(G)$:
$$D_1^{(R)} \otimes D_1^{(R')} = D_1^{(RR')}, \qquad R, R' \in \widehat{G}$$
2. Mixed fusion
$$D_1^{(O,R_O)} \otimes D_1^{(S)} = D_1^{(O,R_O S_O)}$$
where S_O is the image of S under $\widehat{G} \to \widehat{G}_O$.

Note: Fusion is determined from the morphism space:



Then

$$D_1^{(O,R_O)} \otimes D_1^{(O',R'_{O'})} \supset \dim\left(V_{(O,R_O),(O',R_{O'})}^{(O'',R_{O''})}\right) \times D_1^{(O'',R''_{O''})}$$

Example: Gauging Outer Automorphisms

3d Spin$(4N)$ Yang-Mills, and the outer automorphism $G^{(0)} = \mathbb{Z}_2^{(0)}$ that exchanges the two factors in

$$\Gamma^{(1)} = \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)}$$

Lets consider the category formed by the topological lines. Objects in this cat are:

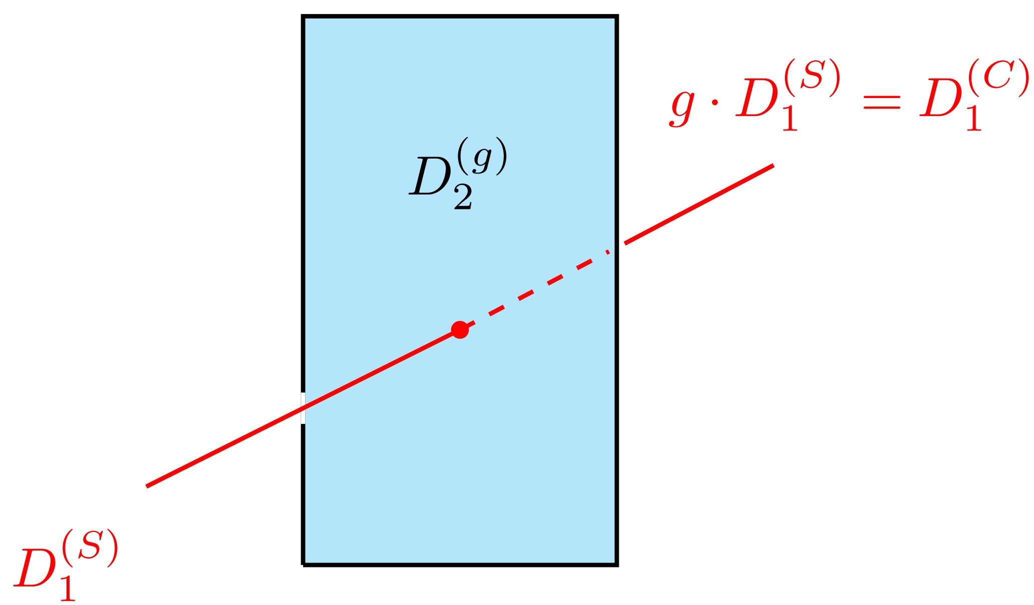
$$\mathcal{C}^{\text{ob}}_{\text{Spin}(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

V is the diagonal of S and C.

$G^{(0)} = \mathbb{Z}_2^{(0)}$ acts as the outer automorphism:

$$D_1^{(S)} \longleftrightarrow D_1^{(C)}, \qquad D_1^{(\text{id})} \text{ and } D_1^{(V)} \text{ are invariant}$$

and gauging results in Pin$^+(4N)$.


$D_2^{(g)}$
$g \cdot D_1^{(S)} = D_1^{(C)}$
$D_1^{(S)}$

Gauged Category

After gauging, the dual to the 0-form symmetry becomes a topological line $D_1^{(-)}$. Objects in $\mathcal{C}_{\mathbb{Z}_2^{(0)}} = \mathcal{C}^{\mathrm{Pin}^+(4N)}$ are one of:

1. Orbits, i.e. invariants:

$$D_1^{(SC)} := \left(D_1^{(S)} \oplus D_1^{(C)}\right)_{\mathcal{C}_{\mathrm{Spin}}}$$

   to be a simple object. Furthermore we have $D_1^{(\mathrm{id})}$ and $D_1^{(V)}$.

2. The topological lines that generate $\mathrm{Rep}(\mathbb{Z}_2)$: $D_1^{(\mathrm{id})}$ and $D_1^{(-)}$.

3. Combination of invariants and $D_1^{(-)}$:

$$D_1^{(V_-)}$$

Thus

$$\mathcal{C}^{\mathrm{ob}}_{\mathrm{Pin}^+(4N)} = \left\{D_1^{(\mathrm{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)}\right\}$$

Fusion in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$

It is useful to compute the fusion in the original $\mathcal{C}_{\mathrm{Spin}(4N)}$:

$$\left(D_1^{(SC)} \otimes D_1^{(V)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} = \left((D_1^{(S)} \oplus D_1^{(C)}) \otimes D_1^{(V)}\right) = \left(D_1^{(C)} \oplus D_1^{(S)}\right) = \left(D_1^{(SC)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}}$$

To see whether $D_1^{(SC)}$ is also present in the gauged category we need to determine the $\mathbb{Z}_2$ transformation properties of the morphisms

$$D_1^{(SC)} \otimes D_1^{(V)} \to D_1^{(SC)}$$

There are two morphisms:

$$\begin{aligned} D_0^{(S\otimes C,V)} &: \qquad D_1^{(S)} \otimes D_1^{(C)} \to D_1^{(V)} \\ D_0^{(C\otimes S,V)} &: \qquad D_1^{(C)} \otimes D_1^{(S)} \to D_1^{(V)} \end{aligned}$$

