Categorification of Chern-Simons link invariants from mirror symmetry

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Based on 2207.14104 , 2105.06039 , 2004.14518

and works to appear with Miroslav Rapcak and Elise LePage,

and with Ivan Danilenko, Yixuan Li, Michael McBreen, Vivek Shende and Peng Zhou

For very many decades,

Hirosi has played a central role in the field.

His discoveries have inspired my research many times.

This talk builds on at least two of of them.

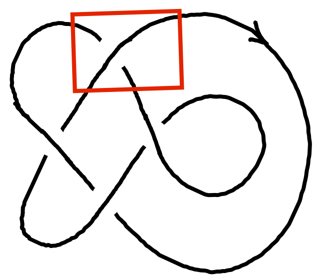
In ’84, Vaughan Jones discovered a polynomial link invariant

$$J_K(q)$$

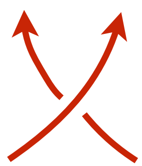
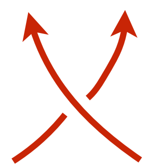
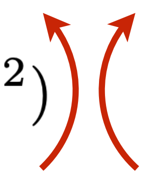
depending on one variable.

The Jones polynomial

is defined by picking a planar projection of the knot

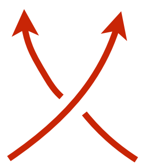
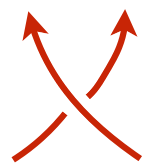
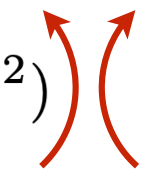


and the "Skein" relation it satisfies as one unties the knot,

$\mathfrak{q}^{n/2}$  $-$ $\mathfrak{q}^{-n/2}$  $= (\mathfrak{q}^{1/2} - \mathfrak{q}^{-1/2})$ 

where one takes $n = 2$.

It turned out other values of n

$\mathfrak{q}^{n/2}$  $-$ $\mathfrak{q}^{-n/2}$  $= (\mathfrak{q}^{1/2} - \mathfrak{q}^{-1/2})$ 

also lead to link invariants.

Taking $n = 0$ one gets the Alexander polynomial,

which was the first know polynomial knot invariant,

dating to 1928.

The proper framework for these

knot invariants was found by Witten in ’89,

who showed that they

originate from a topological quantum field theory,

Chern-Simons theory.

Chern-Simons theory

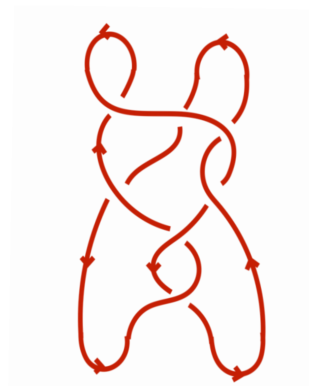
is a rare example of an interacting quantum field theory

which one can explicitly solve.

Witten showed that the Jones polynomial comes from

Chern-Simons theory based on Lie algebra

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$



with knots colored by its spin $\frac{1}{2}$ representation.

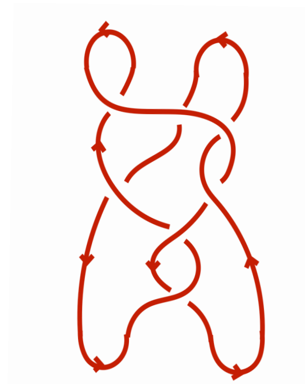
The Alexander polynomial comes from the same setting,

by taking ${}^L\mathfrak{g}$ to be the

Lie superalgebra:

$${}^L\mathfrak{g} = \mathfrak{gl}_{1|1}$$

This placed these link invariants into a more general framework



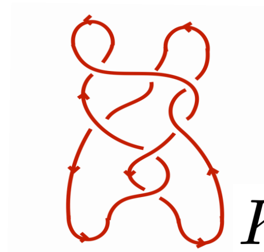
which one gets by

considering Chern-Simons theory based on

different Lie algebras $^L\mathfrak{g}$ and varying the representations

V_i coloring the knots.

In `98 Khovanov showed one can associate to a link



K

a collection of vector spaces

$$\mathcal{H}_K^{*,*} = \bigoplus_{i,j \in \mathbb{Z}} \mathcal{H}_K^{i,j}$$

whose equivariant Euler characteristic is the Jones polynomial

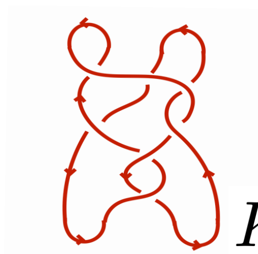
$$J_K(\mathfrak{q}) = \sum_{i,j \in \mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i,j}$$

Khovanov's homology groups

$$\mathcal{H}^{i,j}(K),$$

graded by fermion number i and an "equivariant grading" j

are themselves link invariants,



K

independent of the link projection he used to define them.

Khovanov's construction is part of the

categorification program

pioneered by Crane and Frenkel.

A simple toy model of categorification comes from a Riemannian manifold M whose Euler characteristic

$$\chi(M) = \sum_{k \in \mathbb{Z}} (-1)^k \dim_{\mathbb{Z}} \mathcal{H}^k(M)$$

is categorified by the cohomology groups

$$\mathcal{H}^k(M) = \ker\, d_k / \mathrm{im}\, d_{k-1}$$

of the de Rham complex

$$C^* = \ldots C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} \ldots .$$

From physics perspective,

the Euler characteristic is the partition function

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H}$$

of supersymmetric quantum mechanics with M as a target space.

A collection of vector spaces

$$C^\bullet = \dots C_n \xrightarrow{\partial_n} C_{n+1} \xrightarrow{\partial_{n+1}} \dots$$

is provided by

Morse theory approach to supersymmetric quantum mechanics,

as perturbative supersymmetric ground states,

indexed by the fermion number $F = n$.

The action of the supercharge Q

$$Q = \sum_n \partial_n$$

on the complex

$$C^\bullet = \dots C_n \xrightarrow{\partial_n} C_{n+1} \xrightarrow{\partial_{n+1}} \dots$$

is generated by instantons.

Q defines a differential as it squares to zero,

$$Q^2 = 0$$

Khovanov's categorification of the Jones polynomial

it is explicit and easily calculable.

In 2013 Webster showed that

abstractly,

there exists an algebraic framework

for categorification of

Chern-Simons link invariants of arbitrary simple Lie algebra

$^L\mathfrak{g}$

based on KLRW algebras,

generalizing algebras of Khovanov and Lauda, and of Rouquier.

Unlike Khovanov's theory, Webster's

is a formal construction.

Despite the successes of the knot categorification program

one is missing a fundamental principle that

explains why is categorification possible:

the construction has no right to exist.

Unlike in

our toy example of categorification of the Euler characteristic

of a Riemannian manifold

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H}$$

Khovanov's construction and its generalizations

do not come from either geometry,

or physics in any unified way.

The problem Khovanov initiated is to find

a general framework for construction of link homology groups,

that works uniformly for all Lie algebras,

which explains what link homology groups are,

and why they exist.

In fact, there is a single setting where mathematicians

did find a theory that comes from geometry and which explains

what link homology groups are.

The theory

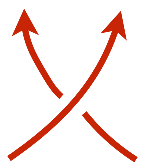
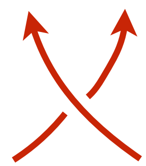
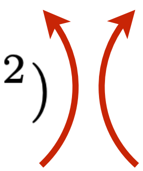
known as Heegard-Floer theory

corresponds to

$$ {}^{L}\mathfrak{g} = \mathfrak{gl}_{1|1} $$

and categorifies the Alexander polynomial.

Despite the fact the Jones and the Alexander polynomial

$\mathfrak{q}^{n/2}$  $-$ $\mathfrak{q}^{-n/2}$  $=$ $(\mathfrak{q}^{1/2} - \mathfrak{q}^{-1/2})$ 

are defined by nearly the same Skein relation,

just changing $n = 2$ to $n = 0$

the theories categorifying them are nothing alike:

one comes from algebra, the other from an A-model.

The solution to the knot categorification problem comes from

mirror symmetry.

Mirror symmetry is a duality discovered by string theorists.

Today, it is a vast subject in mathematics,

connecting two large areas of it,

known as algebraic and symplectic geometry.

Homological mirror symmetry

which describes how mirror symmetry acts on branes,

naturally gives rise to hosts of homological invariants of geometric origin.

Most of the time though, the invariants it gives rise to are not of any particular interest outside of the problem at hand,

e.g. if one studies the quintic.

It turns out that there is a vast new family,

of mirror pairs of manifolds,

where homological mirror symmetry does lead to interesting new invariants.

It solves the knot categorification problem.

Many special features exist in this family,

one of which is the fact that it will involve

hyper-Kahler manifolds.

This will translate to the fact that

problems whose solutions typically only exist formally,

will now have explicit solutions.

One of the sides of mirror symmetry will involve a

theory generalizing Heegard-Floer theory,

from

$$^{L}\mathfrak{g} = \mathfrak{gl}_{1|1}$$

to

an arbitrary Lie (super)algebra.

The story I will describe in this talk

assumes that $^L\mathfrak{g}$ is a simply-laced Lie algebra.

It has a generalization for

$$^L\mathfrak{g}$$

which are non-simply laced Lie algebras and Lie superalgebras,

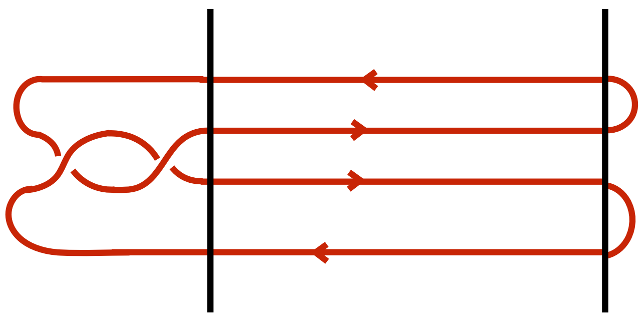
which largely follows the same lines,

but differs in some technical aspects.

Two-dimensional physics enters here because the descriptions we will get

come from two-dimensional theories associated to

$\text{link } K \times \text{time}$ in $\mathbb{R}^3 \times \text{time}$



More precisely, as it comes out of string theory,

the theory is naturally equipped to describe arbitrary links in

$\mathbb{R}^2 \times S^1 \times \text{time}$ and not just $\mathbb{R}^3 \times \text{time}$

The 2d theories are the ones living on

defects in the six dimensional (0,2) conformal field theory

as anticipated from the works of Ooguri and Vafa in ’99,

and Gukov, Schwarz and Vafa in ’04.

Superstring theory will not feature further in this talk,

since the final solution

to the knot categorification problem,

which we could have posed without knowing of it,

may be also phrased without knowing superstring theory,

although it charted the path for us, with no ambiguity along the way.

In the same '89 paper Witten showed that

underlying Chern-Simons theory is a

two-dimensional conformal field theory associated to

${}^L\mathfrak{g}$ and κ

the WZW model

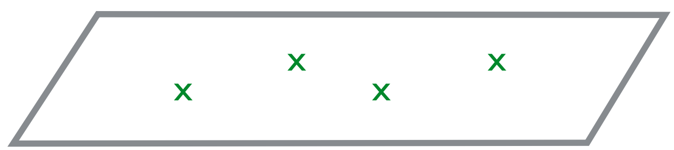
$$\widehat{{}^L\mathfrak{g}}_\kappa$$

with affine Lie algebra symmetry

We will take this as the starting point.

