Topological symmetry in field theory

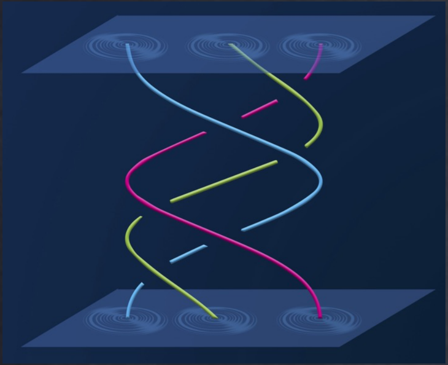
Dan Freed
University of Texas at Austin

October 27, 2022

Joint work with Greg Moore and Constantin Teleman

arXiv:2209.07471

Simons Collaboration (https://scgcs.berkeley.edu)



Global Categorical Symmetry

Symmetry in quantum field theory and quantum gravity

This is a big subject to which Hirosi has made many contributions

Symmetry in quantum field theory and quantum gravity

This is a big subject to which Hirosi has made many contributions

Today I will introduce a framework for internal topological symmetries in QFT

Symmetry in quantum field theory and quantum gravity

This is a big subject to which Hirosi has made many contributions

Today I will introduce a framework for internal topological symmetries in QFT

Leads to a calculus of topological defects which takes full advantage of well-developed theorems and techniques in topological field theory

Symmetry in quantum field theory and quantum gravity

This is a big subject to which Hirosi has made many contributions

Today I will introduce a framework for internal topological symmetries in QFT

Leads to a calculus of topological defects which takes full advantage of well-developed theorems and techniques in topological field theory

Let's begin with some motivation from representation theory of Lie groups and Lie algebras

Computations in... $\mathfrak{sl}_2(\mathbb{R})$

Set

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Computations in... $\mathfrak{sl}_2(\mathbb{R})$

Set

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Simple matrix manipulations verify the identity

$$\boxed{\frac{1}{2}h^2 + ef + fe = \frac{1}{2}h^2 + h + 2fe}$$

Namely, both sides equal

$$\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

In the 3-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ we have

$$h' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad e' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad f' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

In the 3-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ we have

$$h' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad e' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad f' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

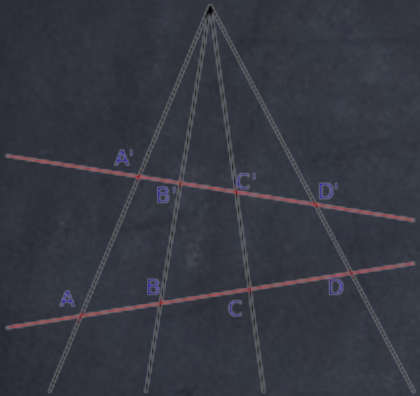
Now slightly less simple matrix manipulations verify the identity

$$\boxed{\frac{1}{2}(h')^2 + e'f' + f'e' = \frac{1}{2}(h')^2 + h' + 2f'e'}$$

Namely, both sides equal

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The Lie group $SL_2(\mathbb{R})$ acts on the projective line $\mathbb{RP}^1$ as fractional linear transformations



The Lie group $\mathrm{SL}_2(\mathbb{R})$ acts on the projective line $\mathbb{RP}^1$ as fractional linear transformations

There is an induced action on differentials $\phi(x)(dx)^\lambda$ for each $\lambda \in \mathbb{C}$

The Lie group $\mathrm{SL}_2(\mathbb{R})$ acts on the projective line $\mathbb{RP}^1$ as fractional linear transformations

There is an induced action on differentials $\phi(x)(dx)^\lambda$ for each $\lambda \in \mathbb{C}$

The infinitesimal action of $\mathfrak{sl}_2(\mathbb{R})$ is:

$$\begin{aligned} \tilde{h} &: \phi \longmapsto -2x\phi' - 2\lambda\phi \\ \tilde{e} &: \phi \longmapsto -\phi' \\ \tilde{f} &: \phi \longmapsto x^2\phi' + 2\lambda x\phi \end{aligned}$$