These are exchanged under $\mathbb{Z}_2^{(0)}$ and thus the morphism space is 2d splitting as $1_+ \oplus 1_-$. Since both $\mathbb{Z}_2$ representations are present:

$$D_1^{(SC)} \otimes D_1^{(V)} = D_1^{(SC)}$$

$$D_1^{(SC)} \otimes D_1^{(V_-)} = D_1^{(SC)}$$

Fusion in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$

$$\left(D_1^{(SC)} \otimes D_1^{(SC)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} = \left(2D_1^{(\mathrm{id})} \oplus 2D_1^{(V)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}}$$

Again determining the morphism spaces and their $\mathbb{Z}_2$ representation decomposition we find

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\mathrm{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

In fact this category is of Tambara-Yamagami type for $\mathbb{Z}_2 \times \mathbb{Z}_2$ and has associators identifying it with

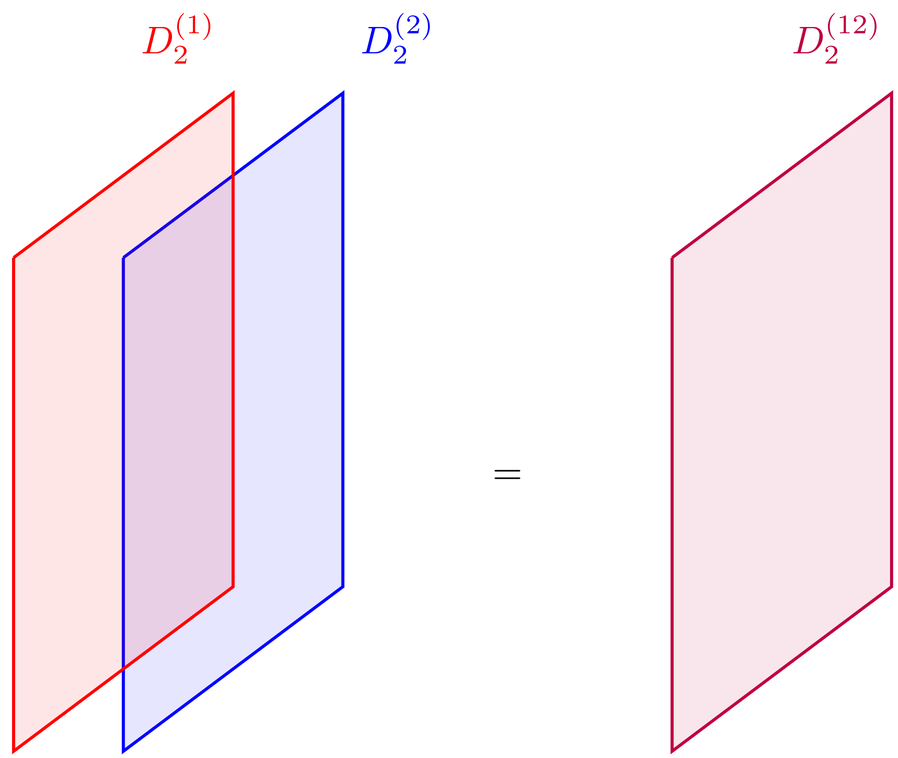
$$\mathcal{C}_{\mathrm{Pin}^+(4N)} = \mathrm{Rep}(D_8)$$

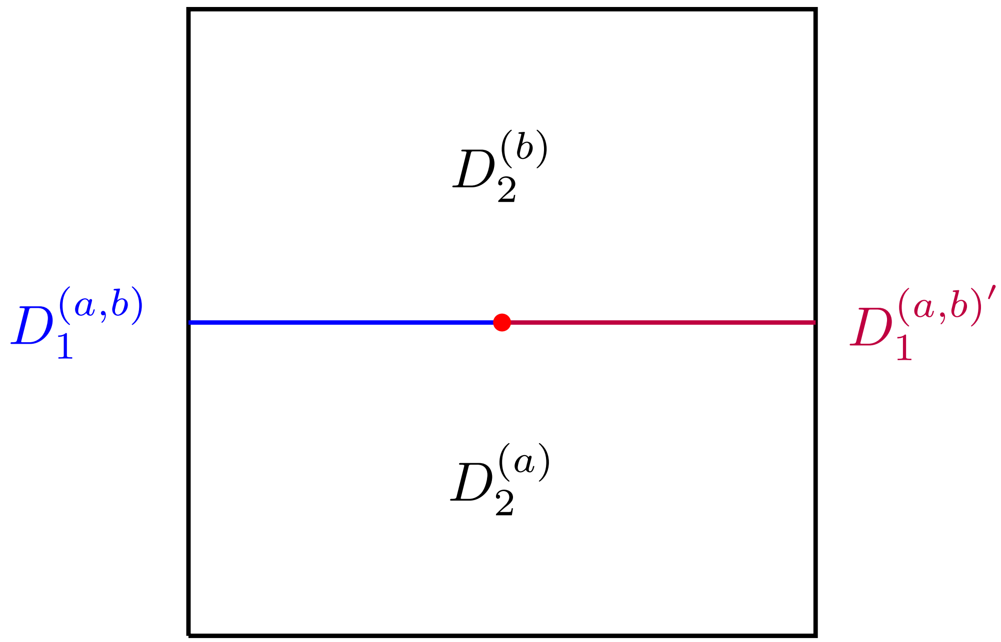
Fusion 2-categories

Objects: D_2 topological surfaces; e.g. $\Gamma^{(p=d-3)}$ form symmetry

1-Morphisms: D_1 topological lines; e.g. $\Gamma^{(p=d-2)}$ form symmetry

2-Morphisms: D_0 topological point operators




$D_2^{(b)}$
$D_1^{(a,b)}$
$D_1^{(a,b)'}$
$D_2^{(a)}$

Gauging 2-categories

Start with a theory with a 2-fusion category symmetry $\mathcal{C}$, with a 0-form symmetry acting on $\mathcal{C}$ [Bhardwaj, Bottini, SSN, Tiwari]

To begin with we start with invertible category:

Objects: D_2 topological surfaces $\Rightarrow$ generates $\Gamma^{(p=d-3)}$ form symmetry

1-Morphisms: D_1 topological lines $\Rightarrow$ generates $\Gamma^{(p=d-2)}$ form symmetry

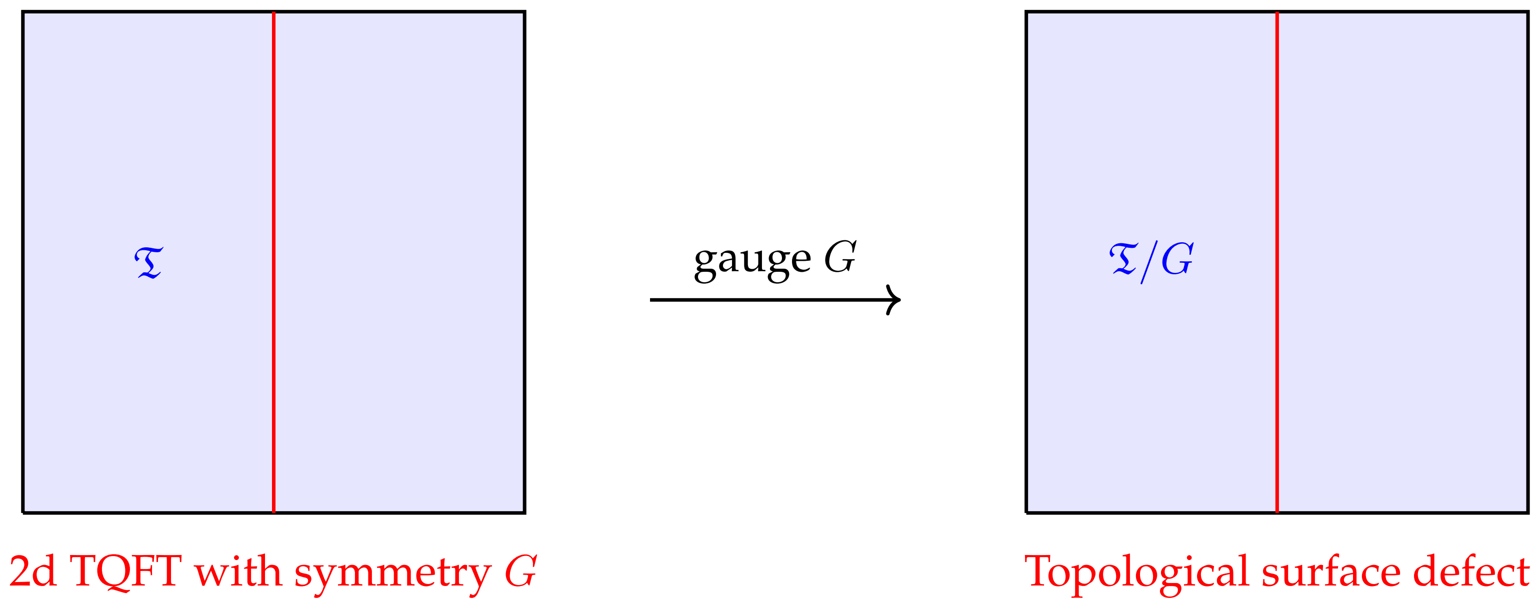
2-Morphisms: D_0 topological point operators

After gauging 0-form symmetry G, we get a new category $\mathcal{C}_G$, whose **objects** are:

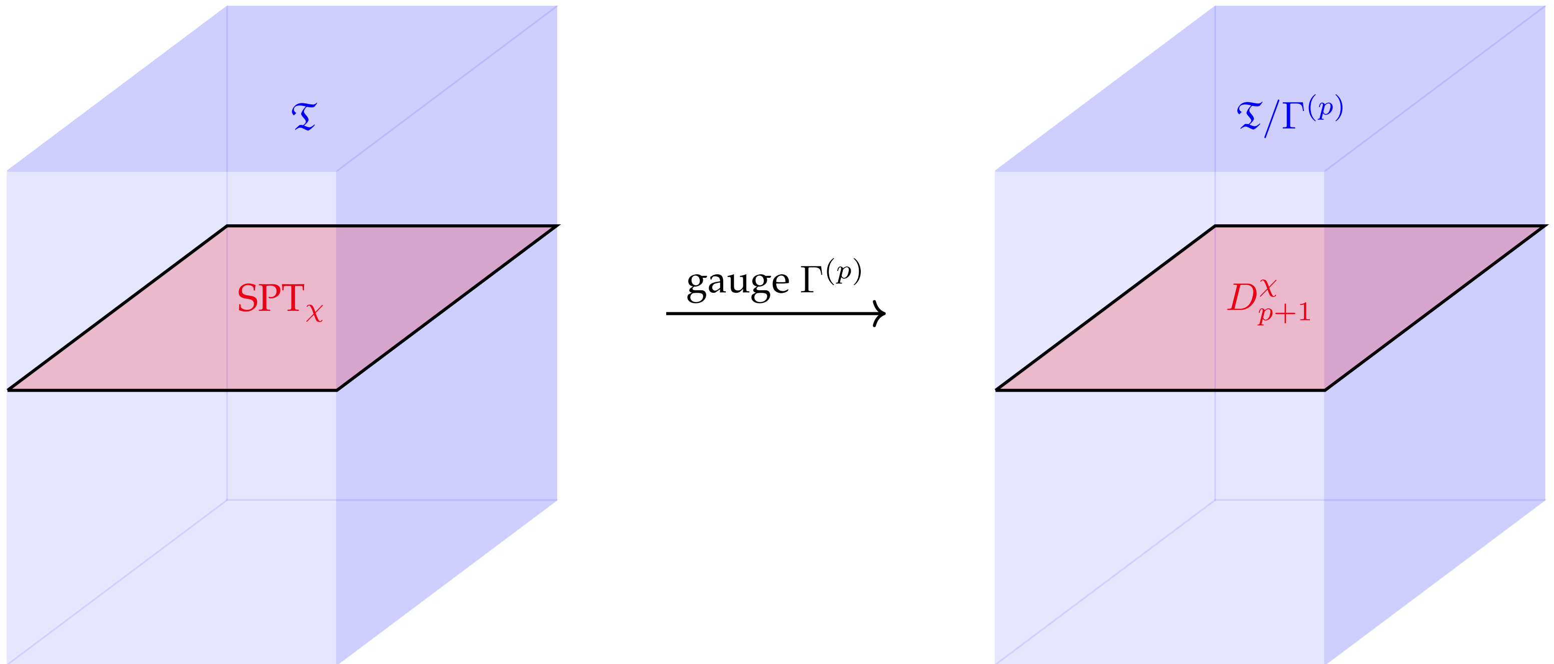
1. G-invariant objects of $\mathcal{C}$, labeled by orbits O.
2. Objects of $2\text{Rep}(G)$ = "module category" of $2\text{Vec}(G)$ (2-representation category)