The Hilbert space of Chern-Simons theory

on a Riemann surface punctured by Wilson lines

$\mathcal{A}$ 

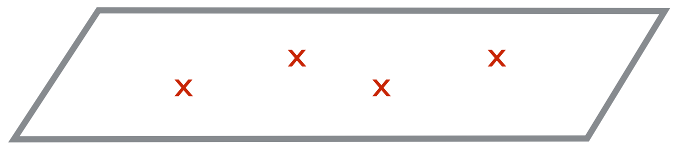
is the space of conformal blocks of

$$\widehat{{}^L\mathfrak{g}_\kappa}$$

Every conformal block of

$$\widehat{L\mathfrak{g}}_{\kappa}$$

$\mathcal{A}$



and hence every state in the Hilbert space,

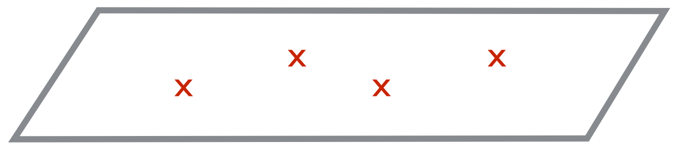
arizes as a solution to a very famous

$$\kappa\,\partial_i \mathcal{V}_\alpha - (r_i)_\alpha^{\,\beta}\, \mathcal{V}_\beta = 0$$

linear differential equation.

The equation solved by conformal blocks of

$$\widehat{L\mathfrak{g}}_\kappa$$

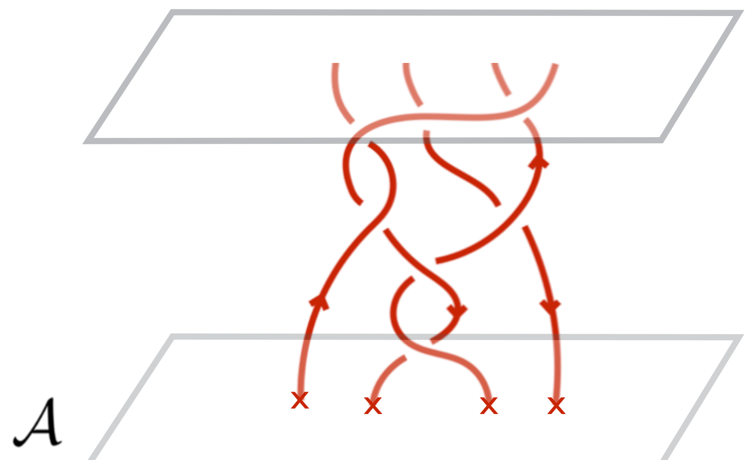
$\mathcal{A}$ 

is the one discovered by Knizhnik and Zamolodchikov in ’84:

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell} \mathcal{V} = \sum_{j \neq \ell} r_{\ell i}(a_\ell / a_j)\, \mathcal{V}.$$

The variables in the equation are the positions of punctures on $\mathcal{A}$.

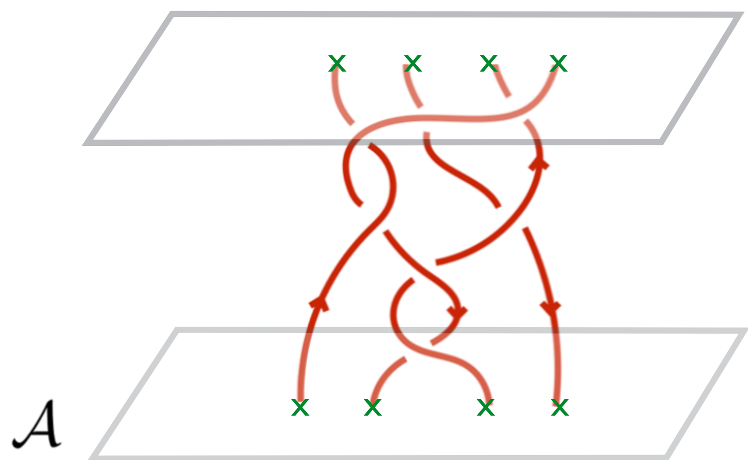
By varying positions of punctures in Chern-Simons time:



we get a colored braid in

$$\mathcal{A} \times [0,1]$$

We also get a monodromy problem,



which is to describe the analytic continuation,

of the fundamental solution of KZ equation

along a path in the parameter space corresponding to the braid.

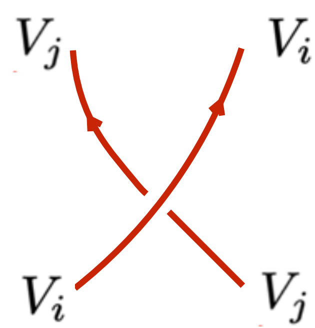
The problem was solved by Drinfeld and by Kazhdan and Lusztig in '89,

who showed that monodromy matrix is a product of

"R-matrices" of the quantum group

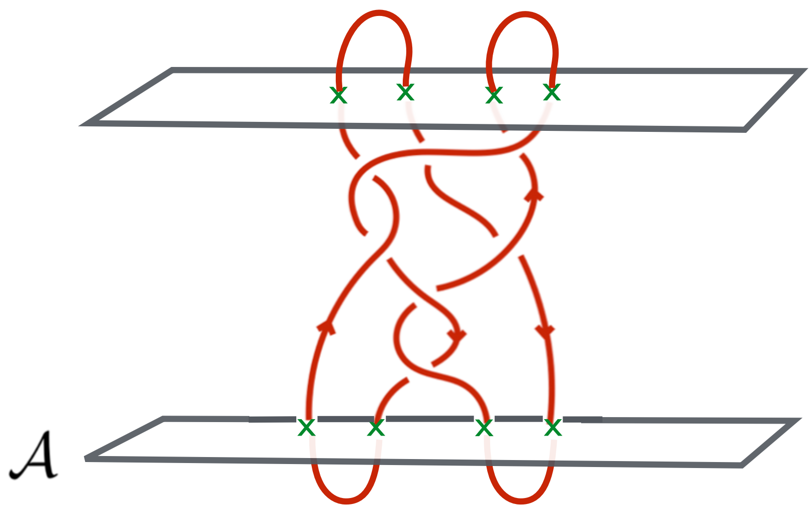
$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

corresponding to ${}^L\mathfrak{g}$



which act by exchanging a neighboring pair of punctures

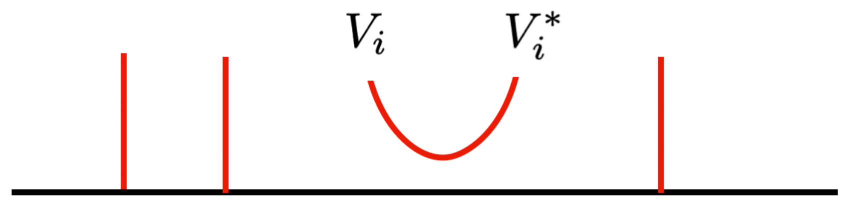
Any link invariant can be represented as a matrix element



of a braiding matrix between conformal blocks

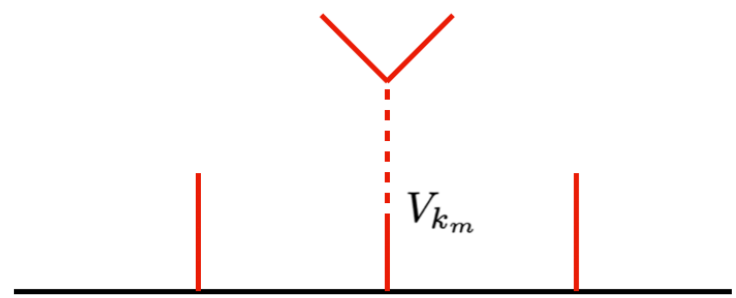
corresponding to cups and caps.

The conformal blocks that describe cups and caps



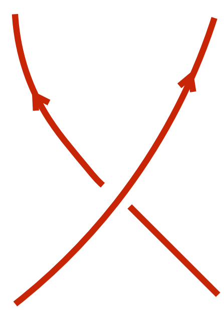
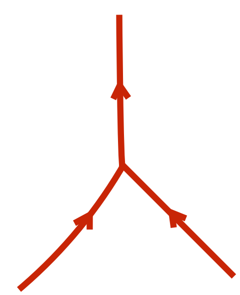
are obtained by bringing together pairs of punctures,

colored by complex conjugate representations,



and fusing them together to $V_{k_m} = \mathbb{1}$

Thus, both braiding and fusion in conformal field theory

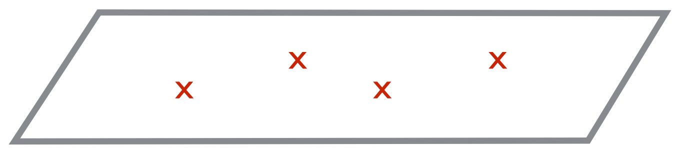
 

play an important role in the story.

To categorify quantum knot invariants,

one would like to associate

to the space conformal blocks one obtains at a fixed time slice

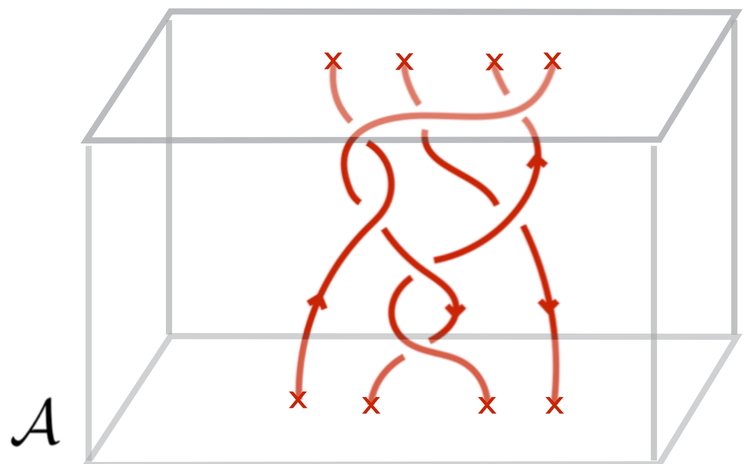
$\mathcal{A}$ 

a bi-graded category.

In addition to the usual fermion number grading

the category should have an additional grading associated to $\mathfrak{q}$

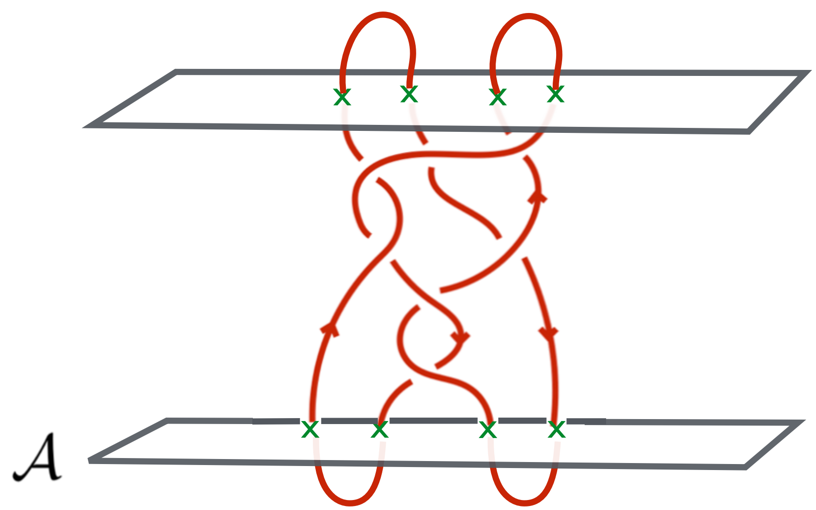
To braids,



one would like to associate functors between the categories

corresponding to the top and the bottom.

To links one would like to associate



a vector space

whose elements are morphisms

between the objects of the categories the top and the bottom,

up to the action of the braiding functor.

Moreover,

one would like to do that in a way that

recovers the quantum knot invariants upon

de-categorification.

One typically proceeds by coming up with a category,

and then one has to work to prove

that de-categorification gives

the quantum knot invariants one set out to categorify.