The Lie group $\mathrm{SL}_2(\mathbb{R})$ acts on the projective line $\mathbb{RP}^1$ as fractional linear transformations

There is an induced action on differentials $\phi(x)(dx)^\lambda$ for each $\lambda \in \mathbb{C}$

The infinitesimal action of $\mathfrak{sl}_2(\mathbb{R})$ is:

$$\begin{aligned} \tilde{h}\colon \phi &\longmapsto -2x\phi' - 2\lambda\phi \\ \tilde{e}\colon \phi &\longmapsto -\phi' \\ \tilde{f}\colon \phi &\longmapsto \quad x^2\phi' + 2\lambda x\phi \end{aligned}$$

Some calculus manipulations verify the identity

$$\boxed{\frac{1}{2}\tilde{h}^2 + \tilde{e}\tilde{f} + \tilde{f}\tilde{e} = \frac{1}{2}\tilde{h}^2 + \tilde{h} + 2\tilde{f}\tilde{e}}$$

Both sides act as multiplication by $4\lambda^2 - 2\lambda$

Instead of each separate computation, we compute *universally* in an abstract algebra

Instead of each separate computation, we compute *universally* in an abstract algebra

Each representation defines a module over the *universal enveloping algebra* $A = U\big(\mathfrak{sl}_2(\mathbb{R})\big)$

Instead of each separate computation, we compute *universally* in an abstract algebra

Each representation defines a module over the *universal enveloping algebra* $A = U\big(\mathfrak{sl}_2(\mathbb{R})\big)$

The identity

$$\boxed{\frac{1}{2}h^2 + ef + fe = \frac{1}{2}h^2 + h + 2fe}$$

holds in A, since $[e, f] = ef - fe = h$, hence it holds in *every* A-module

Instead of each separate computation, we compute *universally* in an abstract algebra

Each representation defines a module over the *universal enveloping algebra* $A = U\big(\mathfrak{sl}_2(\mathbb{R})\big)$

The identity

$$\boxed{\frac{1}{2}h^2 + ef + fe = \frac{1}{2}h^2 + h + 2fe}$$

holds in A, since $[e, f] = ef - fe = h$, hence it holds in *every* A-module

Many recent results about extended notions of symmetry in QFT: Apruzzi, Bah, Benini, Bhardwaj, Bonetti, Bullimore, Córdova, Choi, Cvetič, Del Zotto, Dumitrescu, Frölich, Fuchs, Gaiotto, García Etxebarria, Gould, Gukov, Heckman, Heidenreich, Hopkins, Hosseini, Hsin, Hübner, Intriligator, Ji, Jian, Johnson-Freyd, Jordan, Kaidi, Kapustin, Komargodski, Lake, Lam, McNamara, Minasian, Montero, Ohmari, Pantev, Pei, Plavnik, Reece, Robbins, Roumpedakis, Rudelius, Runkel, Schäfer-Nameki, Scheimbauer, Schweigert, Seiberg, Seifnashri, Shao, Sharpe, Tachikawa, Thorngren, Torres, Vandermeulen, Wang, Wen, Willett, . . . , . . . , . . .

Instead of each separate computation, we compute *universally* in an abstract algebra

Each representation defines a module over the *universal enveloping algebra* $A = U\big(\mathfrak{sl}_2(\mathbb{R})\big)$

The identity

$$\boxed{\frac{1}{2}h^2 + ef + fe = \frac{1}{2}h^2 + h + 2fe}$$

holds in A, since $[e, f] = ef - fe = h$, hence it holds in *every* A-module

Many recent results about extended notions of symmetry in QFT: Apruzzi, Bah, Benini, Bhardwaj, Bonetti, Bullimore, Córdova, Choi, Cvetič, Del Zotto, Dumitrescu, Frölich, Fuchs, Gaiotto, García Etxebarria, Gould, Gukov, Heckman, Heidenreich, Hopkins, Hosseini, Hsin, Hübner, Intriligator, Ji, Jian, Johnson-Freyd, Jordan, Kaidi, Kapustin, Komargodski, Lake, Lam, McNamara, Minasian, Montero, Ohmari, Pantev, Pei, Plavnik, Reece, Robbins, Roumpedakis, Rudelius, Runkel, Schäfer-Nameki, Scheimbauer, Schweigert, Seiberg, Seifnashri, Shao, Sharpe, Tachikawa, Thorngren, Torres, Vandermeulen, Wang, Wen, Willett, ..., ..., ...