Generalized gauging

Again the latter have an interpretation in terms of TQFTs: giving rise to a universal sector of gauging [Bhardwaj, SSN, Wu], related work by [Bartsch, Bullimore, Ferrari, Pearson].



Generalized gauging



Gauging a p-form symmetry of a theory $\mathfrak{T}$:
Stack a $(p+1)$-dim TQFT SPT_χ, protected by $\Gamma^{(p)}$, associated to $\Gamma^{(p)}$ character χ. Gauge diagonal $\Gamma^{(p)}$. The SPT becomes a topological defect in the gauged theory, which generates a $(d-p-2)$-form symmetry.

Symmetry 2-Categories

Lets revisit $\mathfrak{T} = $ 3d Spin$(4N)$ gauge theory. This has $\Gamma^{(1)} = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Gamma^{(0)} = \mathbb{Z}_2$. Strictly speaking, because the 0-form symmetry acts on the 1-form symmetry, the theory has a (split) 2-group: $\rho : \Gamma^{(0)} \to \mathrm{Aut}(\Gamma^{(1)})$. The 2-group is

$$\mathbb{G} = (\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}) \rtimes \mathbb{Z}_2^{(0)}$$

In terms of background fields

$$\delta B_2 = A_1 C_2$$

where B_2, C_2 are backgrounds for $\Gamma^{(1)}$, and A_1 for $\mathbb{Z}_2^{(0)}$.

This collection of surfaces and lines form a 2-category:

$$2\mathrm{Vec}\left((\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}) \rtimes \mathbb{Z}_2^{(0)}\right)$$

Contains $2\mathrm{Rep}(\mathbb{Z}_2^2)$ and $2\mathrm{Vec}(\mathbb{Z}_2)$.

2Rep from 2Vec

After gauging $\mathbb{Z}_2^{(0)}$ the symmetry category is:

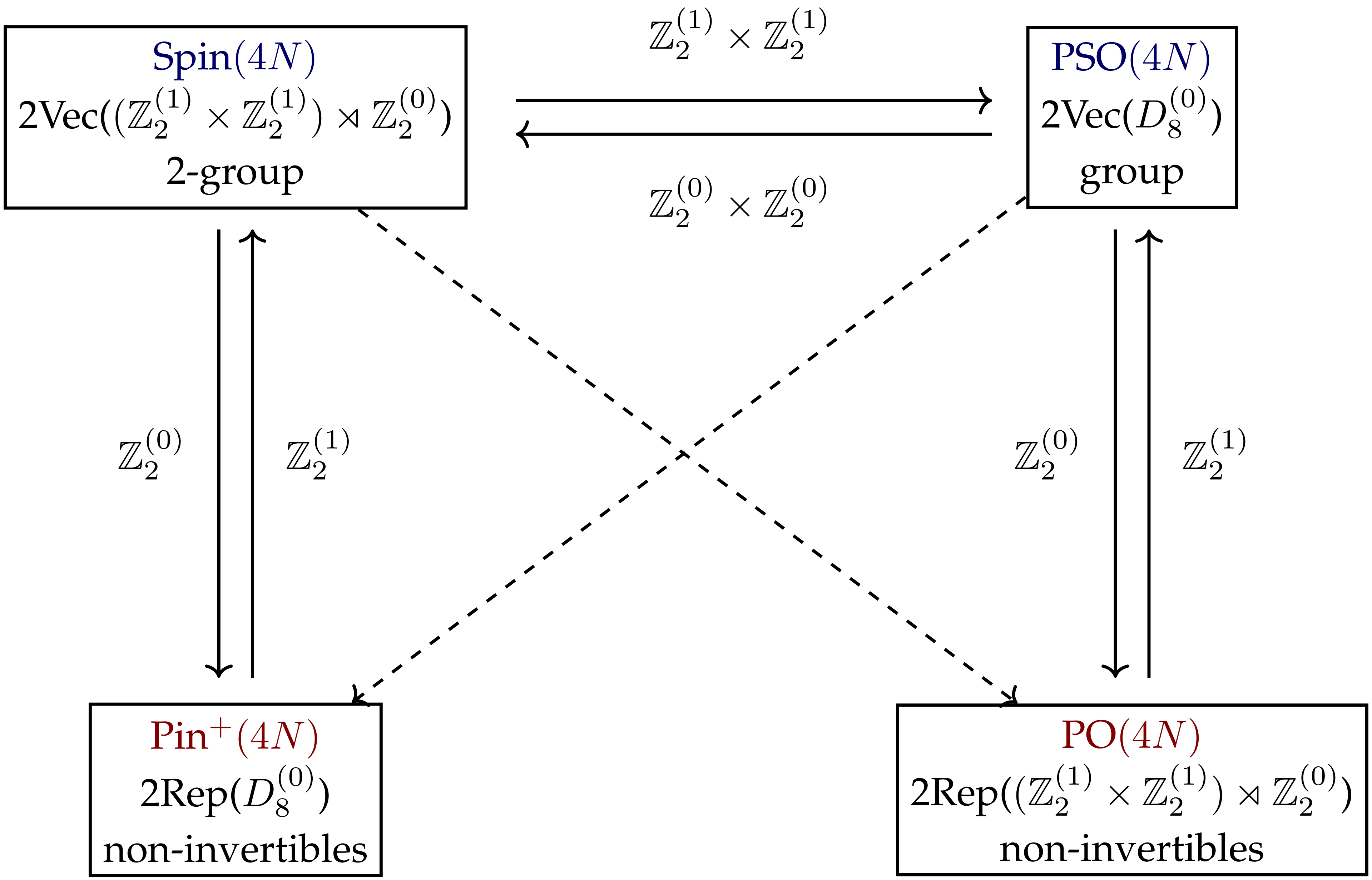
$$\textcolor{darkred}{2\mathrm{Rep}(\widetilde{\Gamma^0})}\,, \qquad \widetilde{\Gamma^{(0)}} = \widehat{\Gamma^{(1)}} \rtimes_{\widehat{\rho}} \Gamma^{(0)}$$