The geometric solutions to the knot categorification problem

I will describe

share the virtue of both that the second step is automatic.

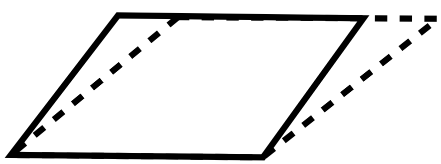
Mirror symmetry,

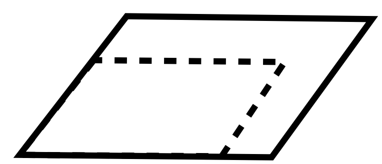
relates pairs of Calabi-Yau manifolds

$\mathcal{X}$ and $\mathcal{Y}$

while exchanging variations of

complex structure "B-type" $\longleftrightarrow$ symplectic or Kahler structures "A-type"





The names are inherited from topological B- and A-type string theory, whose moduli these are.

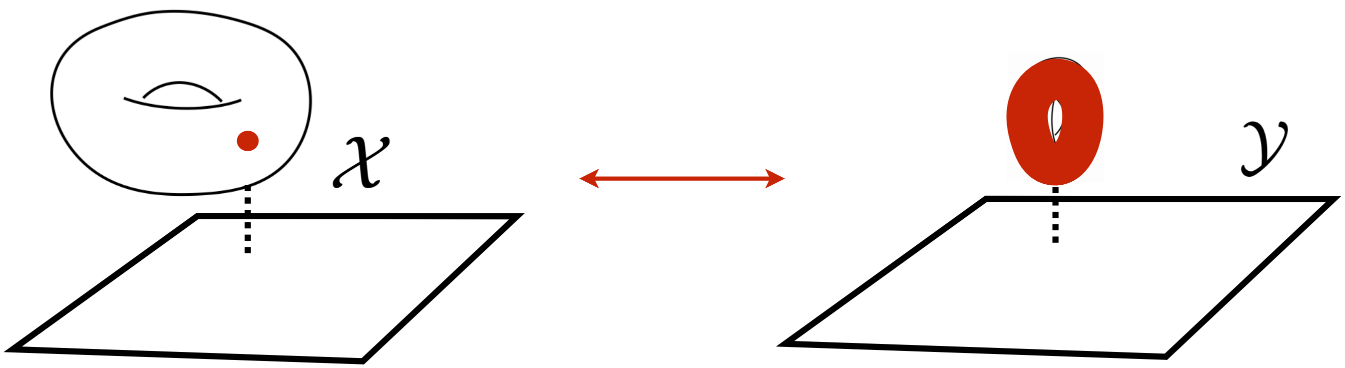
D-branes are key objects in string theory,

and asking how mirror symmetry acts on them

turned out to lead to deep insights

into mirror symmetry.

One such insight was due to Strominger, Yau and Zaslow.



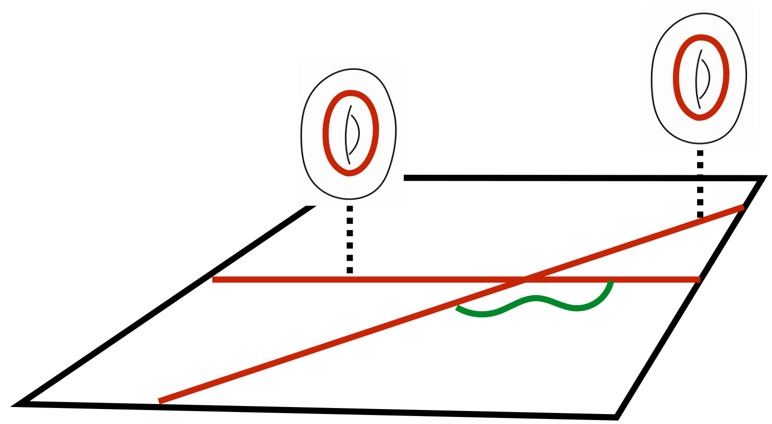
They showed that, in order for every pointlike brane on $\mathcal{X}$

to have a mirror brane on $\mathcal{Y}$,

mirror pairs of manifolds must be fibered by T-dual tori.

One can regard branes on a Calabi-Yau manifold

as objects of a category,



whose morphisms

are open strings stretching between the branes.

There are two natural categories,

corresponding to whether the branes preserve

A-type or B-type sypersymmetry

on the worldsheet.

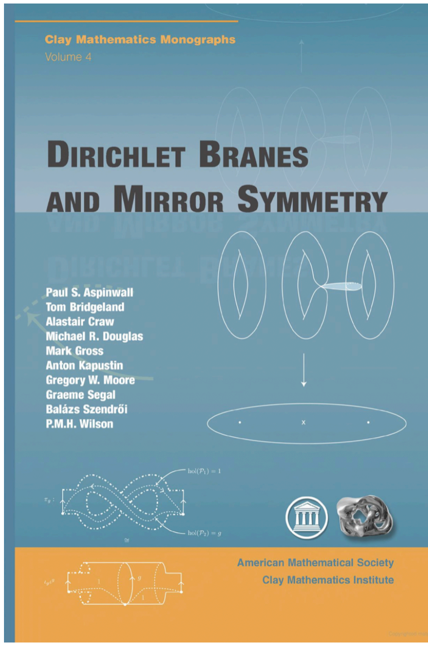
The corresponding morphism spaces

are the cohomology groups of the

supercharge preserved by the branes.

Both have a very well understood and much studied

mathematical formulation.



The category of B-branes

has objects which are branes supported on complex submanifolds of

$$\mathcal{X}$$

It has a mathematical formulation as the

"(derived) category of coherent sheaves."

$$\mathscr{D}_{\mathcal{X}} = D^b Coh(\mathcal{X})$$

The category of A-branes on

$$\mathcal{Y}$$

comes from symplectic geometry, and is known as the

"derived Fukaya category"

$$\mathscr{D}_{\mathcal{Y}} = DFuk(\mathcal{Y})$$

Its objects are branes

supported on real, or Lagrangian submanifolds.

The homological mirror symmetry is an equivalence of categories:

It says that, for any pair of B-branes on

$$\mathcal{X}$$

there is a mirror pair of A-branes on

$$\mathcal{Y}$$

such that the morphism spaces between them agree.

Mirror symmetry thus naturally provides a supply of categories of geometric origin.

In an appropriate setting,

it solves the knot categorification problem.

In parallel to solving the knot categorification problem

we will discover a new family of mirror pairs,

connected to representation theory,

where homological mirror symmetry

can be made as explicit as in the simplest known examples.

The Knizhnik-Zamolodchikov equation

$$\kappa\,\partial_i \mathcal{V}_\alpha - (r_i)_\alpha^{\,\beta}\, \mathcal{V}_\beta = 0$$

which plays a central role in knot theory,

has a geometric counterpart.

In the world of mirror symmetry,

there is an equally fundamental differential equation

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

which is sometimes called

"the quantum differential equation."

The "quantum differential equation"

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

is a linear differential equation for a vector valued function

over the moduli space of either

complexified Kahler moduli (A-type) of

$$\mathcal{X}$$

or complex structures (B-type) on

$$\mathcal{Y}$$

The name

quantum differential equation

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \, \mathcal{V}_\beta = 0.$$

comes from A-model where the connection,

on a vector bundle with fibers $H^{\text{even}}(\mathcal{X})$

over the complexified Kahler moduli space of $\mathcal{X}$

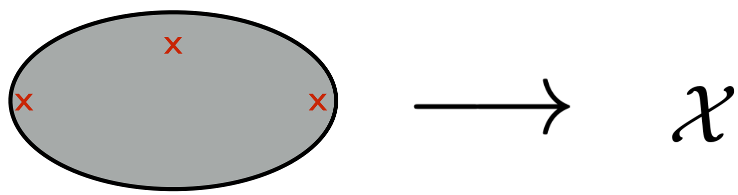
comes from quantum multiplication with divisors

$$C_i \in H^2(\mathcal{X})$$

Quantum product on $H^{\mathrm{even}}(\mathcal{X})$

$$C_{\alpha\beta\gamma} = \sum_{d\geq 0, d\in H_2(\mathcal{X})} (\alpha,\beta,\gamma)_d \; a^d,$$

is a deformation of the classical cup product



defined by Gromov-Witten theory.

In the mirror $\mathcal{Y}$, the connection

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \, \mathcal{V}_\beta = 0.$$

is the classical Gauss-Manin connection

on the vector bundle over the moduli space

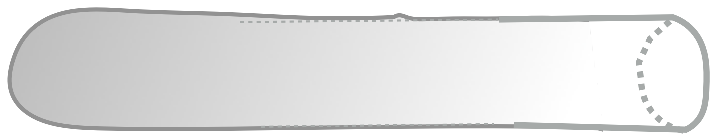
of complex, or B-type, structures,

with fibers the mid-dimensional cohomology

$$H^{\mathrm{mid}}(\mathcal{Y})$$

From perspective of $\mathcal{X}$ the solutions to

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

D  $\dashrightarrow \mathcal{X}$

are obtained as partition functions of an A-twisted sigma model

on a domain curve D , which is best thought of an infinite cigar,

with an S^1 boundary at infinity.

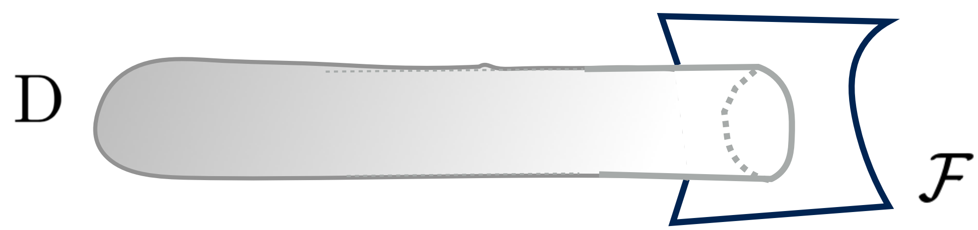
We get a specific solution of the equation

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

by choosing a B-type brane

$$\mathcal{F} \in \mathscr{D}_\mathcal{X} = D^b Coh(\mathcal{X})$$

as the boundary condition at infinity,



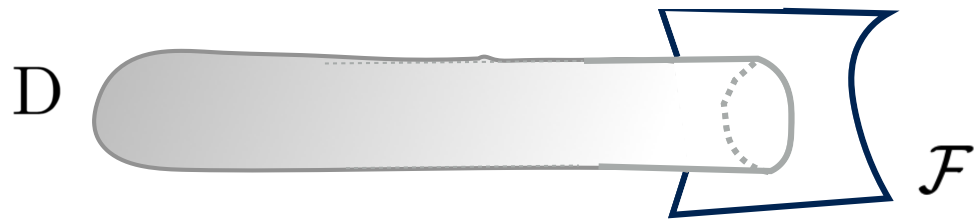
.

The solution depends on the brane only through its K-theory class,

and not the details of its shape.

The fact that the solutions to the equation

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$



mix the A- and B-type structures in

in the interior of the cigar and at the boundary at infinity

is a hallmark of central charge functions,

which $\mathcal{V}_\alpha$ mildly generalize,

by placing insertions at the tip of the cigar.

Both the equation,

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \, \mathcal{V}_\beta = 0.$$

and its monodromy problem,

featured prominently starting with the very first papers on mirror symmetry.