Main idea: Make analogous universal computations with symmetries in QFT

Warning

The word ‘symmetry’ in mathematics usually refers to *groups* (”invertible symmetries”) rather than algebras (“noninvertible symmetries”), but in modern QFT-speak the term ‘symmetry’ is also used for the latter. Algebras of operators, including those that commute with a Hamiltonian, date from the earliest days of quantum mechanics

Motivation: algebras

Abstract symmetry data (for algebras) is a pair (A, R):

A algebra

R right regular module

Definition: Let V be a vector space. An (A, R)-*action on* V is a pair (L, θ) consisting of a left A-module L together with an isomorphism of vector spaces

$$\theta\colon R \otimes_A L \xrightarrow{\cong} V$$

Motivation: algebras

Abstract symmetry data (for algebras) is a pair (A, R):

A algebra

R right regular module

Definition: Let V be a vector space. An (A, R)-*action on* V is a pair (L, θ) consisting of a left A-module L together with an isomorphism of vector spaces

$$\theta\colon R \otimes_A L \xrightarrow{\cong} V$$

R allows us to recover the vector space underlying L—a bit pedantic here; crucial later

Motivation: algebras

Abstract symmetry data (for algebras) is a pair (A, R):

A algebra

R right regular module

Definition: Let V be a vector space. An (A, R)-*action on* V is a pair (L, θ) consisting of a left A-module L together with an isomorphism of vector spaces

$$\theta \colon R \otimes_A L \xrightarrow{\cong} V$$

R allows us to recover the vector space underlying L—a bit pedantic here; crucial later

Elements of A act on all modules; relations in A apply (e.g. Casimirs in $U(\mathfrak{sl}_2(\mathbb{R}))$)

Motivation: algebras

Abstract symmetry data (for algebras) is a pair (A, R):

A algebra

R right regular module

Definition: Let V be a vector space. An (A, R)-*action on* V is a pair (L, θ) consisting of a left A-module L together with an isomorphism of vector spaces

$$\theta \colon R \otimes_A L \xrightarrow{\cong} V$$

R allows us to recover the vector space underlying L—a bit pedantic here; crucial later

Elements of A act on all modules; relations in A apply (e.g. Casimirs in $U(\mathfrak{sl}_2(\mathbb{R}))$)

Analogy:

algebra $\rightsquigarrow$ topological field theory

element of algebra $\rightsquigarrow$ defect in TFT

Field theory

Analogy: field theory $\sim$ module over an algebra OR $\sim$ representation of a Lie group

Warning: This analogy is quite limited

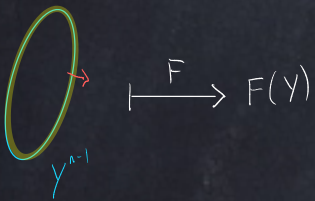
Field theory

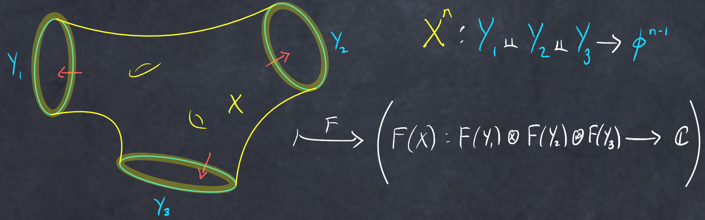
Analogy: field theory ∼ module over an algebra OR ∼ representation of a Lie group

Warning: This analogy is quite limited

Segal Axiom System: A (Wick-rotated) field theory F is a linear representation of a bordism (multi)category $\mathrm{Bord}_n(\mathcal{F})$

n	dimension of spacetime
$\mathcal{F}$	background fields (orientation, Riemannian metric, . . .)