Construction of $\mathfrak{T}/\Gamma^{(0)}$

1. First gauge $\Gamma^{(1)}$: this results in a theory with $\widetilde{\Gamma^{(0)}}$ 0-form symmetry: 2-categorical symmetry is $2\mathrm{Vec}(\widetilde{\Gamma^{(0)}})$
2. Gauge $\widetilde{\Gamma^{(0)}}$: 2-categorical symmetry is $2\mathrm{Rep}(\widetilde{\Gamma^{(0)}})$

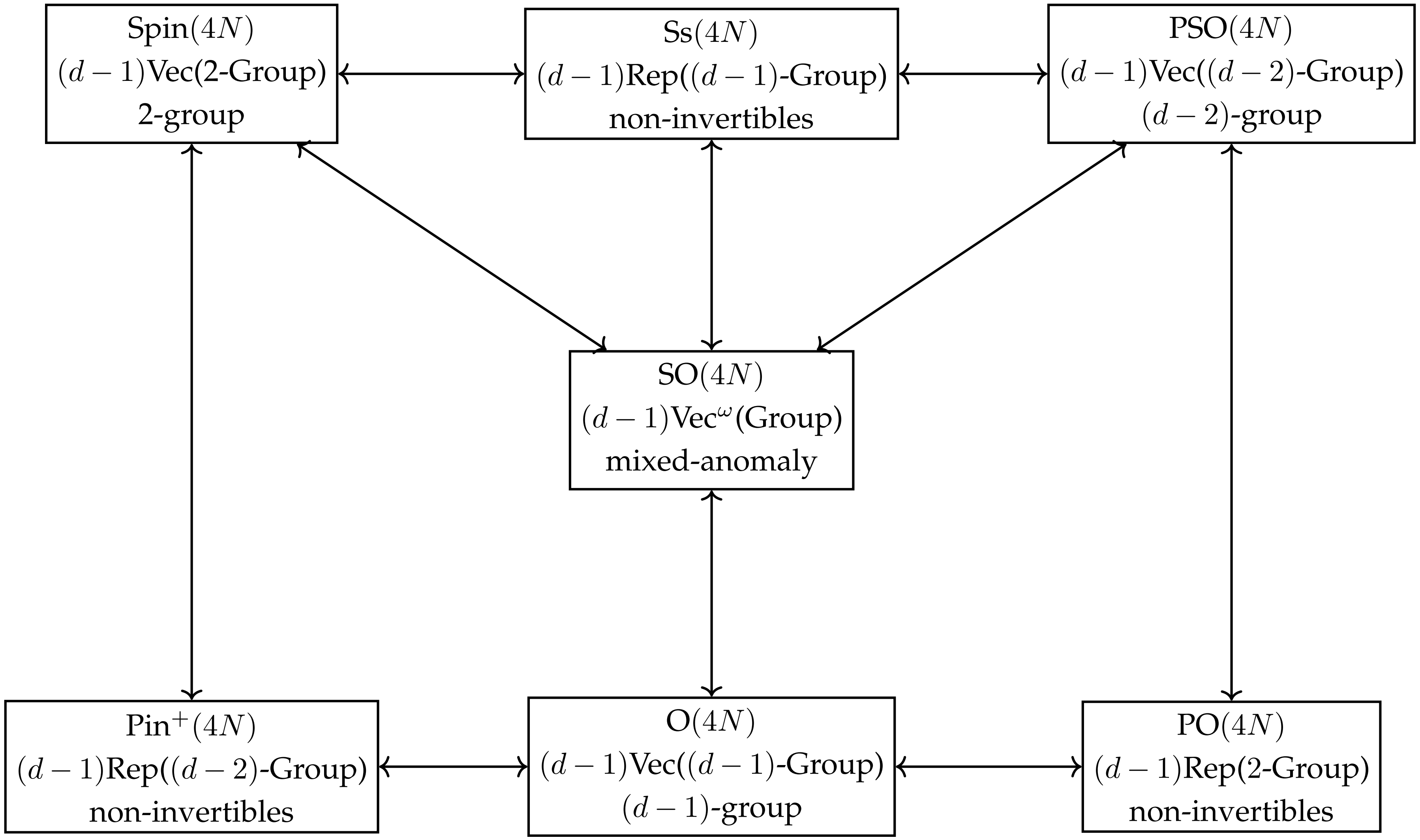
In the example: $\widetilde{\Gamma^{(0)}} = (\mathbb{Z}_{\times}\mathbb{Z}_2) \rtimes \mathbb{Z}_2 = D_8$.