The Knizhnik-Zamolodchikov equation

$$\kappa\,\partial_i \mathcal{V}_\alpha - (r_i)_\alpha^\beta\,\mathcal{V}_\beta = 0$$

not only has the same flavor

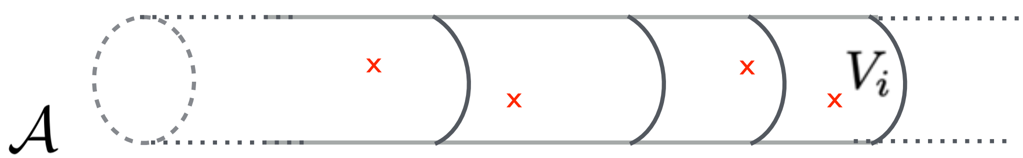
as the quantum differential equation:

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta\,\mathcal{V}_\beta = 0.$$

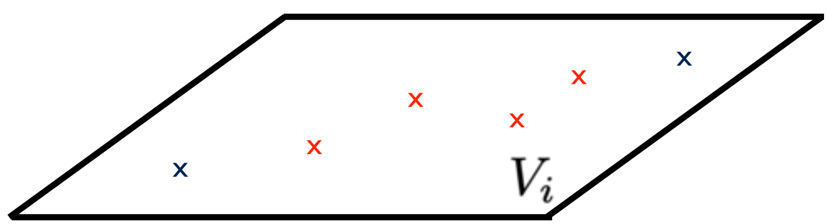
under certain conditions, they coincide.

On the knot theory side, we want to take

the Riemann surface to be a punctured infinite cylinder,



rather than a complex plane with punctures.



This enriches the theory, allowing it to describe

invariants of knots in

$$M_3 = \mathbb{R}^2 \times S^1$$

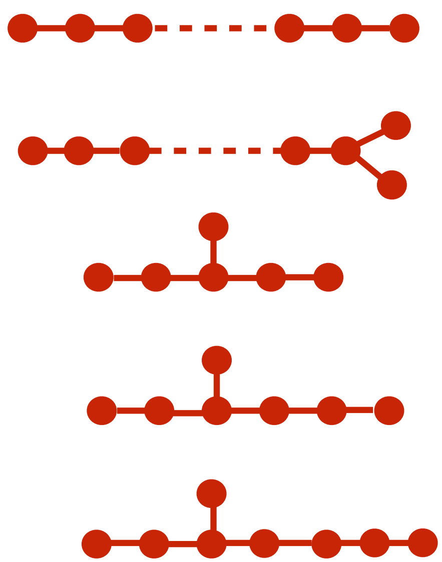
and not only in

$$M_3 = \mathbb{R}^3$$

We will also specialize, for the time being, to

$${}^L\mathfrak{g}$$

which is a simply laced Lie algebra



Throughout, we will restrict representations V_i to be minuscule.

On the geometric side, we want to take the target manifold

$$\mathcal{X}$$

to be a very special Calabi-Yau manifold:

one which is best described as the

moduli space of G -monopoles on

$$\mathbb{R}^3$$

with prescribed singularities.

The monopole group

$$G$$

is related to the Chern-Simons gauge group

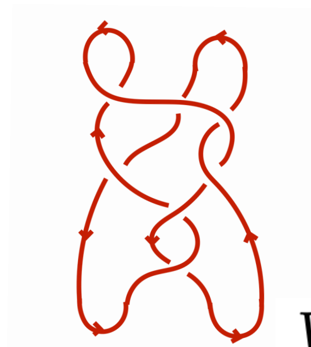
$$^L G$$

(whose Lie algebra is $^L\mathfrak{g}$)

by Langlands, or electric-magnetic duality.

In Chern-Simons theory,

view the knots in three dimensional space

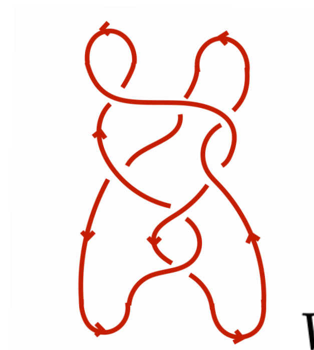


V_i

as paths of heavy particles, electrically charged under

^{L}G

In the geometric description, the same heavy particles



V_i

appear as singular, Dirac monopoles of the Langlands dual group

$$G$$

This magnetic description is what is needed

to understand categorification,

as anticipated in works of Witten.

The manifold

$$\mathcal{X}$$

has played an important role in mathematics before,

in the geometric Langlands correspondence,

where it is known as a

resolution of a transversal slice to affine Grassmannian of G

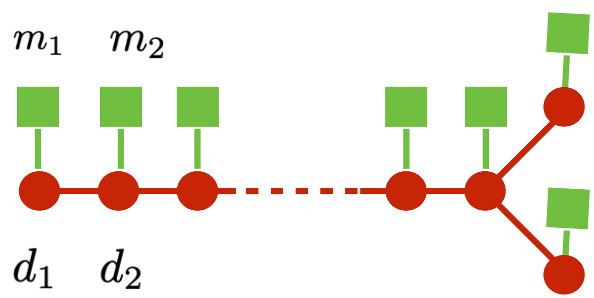
The same manifold

$$\mathcal{X}$$

is also the Coulomb branch

of a certain three dimensional quiver gauge theory

$\mathfrak{g} = D_n$



with N=4 supersymmetry.

The monopole moduli space

$$\mathcal{X}$$

is parameterized in part by positions of some number of smooth

't Hooft-Polyakov type monopoles on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

whereas positions of singular, Dirac-type monopoles are fixed,

and determine the metric on $\mathcal{X}$

The split,

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

reflects a choice of complex structure on

$$\mathcal{X}$$

in which the positions of singular monopoles on $\mathbb{R}$

$\mathbb{R}$ 

are the real Kahler moduli,

and their positions on $\mathbb{C}$ the complex structure moduli.

The monopole moduli space $\mathcal{X}$ has more symmetries than

a typical Calabi-Yau,

since it is hyper-Kahler.

As a consequence, its quantum cohomology of

$$\mathcal{X}$$

differs from classical,

only if we work equivariantly with respect to a torus action

$$\mathbb{C}^\times_{\mathfrak{q}} \subset \mathrm{T}$$

that scales its holomorphic symplectic form

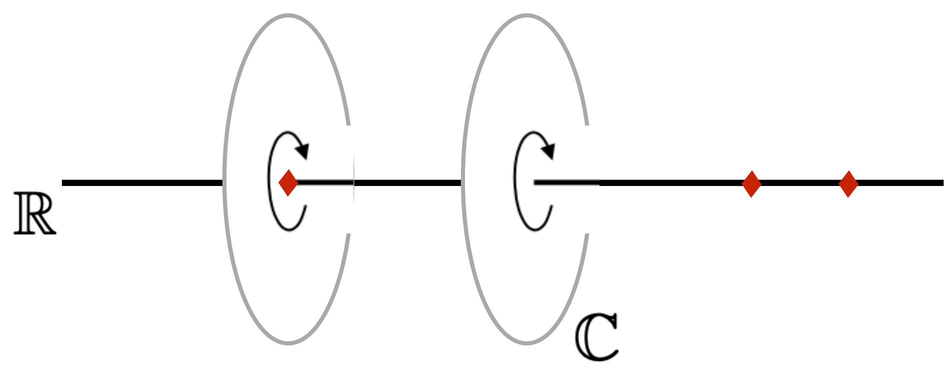
$$\omega^{2,0} \to \mathfrak{q}\, \omega^{2,0}$$

The symmetry

$$\omega^{2,0} \to \mathbf{q}\,\omega^{2,0}$$

comes from rotations $z \to \mathbf{q} z$ of the complex plane in

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

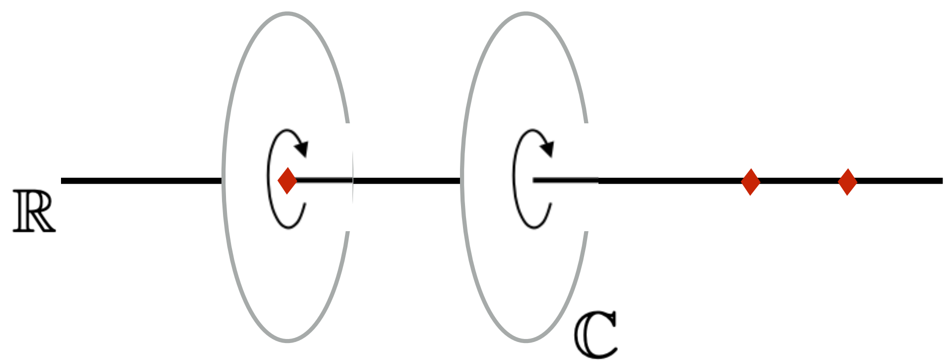


which is an isometry of $\mathcal{X}$ if we place all the singular, Dirac-type monopoles at the origin.

The parameter

$\mathfrak{q}$

of knot theory will be related to keeping track of charges of states



under this symmetry.

The fact that trigonometric Knizhnik-Zamolodchikov equation of

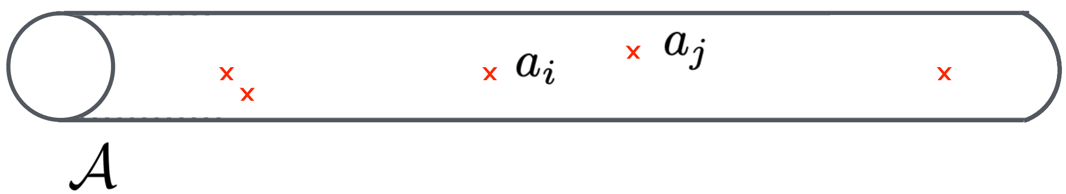
$$\widehat{L\mathfrak{g}}$$

has a geometric interpretation as the quantum differential equation of

$$\mathcal{X}$$

is a recent theorem by Ivan Danilenko.

We took the Riemann surface $\mathcal{A}$ to be a cylinder rather than a plane,



because the B-fields,

that pair with the real Kahler moduli

to get the complex ones, are periodic.

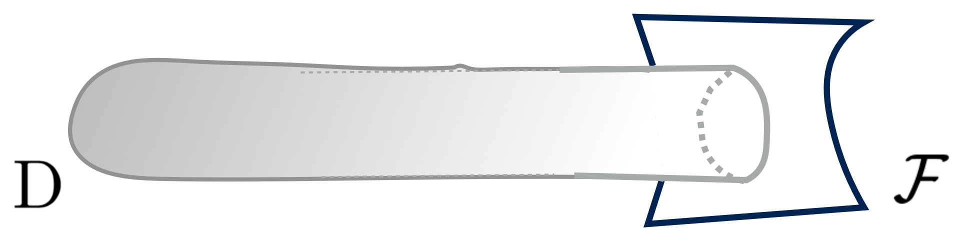
It follows that which solution of the KZ equation we get

$$\partial_i \mathcal{V}_\alpha - (C_i)_\alpha^\beta \mathcal{V}_\beta = 0.$$

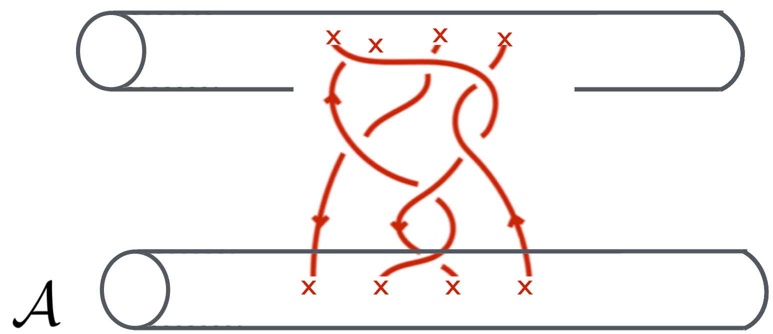
is determined by the choice of a B-type brane

$$\mathcal{F} \in \mathscr{D}_{\mathcal{X}}$$

as the boundary condition at infinity.