Field theory

Analogy: field theory $\sim$ module over an algebra OR $\sim$ representation of a Lie group

Warning: This analogy is quite limited

Segal Axiom System: A (Wick-rotated) field theory F is a linear representation of a bordism (multi)category $\mathrm{Bord}_n(\mathcal{F})$

n	dimension of spacetime
$\mathcal{F}$	background fields (orientation, Riemannian metric, ...)

Fully local theory for *topological* theories; full locality in principle for general theories

Field theory

Analogy: field theory $\sim$ module over an algebra OR $\sim$ representation of a Lie group

Warning: This analogy is quite limited

Segal Axiom System: A (Wick-rotated) field theory F is a linear representation of a bordism (multi)category $\mathrm{Bord}_n(\mathcal{F})$

n	dimension of spacetime
$\mathcal{F}$	background fields (orientation, Riemannian metric, . . .)

Fully local theory for *topological* theories; full locality in principle for general theories

Kontsevich-Segal: Axioms for 2-tier nontopological theory $F\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{F}) \to t\,\mathrm{Vect}$

Domain walls, boundary theories, defects

$\sigma, \sigma_1, \sigma_2$	$(n+1)$-dimensional theories
$\delta\colon\ \sigma_1 \to \sigma_2$	domain wall
$\rho\colon\ \sigma \to \mathbb{1}$	right boundary theory
$\widetilde{F}\colon \mathbb{1} \to \sigma$	left boundary theory

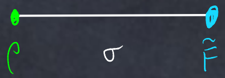
Domain walls, boundary theories, defects

$\sigma, \sigma_1, \sigma_2$	$(n+1)$-dimensional theories	
$\delta\colon\ \sigma_1 \to \sigma_2$	domain wall	(σ_2, σ_1)-bimodule
$\rho\colon\ \sigma \to \mathbb{1}$	right boundary theory	right σ-module
$\widetilde{F}\colon \mathbb{1} \to \sigma$	left boundary theory	left σ-module

Domain walls, boundary theories, defects

$\sigma, \sigma_1, \sigma_2$	$(n+1)$-dimensional theories	
$\delta\colon\ \sigma_1 \longrightarrow \sigma_2$	domain wall	(σ_2, σ_1)-bimodule
$\rho\colon\ \sigma \longrightarrow \mathbb{1}$	right boundary theory	right σ-module
$\widetilde{F}\colon \mathbb{1} \longrightarrow \sigma$	left boundary theory	left σ-module

The "sandwich" $\rho \otimes_\sigma \widetilde{F}$ is an (absolute) n-dimensional theory

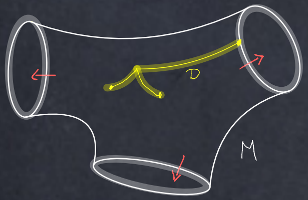


Domain walls, boundary theories, defects

$\sigma, \sigma_1, \sigma_2$	$(n+1)$-dimensional theories
$\delta\colon\ \sigma_1 \to \sigma_2$	domain wall
$\rho\colon\ \sigma \to \mathbb{1}$	right boundary theory
$\widetilde{F}\colon \mathbb{1} \to \sigma$	left boundary theory

The "sandwich" $\rho \otimes_\sigma \widetilde{F}$ is an (absolute) n-dimensional theory

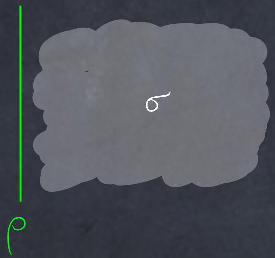
More generally, one can have *defects* supported on any (stratified) manifold $D \subset M$



Main definition: abstract symmetry data

Fix a dimension n and background fields $\mathcal{F}$ (which we keep implicit)

Definition: A *quiche* is a pair (σ, ρ) in which $\sigma\colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is an $(n+1)$-dimensional topological field theory and ρ is a right topological σ-module.



Main definition: abstract symmetry data

Fix a dimension n and background fields $\mathcal{F}$ (which we keep implicit)

Definition: A *quiche* is a pair (σ, ρ) in which $\sigma\colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is an $(n+1)$-dimensional topological field theory and ρ is a right topological σ-module.