Categorical Symmetry Webs: 3d $\mathfrak{so}(4N)$



d-dim Categorical Symmetry Web

[Bhardwaj, Bottini, SSN, Tiwari, wip]



Drinfeld Center and SymTFT

n-categorical webs characterize all symmetries of different ”generalized global forms” of the gauge theory. Mathematically manifested in the statement that the n-categories in a given web share the same Drinfeld center.

Well studied case in 2d: Turaev-Viro $\mathrm{TV}_{\mathcal{C}}$ 3d TQFTs characterize the set of lines of the Drinfeld center of 2d theories with fusion symmetry $\mathcal{C}$. Long history in cond-mat, math. See recent work by [Kaidi, Ohmori, Zheng] for $\mathcal{C} = TY(\mathbb{Z}_N)$.

In $d > 2$: generate the categorical symmetry webs through different b.c. on the Drinfeld center. [Bhardwaj, SSN; wip].

We don’t need to go all categorical: this is in fact a very familiar concept to QFT’ers, string theorists, holographers, and nowadays goes under the name of

$\Rightarrow$ Symmetry TFT

Anomaly Theories

$\mathfrak{T}$ be a d-dimensional QFT, with global symmetries. 't Hooft anomalies are detected by coupling the theory to background fields B_{p+1} for a p-form symmetry, and perform background gauge transformations.

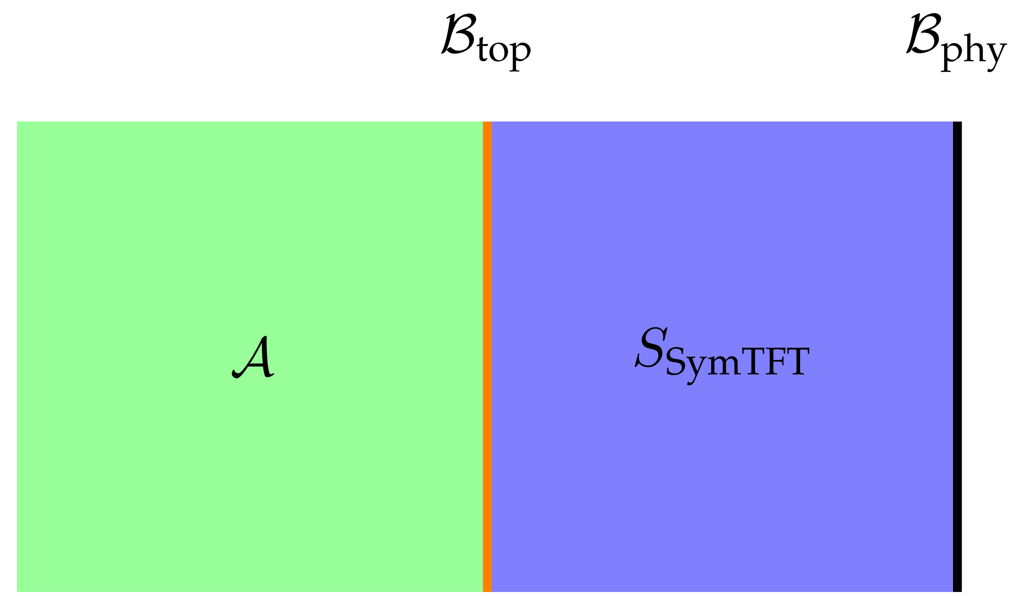
$$Z(B_{p+1} + \delta\lambda_p) = \varphi(\lambda_p, B_{p+1}) \times Z(B_{p+1}) \,,$$

where $\varphi(\lambda_p, B_{p+1}) \in U(1)$ is the partition function of the anomaly theory $\mathcal{A}_{d+1}(B_{p+1})$, which is an SPT protected by p-form symmetry.

Symmetry TFT or "The Sandwich"

[Freed, Teleman][Gaiotto, Kulp][Apruzzi, Bonetti, Garcia-Extebarria, Hosseini, SSN][Freed, Moore, Teleman][Kaidi, Ohmori, Zheng]

The Symmetry TFT or SymTFT S_{SymTFT} is a $(d+1)$-dimensional topological field theory, which admits gapped boundary conditions:



$\mathcal{A}$ is the anomaly theory, and the theory $\mathfrak{T}$ is obtained after collapsing the interval. Different choices of b.c. result in different "global forms" of the theory.

Example: BF-Theories as Symmetry TFTs

Consider a 5d BF-theory

$$N \int_{M_5} B_2 \wedge dC_2$$

This could e.g. be the SymTFT for 4d pure YM with gauge algebra $\mathfrak{su}(N)$, as discussed in the context of holography by [Witten '98].

The gapped, topological b.c. are given in terms of B_2 and C_2. E.g. Dirichlet b.c. for B_2. This results in line operators (e.g. Wilson or 't Hooft) in the dual CFT [Gross, Ooguri].

B.c. are picking out maximal commuting set of (parital) Wilson surfaces $W_n(B_2, M_2)$, $W_m(C_2, M_2')$, which satisfy the flux non-commutativity in 5d [Maldacena, Moore, Seiberg][Witten]

$$W_n(B_2, M_2) W_m(C_2, M_2') = W_m(C_2, M_2') W_n(B_2, M_2) e^{2\pi i nm M_2 \cdot M_2'}$$

SymTFT from String Theory/Holography

Two conjectures/observations:

1. Geometric Engineering:
   Consider M-/string theory compactified on a non-compact, special holonomy space X to d-dim QFT $\mathfrak{T}_X$. The reduction of the topological couplings in 11d/10d on ∂X result in the SymTFT of $\mathfrak{T}_X$

2. Holography:
   The bulk topological couplings in the supergravity on AdS_{d+1} or M_{d+1} (e.g. for Klebanov Strassler) give rise to the SymTFT of the holographically dual boundary theory.