It also follows that a braid in $\mathcal{A} \times [0,1]$



has a geometric interpretation in terms of

$$\mathcal{X}$$

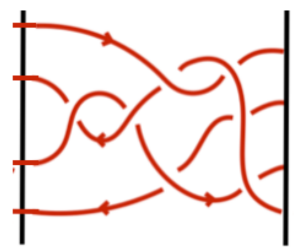
as a path in (complexified) Kahler moduli that avoids sungularities.

A central expectation in mirror symmetry,

is the fact that

monodromy of the quantum differential equation

$\mathfrak{B}$



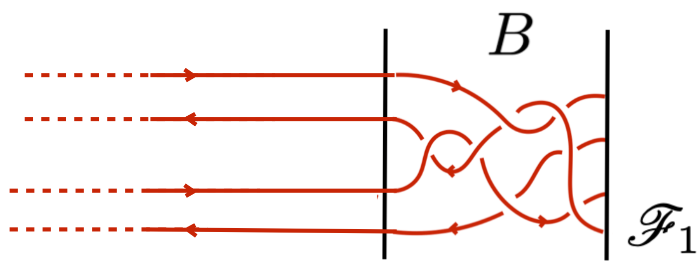
is categorified by a functor acting on the category of branes,

$$\mathscr{B} : \mathscr{D}_{\mathcal{X}} \to \mathscr{D}_{\mathcal{X}'}$$

which "transports" the category along the path B

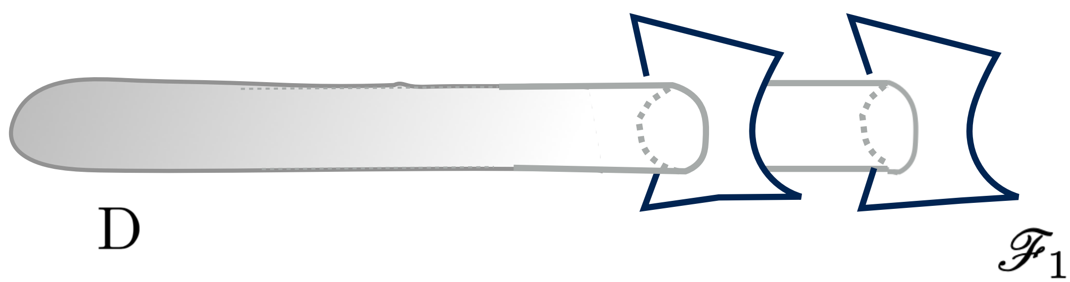
and which is an equivalence.

Physically, braid group



acts, in the sigma model on the cigar,

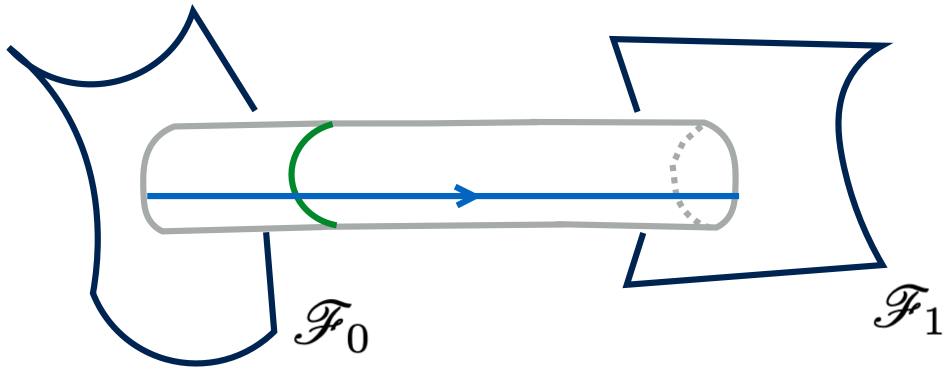
by letting the moduli of the theory vary near the boundary at infinity.



The direction along the (Euclidian) cigar

coincides with the "time" along the braid.

It follows sigma model on the annulus



with moduli that vary according to the braid computes

the matrix element of the monodromy

$$\mathfrak{B} \in U_{\mathfrak{q}}({}^L\mathfrak{g})$$

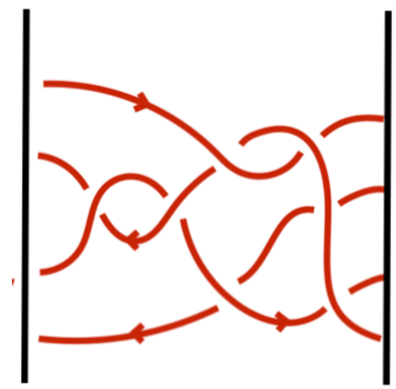
between a pair of branes

For this, we view the time to run along the annulus.

The category of B-branes

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

manifestly categorifies



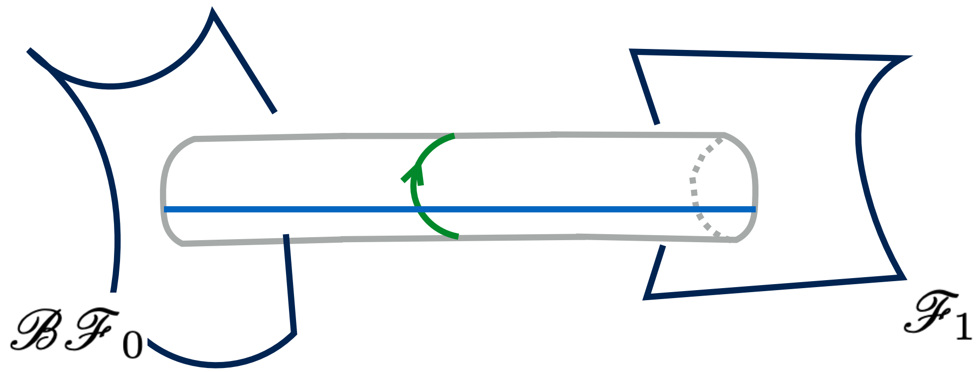
the matrix elements of

$$\mathfrak{B} \in U_{\mathfrak{q}}({}^L\mathfrak{g})$$

between pairs of conformal blocks.

Sigma model on the same Euclidian annulus

with "time" viewed as running around the S^1 ,



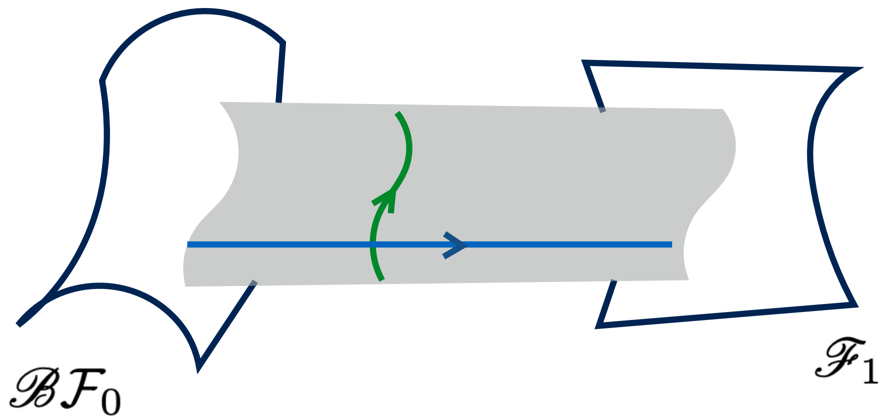
computes the supertrace

$$\mathrm{Tr}(-1)^F e^{-\beta H}$$

which is the Euler characteristic

of the supercharge Q preserved by the two branes.

The cohomology of the supercharge Q



is the basic ingredient in the category of branes

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

a graded vector space,

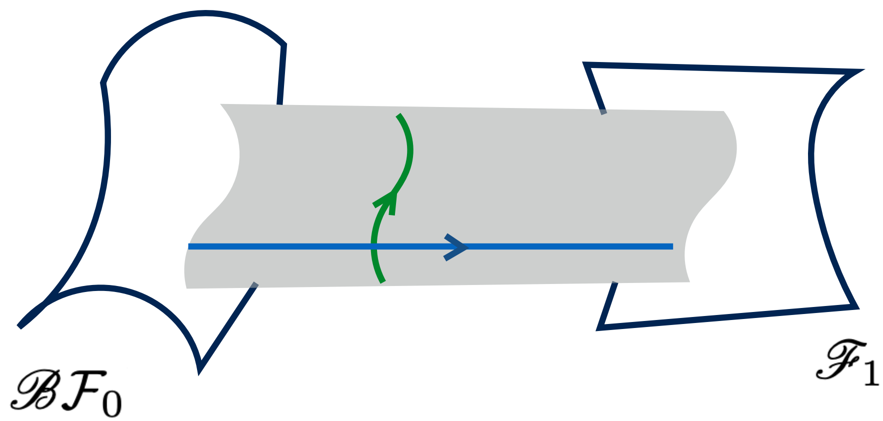
$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

the space of morphisms between a pair of branes.

So, for any pair of branes, we get collection of cohomology groups,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

the space of morphisms between the branes



whose Euler characteristic

$$\chi(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

is the braiding matrix element.

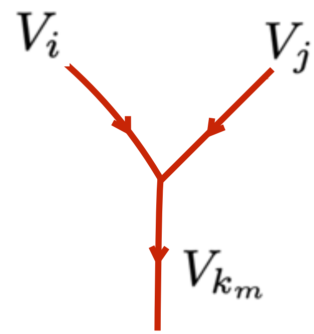
To extend this to link invariants, we need objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

which lead to conformal blocks

in which pairs of vertex operators fuse to trivial representation.

For this, we need to understand

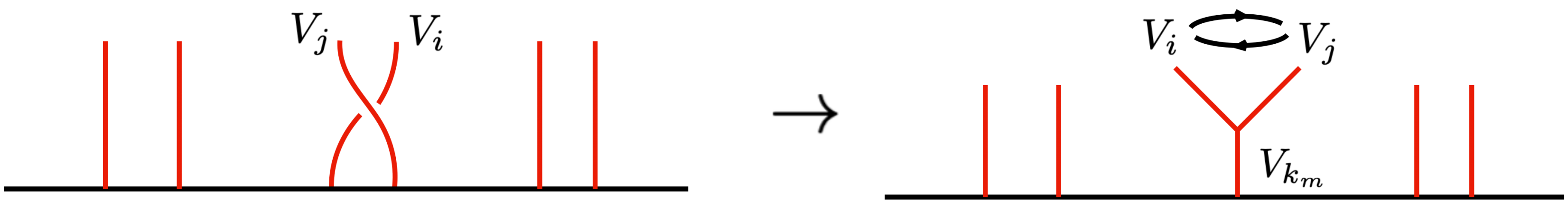


the geometric interpretation of fusion in terms of

$$\mathcal{X}$$

and its category of B-type branes.

In conformal field theory fusion diagonalizes braiding.



The analogue of this in the category of branes is

turns out to be existence of a "perverse filtration"

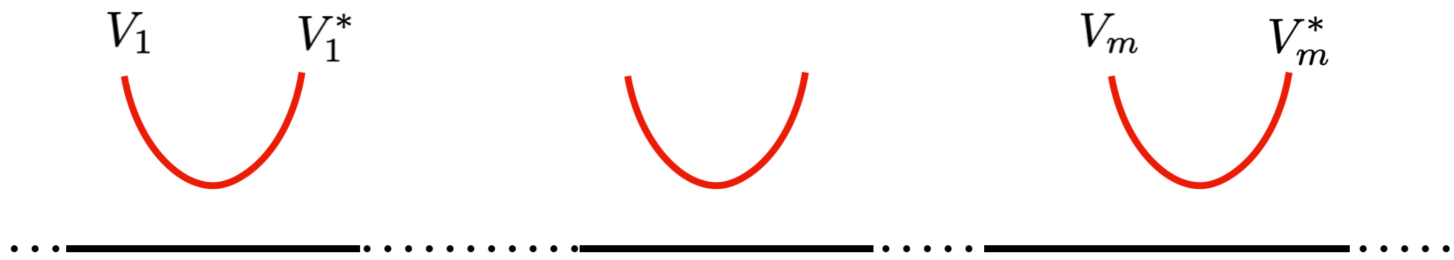
envisioned by Chuang and Rouquier

in abstract terms,

as the right structure to describe

the action of braiding on derived categories.