Example: Let G be a finite group. Then for a G-symmetry we let σ be finite gauge theory in dimension $n+1$. Note this is the *quantum* theory which sums over principal G-bundles

Main definition: abstract symmetry data

Fix a dimension n and background fields $\mathcal{F}$ (which we keep implicit)

Definition: A *quiche* is a pair (σ, ρ) in which $\sigma\colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is an $(n+1)$-dimensional topological field theory and ρ is a right topological σ-module.

Example: Let G be a finite group. Then for a G-symmetry we let σ be finite gauge theory in dimension $n+1$. Note this is the *quantum* theory which sums over principal G-bundles

Regular ρ: Suppose $\mathcal{C}'$ is a symmetric monoidal n-category and σ is an $(n+1)$-dimensional topological field theory with codomain $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Let $A = \sigma(\mathrm{pt})$. Then A is an algebra in $\mathcal{C}'$ which, as an object in $\mathcal{C}$, is $(n+1)$-dualizable. Assume that the right regular module A_A is n-dualizable as a 1-morphism in $\mathcal{C}$. Then the boundary theory ρ determined by A_A is the *right regular boundary theory* of σ, or the *right regular σ-module.*

Main definition: abstract symmetry data

Fix a dimension n and background fields $\mathcal{F}$ (which we keep implicit)

Definition: A *quiche* is a pair (σ, ρ) in which $\sigma\colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is an $(n+1)$-dimensional topological field theory and ρ is a right topological σ-module.

Example: Let G be a finite group. Then for a G-symmetry we let σ be finite gauge theory in dimension $n+1$. Note this is the *quantum* theory which sums over principal G-bundles

Regular ρ: Suppose $\mathcal{C}'$ is a symmetric monoidal n-category and σ is an $(n+1)$-dimensional topological field theory with codomain $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$. Let $A = \sigma(\mathrm{pt})$. Then A is an algebra in $\mathcal{C}'$ which, as an object in $\mathcal{C}$, is $(n+1)$-dualizable. Assume that the right regular module A_A is n-dualizable as a 1-morphism in $\mathcal{C}$. Then the boundary theory ρ determined by A_A is the *right regular boundary theory* of σ, or the *right regular σ-module.*

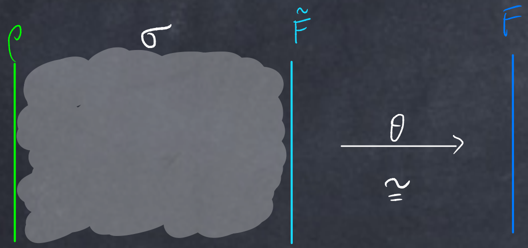
A regular boundary theory is also sometimes called *Dirichlet*

Main definition: concrete realization of symmetry

Definition: Let (σ, ρ) be an n-dimensional quiche. Let F be an n-dimensional field theory. A (σ, ρ)-*module structure* on F is a pair $(\widetilde{F}, \theta)$ in which $\widetilde{F}$ is a left σ-module and θ is an isomorphism

$$\theta \colon \rho \otimes_{\sigma} \widetilde{F} \xrightarrow{\cong} F$$

of absolute n-dimensional theories.



Main definition: concrete realization of symmetry

Definition: Let (σ, ρ) be an n-dimensional quiche. Let F be an n-dimensional field theory. A (σ, ρ)-*module structure* on F is a pair $(\widetilde{F}, \theta)$ in which $\widetilde{F}$ is a left σ-module and θ is an isomorphism

$$\theta \colon \rho \otimes_{\sigma} \widetilde{F} \xrightarrow{\cong} F$$

of absolute n-dimensional theories.

- The theory F and so the boundary theory $\widetilde{F}$ may be topological or nontopological

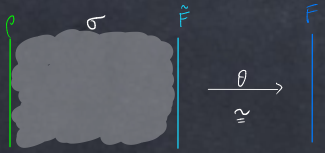
Main definition: concrete realization of symmetry

Definition: Let (σ, ρ) be an n-dimensional quiche. Let F be an n-dimensional field theory. A (σ, ρ)-*module structure* on F is a pair $(\widetilde{F}, \theta)$ in which $\widetilde{F}$ is a left σ-module and θ is an isomorphism

$$\theta \colon \rho \otimes_{\sigma} \widetilde{F} \xrightarrow{\cong} F$$

of absolute n-dimensional theories.