By now there is a huge amount of evidence for this:
including the construction of non-invertible symmetries in holography
[Apruzzi, Bah, Bonetti, SSN], [Gracia-Extebarria][Antinucci, Benini, Copetti, Galati, Rizzi]

Holographic dual to 4d $\mathcal{N}=1$ SYM and Symmetries from Branes

[Apruzzi, Bah, Bonetti, SSN]

- D3s at the conifold $C(T^{1,1})$ are dual to IIB on $\mathrm{AdS}_5 \times T^{1,1}$, $\int F_5 = N$.
- $T^{1,1} \sim S^3 \times S^2$: wrap D5-branes on S^2, inducing $\int_{S^3} F_3 = M$
  $\Rightarrow$ breaks conformal invariance

[Klebanov-Strassler] (KS) solution: Dual to a cascade of Seiberg dualities, which end in pure $\mathfrak{su}(M)$ $\mathcal{N}=1$ SYM:

$$ds^2 = \underbrace{\frac{r^2}{R^2} d\mathbf{x}^2 + \frac{R^2}{r^2} dr^2}_{M_5} + R^2 ds^2_{T^{1,1}} \,.$$

r= radial direction, RG-flow; $R(r) \sim \ln(\frac{r}{r_s})^{1/4}$, $r_s = r_0 e^{-N/gM^2 - 1/4}$.

Near horizon limit: $r \to r_0$. Global form of gauge group is not fixed by this data alone.

QFT Interlude: Non-Invertibles in 4d SYM

$SU(M)$ pure SYM has M confining vacua, and invertible symmetries.
$\Rightarrow$ What about $PSU(M)$?

4d $\mathcal{N}=1$ $SU(M)$ SYM has $\Gamma^{(0)}=\mathbb{Z}_{2M}$, whose background field is A_1, and 1-form symmetry $\Gamma^{(1)}=\mathbb{Z}_M$ with background field B_2 with a mixed anomaly:

$$\mathcal{A}=-2\pi\frac{1}{M}\int A_1\cup\frac{\mathfrak{P}(B_2)}{2}\,,$$

Gauge $\Gamma^{(1)}$ to get to $PSU(M)$. However this is precisely a KOZ [Kaidi, Ohmori, Zheng]-applicable setup:

$\Gamma^{(0)}$-generator transforms in presence of background fields for $\Gamma^{(1)}$:

$$D_3^g(M_3)\to D_3^g(M_3)\exp\left(\int_{M_4}-\frac{2\pi i}{M}\frac{\mathfrak{P}(B_2)}{2}\right)$$

for $\partial M_4=M_3$.

Non-Invertible Symmetries in 4d SYM

Gauging $\mathbb{Z}_M^{(1)}$ to $PSU(M)$ is consistent after dressing D_3^g with a (e.g. minimal) TQFT that which has 1-form symmetry $\mathbb{Z}_M$ and cancels the anomaly.

For $\mathbb{Z}_M$ the minimal (spin) TQFT is $\mathcal{A}^{M,1} = U(1)_M$ [Hsin, Lam, Seiberg]. The dressed defect is

$$\mathcal{N}_3^{(1)} = D_3^{(1)} \otimes \mathcal{A}^{M,1}$$

For M odd the TQFTs obey $\mathcal{A}^{M,1} \otimes \mathcal{A}^{M,1} = \mathcal{A}^{M,2} \otimes \mathcal{A}^{M,2}$. Results in non-invertibles in the $PSU(M)$ theory

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = \mathcal{A}^{M,2} \mathcal{N}_3^{(2)}$$

Defining the conjugate $\mathcal{N}_3^{(1)\dagger} = D_3^{-1} \otimes \mathcal{A}^{M,-1}$ results in

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)\dagger} = \sum_{M_2 \in H_2(M_3, \mathbb{Z}_M)} \frac{(-1)^{Q(M_2)} D_2(M_2)}{|H^0(M_3, \mathbb{Z}_M)|}$$

which is the condensation defect of the 1-form symmetry on M_3 with $D_2(M_2) = e^{i2\pi \int_{M_2} b_2/M}$, where b_2 is the gauge field for the 1-form symmetry. The RHS is a condensation defect of the 1-form symmetry. [Gaiotto, Johnson-Freyd][Choi, Cordova, Hsin, Lam Shao][Rumpedakis, Seifnashri, Shao]

SymTFT from Sugra

The full 5d topological action for this background is, in the near horizon limit:
[Cassani, Faedo][Apruzzi, van Beest, Gould, SSN][Apruzzi, Bah, Bonetti, SSN]

$$S_{\text{SymTFT}} = 2\pi \int_{M_5} \left(\frac{1}{2} N(b_2 dc_2 - c_2 db_2) + M(A_1 dc_3 + c_3 dA_1) + N b_2 f_3^b + A_1 (g_2^b)^2 \right)$$

$\Rightarrow$ structure of BF-couplings for 1-form fields and 0-/2-form symmetries, as well as mixed anomalies.

A_1 R-symmetry background
b_2, c_2 come from H_3 and F_3
f_1^b, g_2^b, integral lifts of classes in $H^1(M, \mathbb{Z}_{2M})$ and $H^2(M, \mathbb{Z}_M)$: gauge fields for 0-form and 1-form symmetries.

Symmetries can be extract using the Gauss law constraints, generalizing [Belov, Moore]. This reconstructs the expected invertible ($SU(M)$) and non-invertible ($PSU(M)$) symmetries generated by:

$$\mathcal{N}_3^{(1)}(M_3) = \int Da e^{2\pi i \int_{M_3} \left(c_3 + \frac{1}{2} M a da + a g_2^b \right)}$$

Alternative: Symmetries from Branes

[Apruzzi, Bah, Bonetti, SSN]

Observation: in the near horizon limit, branes inserted in a holographic setup furnish symmetry generators, with the topological couplings remaining.
Close to the boundary $r \to \infty$:
$T_{\mathrm{Dp}} \sim r^p$ $(p > 0)$, such that the DBI part of the action decouples.