One learns from this that cups and caps



come from branes on

$$\mathcal{X}$$

with simple geometric meaning.

These branes are structure sheaves of vanishing cycles in $\mathcal{X}$

$$\mathcal{U} = \mathcal{O}_U$$

known as "minuscule Grassmanians"

$$U = G/P_1 \times \ldots \times G/P_d$$

The cycles shrink to a point as punctures come together in pairs.

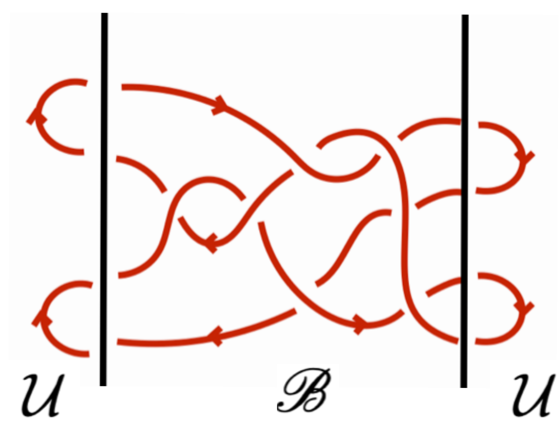
Using very special properties of perverse filtrations

and these vanishing cycle branes,

I proved that, not only do the homology groups

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U},\mathcal{U})$$

manifestly categorify the corresponding $U_{\mathfrak{q}}(^L\mathfrak{g})$ link invariants,



they are themselves link invariants.

The approach by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

explains the origin of homological link invariants

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U},\mathcal{U})$$

in geometry and physics,

in a manner that works uniformly for all groups.

Recently, Ben Webster proved that homological link invariants

$$Hom^{*,*}_{\mathscr{D}\mathcal{X}}(\mathscr{B}\mathcal{U},\mathcal{U})$$

that come from B-type branes on

$$\mathcal{X}$$

are equivalent to algebraic invariants he defined in '13,

using a (cylindrical version) of the KLRW algebra

$$\mathscr{A}$$

and moreover generalize them to links in

$$\mathbb{R}^2 \times S^1$$

As stated, the approach by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

is not much more explicit than that by the KLRW algebra.

Both exist only abstractly.

To give a solution of the theory,

by which I mean an explicit description of it,

we will make use of homological mirror symmetry,

or more precisely,

an equivariant version of it.

In the very best instances of homological mirror symmetry,

$$\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathcal{Y}}$$

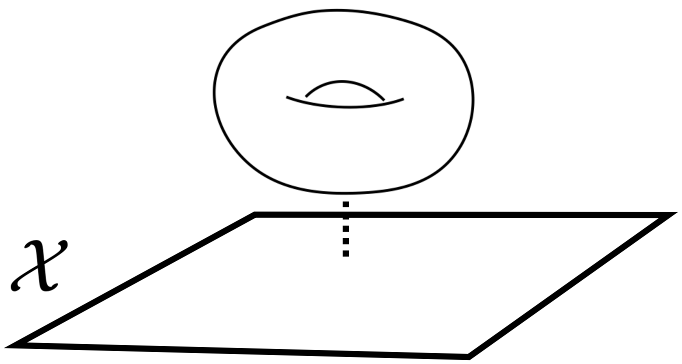
one learns how to make the equivalence of two categories manifest,

and both theories based on

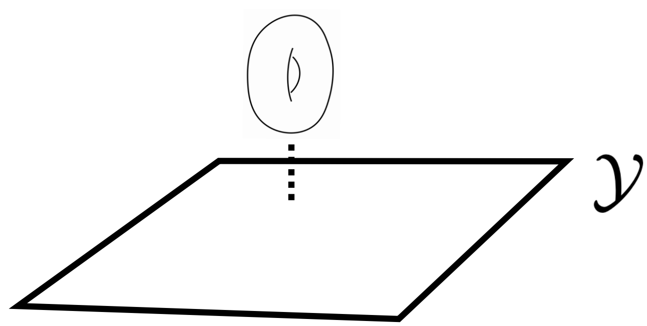
$\mathcal{X}$ and on $\mathcal{Y}$

become solvable exactly.

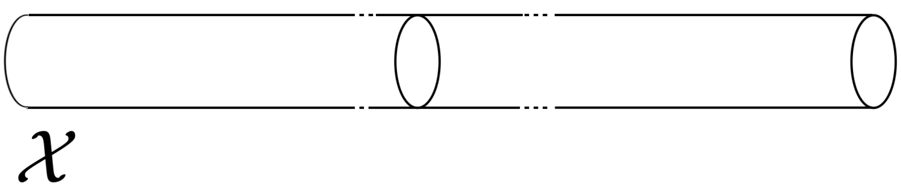
One of the very simplest examples of homological mirror symmetry is when

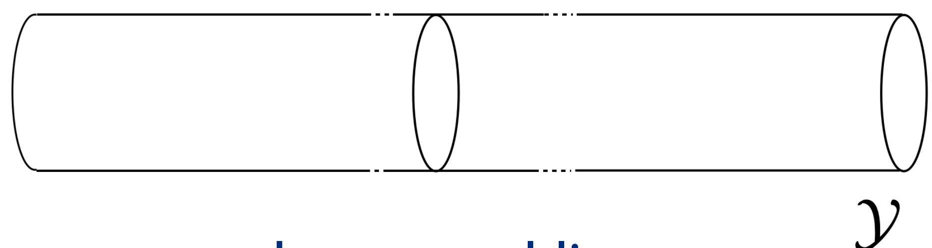


and



are taken to be simply a pair of infinite cylinders,

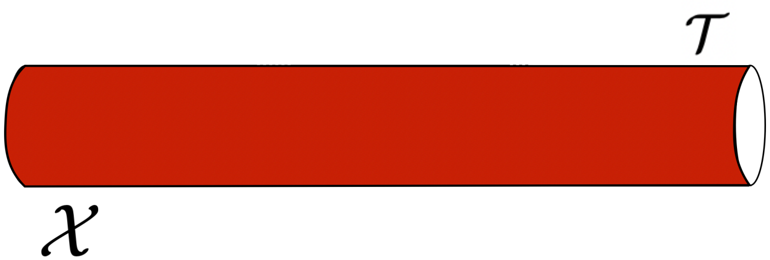


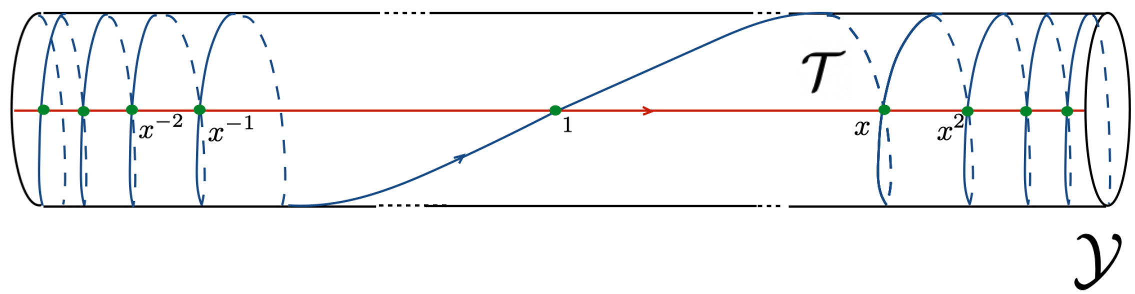


their torus fibers being circles, the common base a real line.

Categories of branes on the two sides turn out to be

each generated by a single brane:





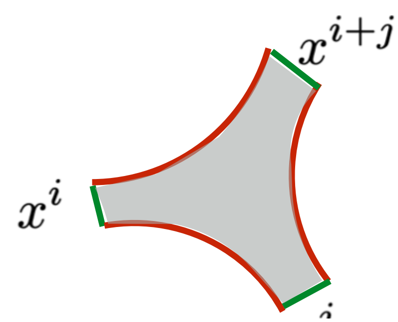
While the branes look different,

their algebras of open strings turn out to be the same,

$$\mathscr{A} = Hom^*_{\mathscr{D}_{\mathcal{X}}}(\mathcal{T}, \mathcal{T})$$

both equal to the algebra of functions on the complex cylinder

$$\mathscr{A} = \mathbb{C}[x, x^{-1}]$$



The homological mirror symmetry,

$$\mathcal{X} \overset{\text{mirror}}{\longleftrightarrow} \mathcal{Y}$$

is a consequence of a pair of equivalences

$$\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathcal{Y}}$$

where $\mathscr{D}_{\mathcal{Y}}$ is the category of A-branes on $\mathcal{Y}$

and $\mathscr{D}_{\mathscr{A}}$ is the (derived) category of representations

of the open string algebra.

This simple example is the model for

how one hopes to understand homological mirror symmetry in all cases.

Webster's proof of equivalence of categorification

of $U_{\mathfrak{q}}({}^L\mathfrak{g})$ link invariants

via B-type branes on

$$\mathcal{X}$$

and via KLRW algebra

$$\mathscr{A}$$

is really the first of the two equivalences in homological mirror symmetry:

$$\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathcal{Y}}$$

The ordinary, non-equivariant mirror of our

$$\mathcal{X}$$

is a hyper-Kahler manifold

$$\mathcal{Y}$$

which is, to a first approximation,

given by a hyper-Kahler rotation of $\mathcal{X}$

($\mathcal{Y}$ is also a moduli space of G monopoles, just on $\mathbb{R}^2 \times S^1$)

We will not describe

$$\mathscr{D}y$$

directly, but rather we will make use of a further simplification,

available in the present setting.

Recall that

$$\mathcal{X}$$

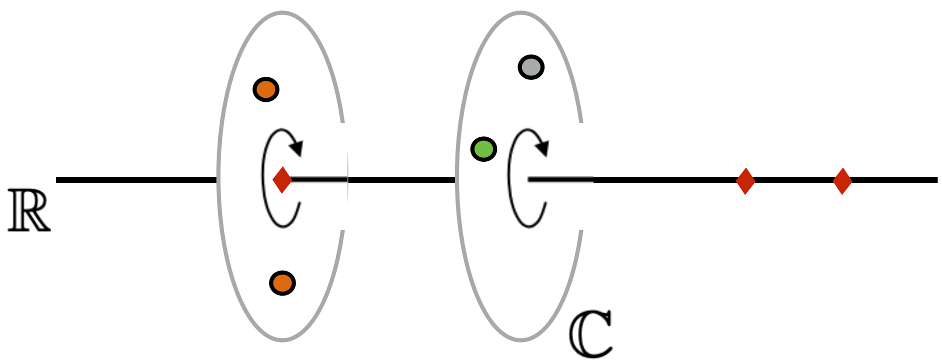
is a moduli space of monopoles on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

where a symmetry that corresponds to rotations of

$$\mathbb{C}$$

plays a key role — this is how we got $\mathfrak{q}$ into the problem.