- The theory F and so the boundary theory $\widetilde{F}$ may be topological or nontopological
- The sandwich picture of F as $\rho \otimes_{\sigma} \widetilde{F}$ separates out the topological part (σ, ρ) of the theory from the potentially nontopological part $\widetilde{F}$ of the theory.



Main definition: concrete realization of symmetry

Definition: Let (σ, ρ) be an n-dimensional quiche. Let F be an n-dimensional field theory. A (σ, ρ)-*module structure* on F is a pair $(\widetilde{F}, \theta)$ in which $\widetilde{F}$ is a left σ-module and θ is an isomorphism

$$\theta\colon \rho \otimes_{\sigma} \widetilde{F} \xrightarrow{\cong} F$$

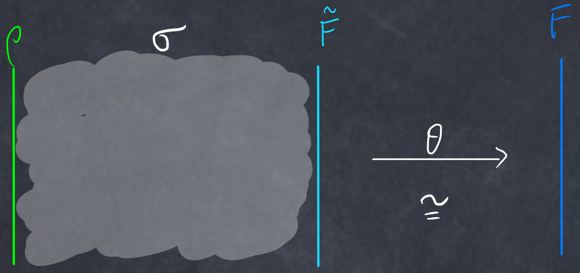
of absolute n-dimensional theories.

- The theory F and so the boundary theory $\widetilde{F}$ may be topological or nontopological
- The sandwich picture of F as $\rho \otimes_{\sigma} \widetilde{F}$ separates out the topological part (σ, ρ) of the theory from the potentially nontopological part $\widetilde{F}$ of the theory.
- Symmetry persists under renormalization group flow, hence a low energy approximation to F should also be an (σ, ρ)-module. If F is gapped, then we can bring to bear powerful methods and theorems in topological field theory to investigate *topological* left σ-modules. This leads to dynamical predictions

Example: quantum mechanics with G-symmetry

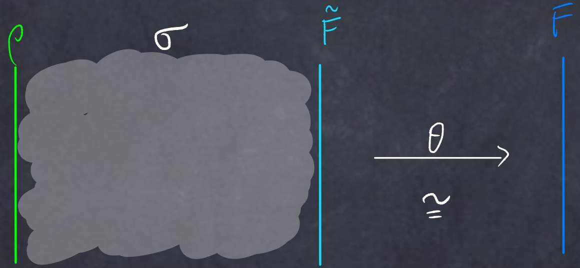
$n = 1$

$\mathcal{F}$	{orientation, Riemannian metric} for F and $\widetilde{F}$
$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group
$S\colon G \to \mathrm{Aut}(\mathcal{H})$	action on $\mathcal{H}$
$\sigma(\mathrm{pt})$	$\mathbb{C}[G]$
$F(\mathrm{pt})$	$\mathcal{H}$
$\widetilde{F}(\mathrm{pt})$	${}_{\mathbb{C}[G]}\mathcal{H}$ (left module)



Example: gauge theory with BA-symmetry

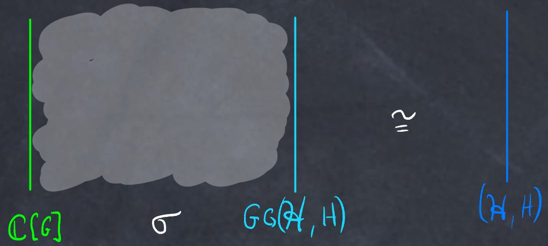
n	any dimension
A	finite abelian group $A = \mu_2$
BA	a homotopical/shifted A ("1-form A-symmetry")
H	Lie group with $A \subset Z(H)$ $H = \mathrm{SU}_2$
$\overline{H} = H/A$	$\overline{H} = \mathrm{SO}_3$
F	H-gauge theory
$\widetilde{F}$	$\overline{H}$-gauge theory