In the KS setup:
D5-branes on $S^3 \times M_3 \subset T^{1,1} \times M_4$ give rise to

$$S_{\mathrm{D5}} = 2\pi \int_{M_3} \left(c_3 + \frac{M}{2} a da + a db_1 \right)$$

Origin of fields: c_3 (from C_6 on S^3), b_1 (from C_4 on S^3, and $db_1 = g_2^b$), $U(1)$ gauge field a on the brane.

What happens when we fuse two branes?

Naively: $U(2)$ non-abelian gauge theory on the world-volume. However, in the presence of B_2 flux in the transverse S^2 gives rise to the Myers effect: D5s puff up to a D7

$$2 \times \mathrm{D5} + B_2 \quad \to \quad \text{D7 with flux} \int_{S^2} f_2 = 2$$

The theory on the 7-brane on $S^2 \times S^3 \times M_3$ is

$$S_{\mathrm{D7}} = 2\pi \int_{M_3} (2c_3 + M ada + 2adb_1)$$

This precisely reproduces the non-invertible fusion in the field theory

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = A^{M,2} \mathcal{N}_2^{(2)}$$

The $\mathcal{N}_3^{(1)} \otimes (\mathcal{N}_3^{(1)})^\dagger =$ condensation defect for the 1-form symmetry on M_3, *has to* come from D5-$\overline{\text{D5}}$ via tachyon condensation [Sen] resulting in the condensation defect!

The holographic interpretation of branes as symmetry generators has been extended to geometric engineering [Heckman, Hubner, Thorres, Zheng], realizing symmetries e.g. in F-theory geometric engineering.

Disorder Operators from Hanany-Witten transition

[Apruzzi, Bah, Bonetti, SSN]

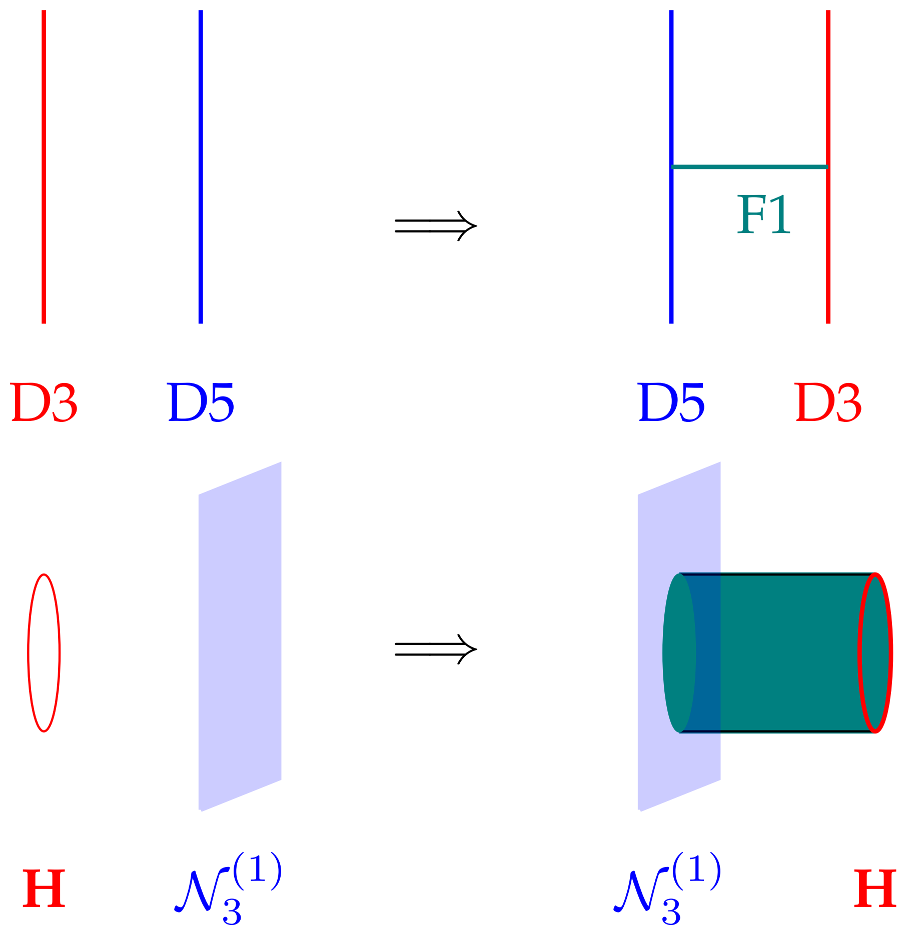
The cherry on top is the implementation of the action of the non-invertible symmetry on the ’t Hooft lines:

- Charged line operators:
  D3s stretching along the radial direction and wrapped on $S^2 \times S^1$ give rise to ’t Hooft lines.
- Topological defects:
  D5s on $S^3 \times M_3$ generate the non-invertible codim 1 topological defects.

Brane	x_0	x_1	x_2	x_3	r	z_1	z_2	w_1	w_2	w_3
D3	X				X	X	X			
D5	X	X	X					X	X	X
F1	X			X						

D3 and D5 link and undergo a Hanany-Witten transition when the D3 is passed through along x_3:

Preserving the linkning requires the creation of an F1:


F1
D3
D5
D5
D3
H
$\mathcal{N}_3^{(1)}$
$\mathcal{N}_3^{(1)}$
H

Outlook

Generalized symmetries – higher form, higher group and non-invertible symmetries – are ubiquitous in QFTs

1. Learn to gauge higher-categorical symmetries; 't Hooft anomalies
2. Physical implications of these symmetries (confinement, pion decay etc)
3. Develop a mathematically sound framework for higher fusion categories (higher meaning ≥ 2)
4. Is this the most general "symmetry structure" for QFTs?
5. Implications for quantum gravity, weak gravity conjecture, no-global symmetry conjectures
6. Constraints on asymptotic growth of state from non-invertibles in higher dim [Harlow, Ooguri][Lin, Okada, Seifnashri, Tachikawa]

Thank you and Happy Birthday, Hirosi!

And many happy returns!