This means that all the relevant information about the geometry of the "big"

$$\mathcal{X}$$

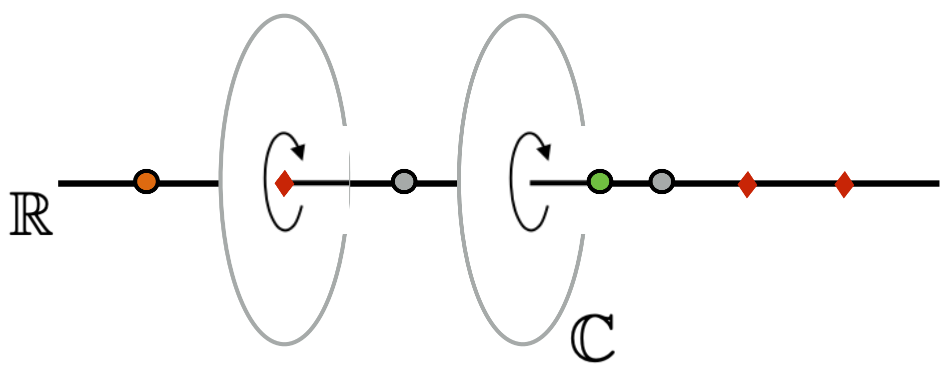
is much more efficiently contained in the "small"

$$X \subset \mathcal{X}$$

a subspace of half the dimension

where all monopoles,

singular or not, are at the origin of $\mathbb{C}$ and at points in $\mathbb{R}$



We will define the equivariant mirror of the "big" $\mathcal{X}$, call it "small"

$$Y$$

to be the ordinary mirror of the "small" X :

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, leftrightarrow, "\text{mirror}"] \arrow[drr, leftrightarrow, "\text{equivariant mirror}"] & & \mathcal{Y} \arrow[d] \\
X \arrow[u] \arrow[rr, leftrightarrow, "\text{mirror}"] & & Y
\end{tikzcd}$$

The key fact is that

the bottom row has as much information about the geometry as the top:

$$\begin{tikzcd}
\mathcal{X} \arrow[r, leftrightarrow, "\text{mirror}"] \arrow[dr, leftrightarrow, "\text{equivariant}", "\text{mirror}"'] & \mathcal{Y} \arrow[d] \\
X \arrow[u] \arrow[r, leftrightarrow, "\text{mirror}"] & Y
\end{tikzcd}$$

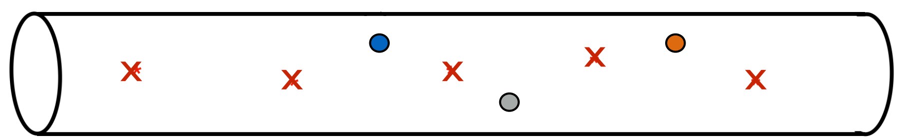
it is just more efficiently packaged.

The equivariant mirror

$$Y$$

is (a cousin of) configuration space of points on

$\mathcal{A}$



the Riemann surface with punctures

where conformal blocks live,

"colored" by simple roots of ${}^L\mathfrak{g}$ but otherwise indistinguishable,

(with some locus deleted and singularities resolved).

There is a potential on

$$Y$$

which is a multi-valued holomorphic function,

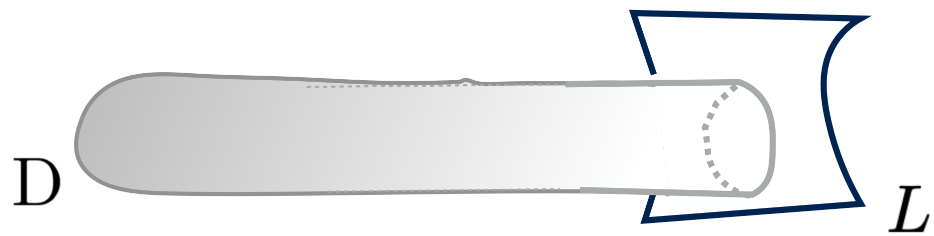
$$W = \lambda_0 W_0 + \sum_{a=1}^{\mathrm{rk}} \lambda_a W^a$$

which makes the mirror theory into a "Landau-Ginzburg" model.

Corresponding to a solution of the

Knizhnik-Zamolodchikov equation

is an A-brane at the boundary of D at infinity,



The brane is an object of the category of A-branes

$$\mathscr{D}_Y$$

the "derived Fukaya-Seidel category" of Y with potential W .

The mirror description based on

$$Y$$

leads to "integral" formulation of conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

as period integrals:

$$\mathcal{V}_\alpha[L] = \int_L \Phi_\alpha \, \Omega \, e^{-W}$$

explaining works of Feigin and E.Frenkel in the '80's

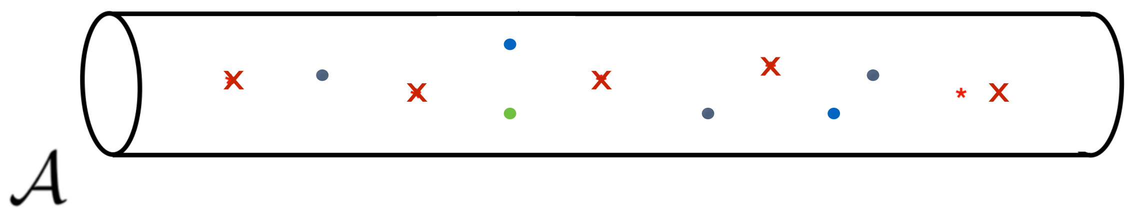
and Schechtman and Varchenko.

One can describe this category very explicitly

thanks to the fact

Y

a (close cousin of) configuration space of colored points

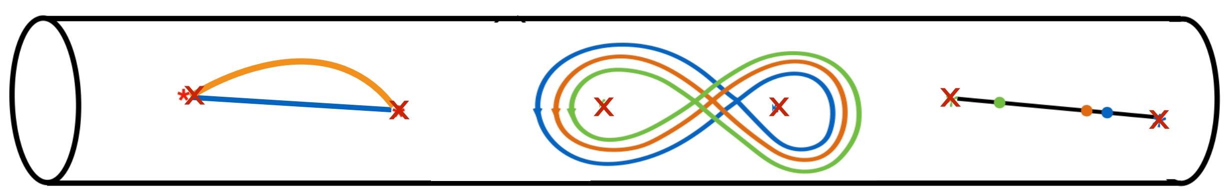


on the punctured Riemann surface.

Objects of the category of A-branes on

$$Y$$

have a description in terms of the Riemann surface,



$\mathcal{A}$

as products of one dimensional curves

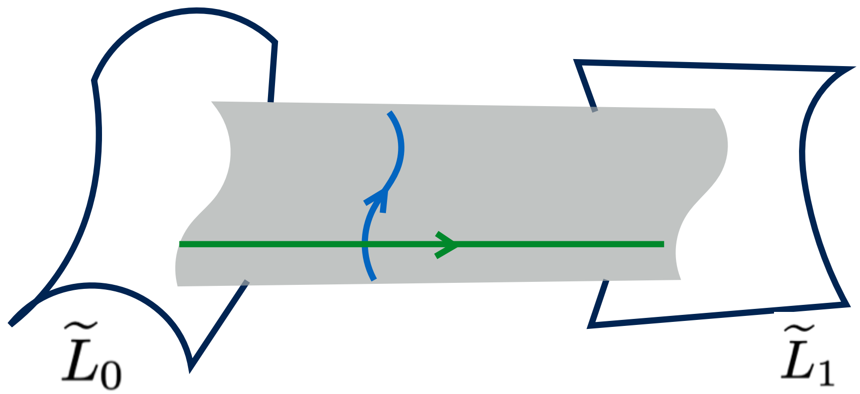
colored by simple roots, or generalized intervals.

In any category of A-branes

the spaces of morphisms between a pair of branes

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \mathrm{Ker}\, Q/\mathrm{Im}\, Q.$$

are defined by Floer theory,



which is modeled after Morse theory approach to supersymmetric

quantum mechanics.

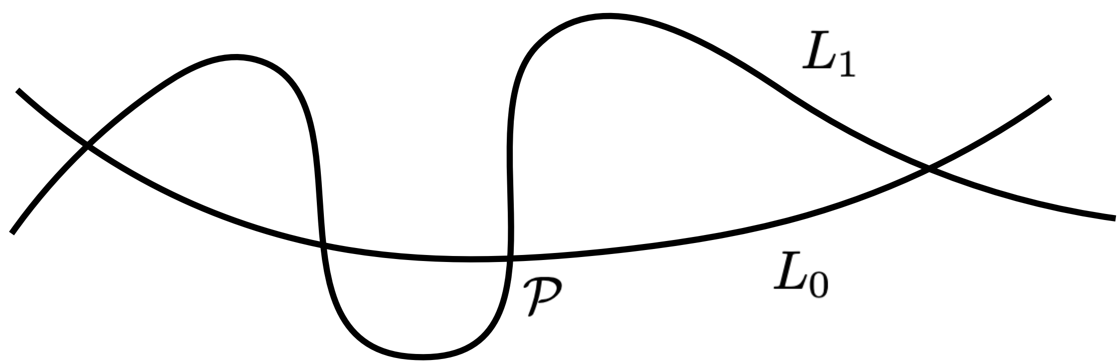
The role of the Morse complex from the beginning of the talk

is taken by the Floer complex, which is a vector space

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

spanned by the intersection points of the two Lagrangians,

and graded by the fermion number



and $\mathfrak{q}$ -degrees, the latter is thanks to the fact W is multi-valued.

The action of the differential Q on this space

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

of perturbative supersymmetric ground states

$$Q : CF^{*,*} \to CF^{*,*}$$

is generated by instantons.

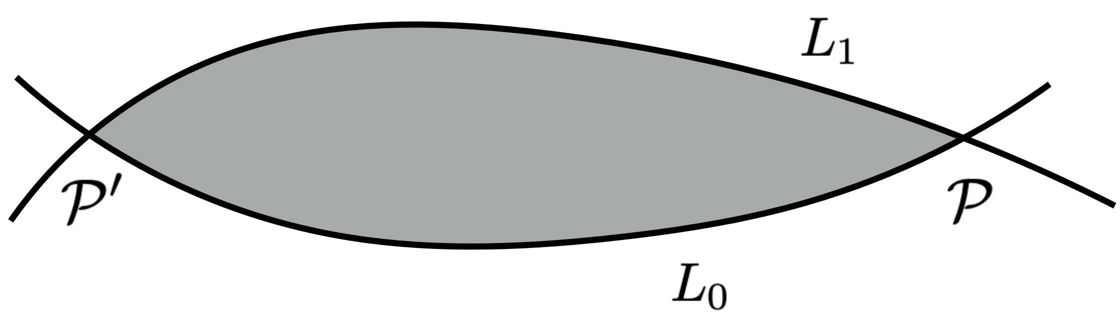
In Floer theory,

the coefficient of $\mathcal{P}'$ in $Q\,\mathcal{P}$

is obtained by counting holomorphic disk instantons in Y

interpolating from $\mathcal{P}$ to $\mathcal{P}'$, of Femion number one

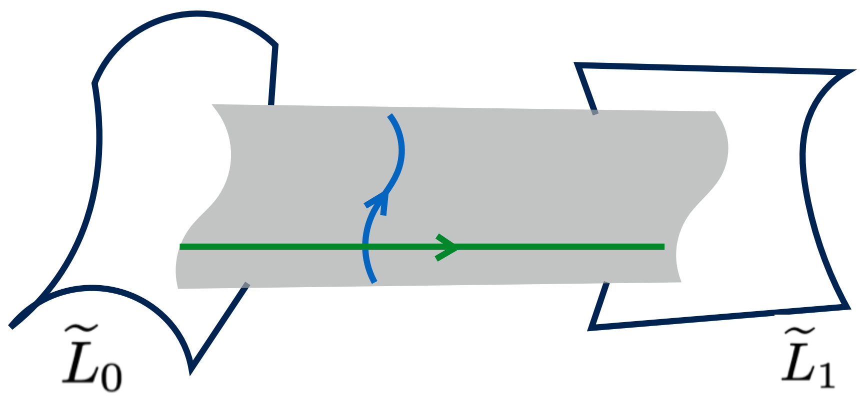
and $\mathfrak{q}$-degree zero.



The cohomology of the resulting complex

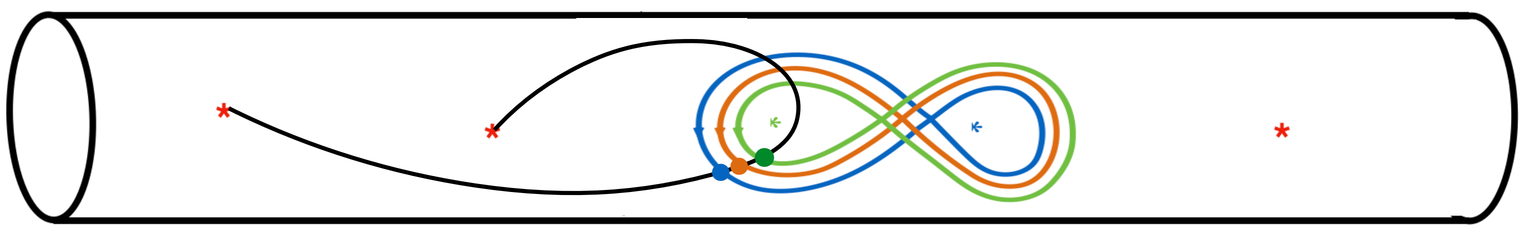
$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \mathrm{Ker}\, Q / \mathrm{Im}\, Q.$$

is the space of morphisms between the branes in $\mathscr{D}_Y$.



A vast simplification in the present case is that

just as the branes have a description in terms of the Riemann surface



so do their intersection points,

as well as the maps between them.

The theory which results is a generalization,

to arbitrary simply laced Lie algebra

$$^L\mathfrak{g}$$

of "Heegard-Floer" theory.

The latter is associated to

$$^L\mathfrak{g} = \mathfrak{gl}_{1|1}$$

and categorifies the Alexander polynomial.

Heegard-Floer theory is phrased in the same, one dimensional, terms.

Mirror symmetry

helps us understand exactly which questions we need to ask

$$
\begin{tikzcd}
\mathcal{X} \arrow[rr, leftrightarrow, "\text{mirror}"] \arrow[drr, leftrightarrow, "\text{equivariant mirror}"] & & \mathcal{Y} \arrow[d] \\
X \arrow[u] \arrow[rr, leftrightarrow, "\text{mirror}"] & & Y
\end{tikzcd}
$$

to recover homological knot invariants from Y ,

for an arbitrary simply laced Lie algebra

${}^{L}\mathfrak{g}$

For any simply laced Lie algebra

$${}^{L}\mathfrak{g}$$

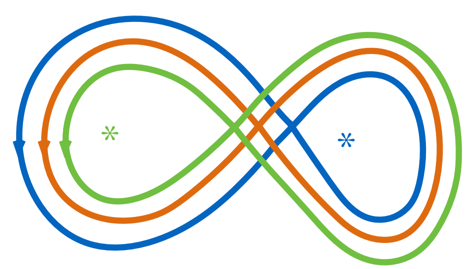
branes which serve as "cups" and "caps" "upstairs" on

$$\mathcal{X}$$

(associated to "minuscule Grassmanians")

are respectively, the generalized intervals

and the generalized figure eight branes.

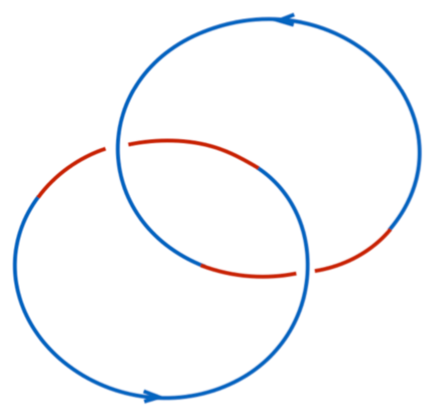


In the description based on

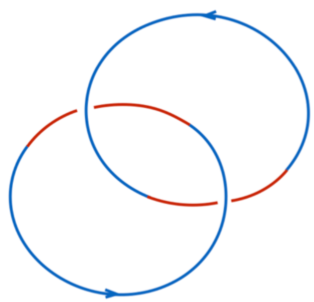
$$Y$$

both the Lagrangians and the action of braiding on them are geometric.

Start with a projection of a link to a the surface $\mathcal{A}$:



To translate it to a pair of A-branes by choosing a bicoloring,

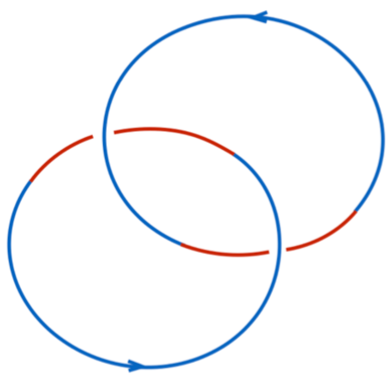


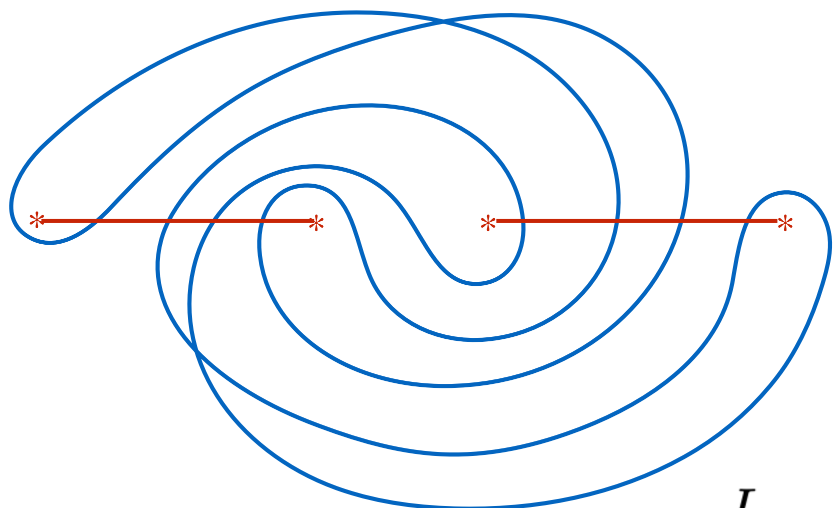
of every link component

by an equal number of segments of each color,

such that red always underpasses the blue.

The mirror Lagrangians $I_{\mathcal{U}}$ and $\mathscr{B}E_{\mathcal{U}}$ are obtained by replacing all the red segments by the interval-type branes and the blue segments by figure eight-type branes:





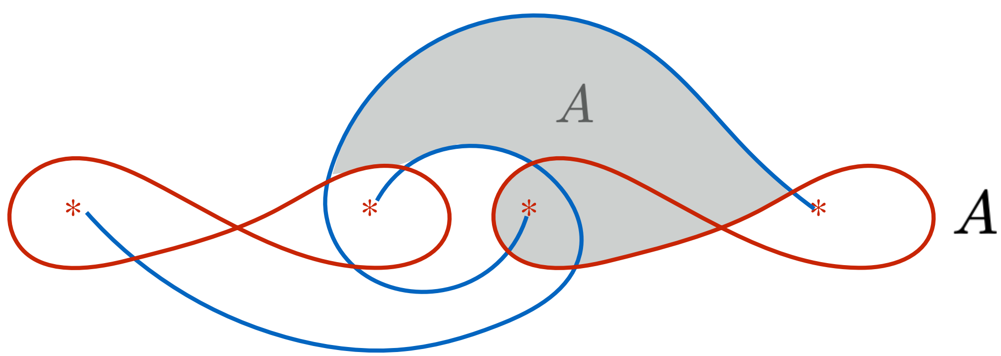
$^L\mathfrak{g} = \mathfrak{su}_2$

As in Heegard-Floer theory,

computing the action of the differential

can be translated to a sequence well defined, but hard

problems in complex analysis in one dimension.



Surprisingly, this problem can be solved.

One solves all the disk counting problems

at once,

by making the homological mirror symmetry that relates

“downstairs” mirror pair

$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_Y$$

manifest.

The fact mirror symmetry can sum up curve counts is its basic property.

As in the simplest examples of homological mirror symmetry,

the categories on the two sides

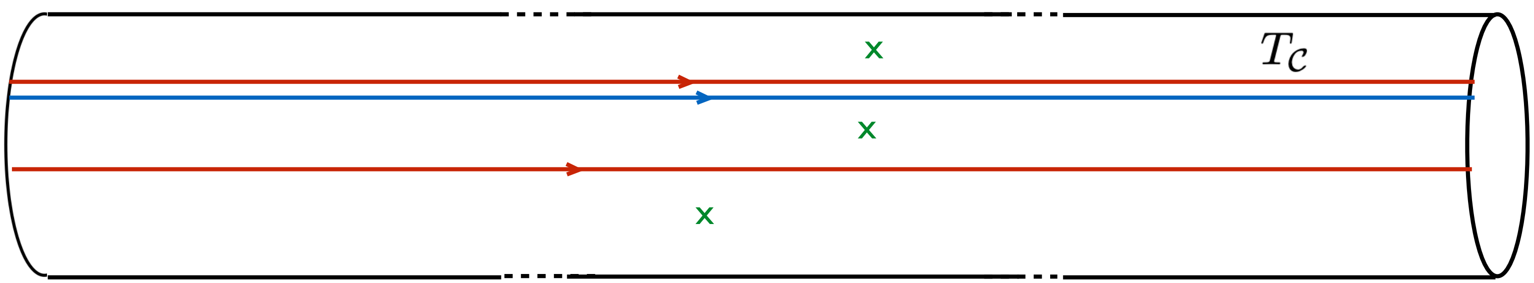
$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_Y$$

are generated by a finite number of branes.

From perspective of

$$Y$$

the generating set of branes



are products of real line Lagrangians,

$$T_{\mathcal{C}} = T_{i_1} \times T_{i_2} \times \ldots \times T_{i_D}$$

colored by simple roots.

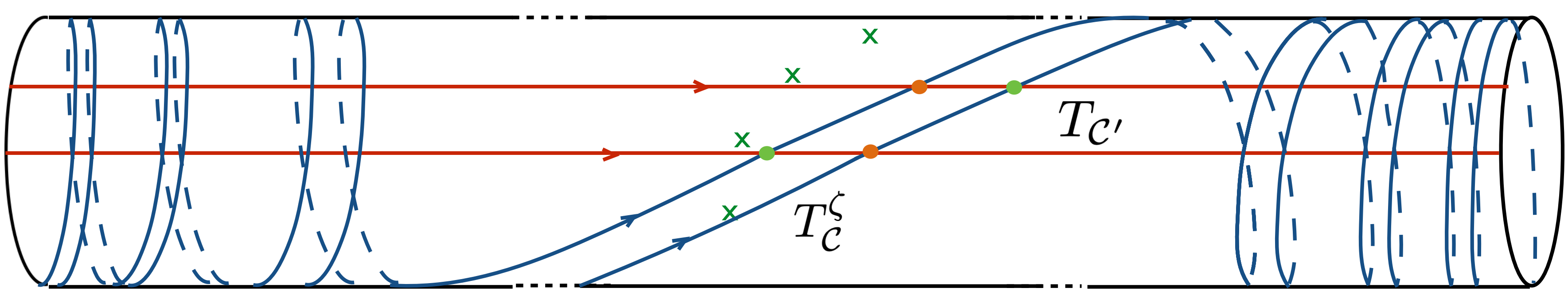
This is a simple generalization of our very simplest example.

The associated "downstairs" algebra of open strings

$$A$$

is a smaller cousin of the "upstairs" KLRW algebra.

$$\mathscr{A}$$



The result is an explicit algorithm for

computing link homologies for arbitrary

whose input is an arbitrary

link, a choice of a Lie algebra and its representations,

and whose output is a bigraded link homology.