Defects: quantum mechanics

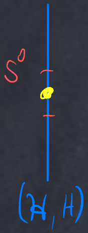
$n = 1$

$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group



Defects: quantum mechanics

$n = 1$	
$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group



Consider a point defect in F. The link of a point in a 1-manifold (imaginary time) is S^0, a 0-sphere of radius ϵ, and the vector space of defects is

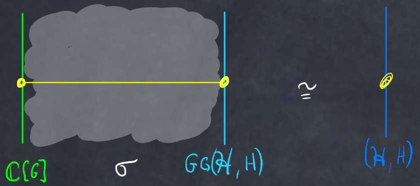
$$\varprojlim_{\epsilon \to 0} \operatorname{Hom}\big(1, F(S^0_\epsilon)\big)$$

which is a space of singular operators on $\mathcal{H}$. To focus on formal aspects we write '$\operatorname{End}(\mathcal{H})$'

Defects: quantum mechanics

$n = 1$

$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group



Consider a point defect in F. The link of a point in a 1-manifold (imaginary time) is S^0, a 0-sphere of radius ϵ, and the vector space of defects is

$$\varprojlim_{\epsilon \longrightarrow 0} \operatorname{Hom}\big(1, F(S^0_\epsilon)\big)$$

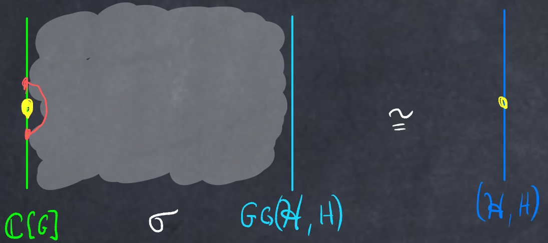
which is a space of singular operators on $\mathcal{H}$. To focus on formal aspects we write 'End$(\mathcal{H})$'

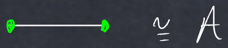
We now consider defects in $(\rho, \sigma, \widetilde{F})$ which transport to point defects in F

Point ρ-defects

The link is a closed interval with ρ-colored boundary. It evaluates under (σ, ρ) to the *vector space* $A = \mathbb{C}[G]$. The "label" of the defect is therefore an element of A. Note $G \subset A$ labels invertible defects.

ρ-defects are topological

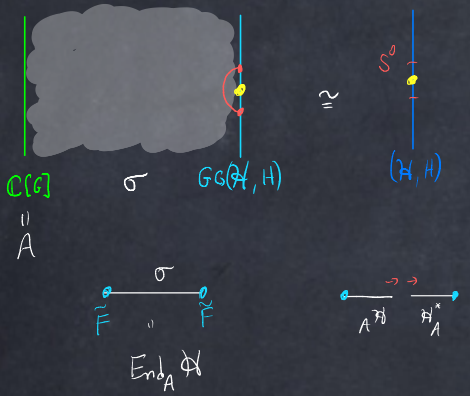




Point $\widetilde{F}$-defects

The link is again a closed interval, but now with $\widetilde{F}$-colored boundary. The value under $(\sigma, \widetilde{F})$ is $\mathrm{End}_A(\mathcal{H})$, the space of observables that commute with the G-action

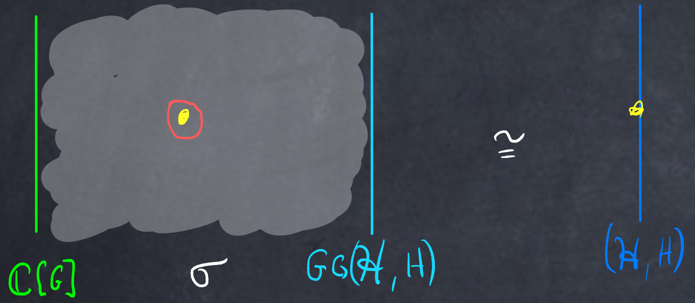
$\widetilde{F}$-defects are typically not topological



Point σ-defects: central defects

The link is S^1, and the value under σ is the vector space which is the center of the group algebra $A = \mathbb{C}[G]$.

σ-defects are topological

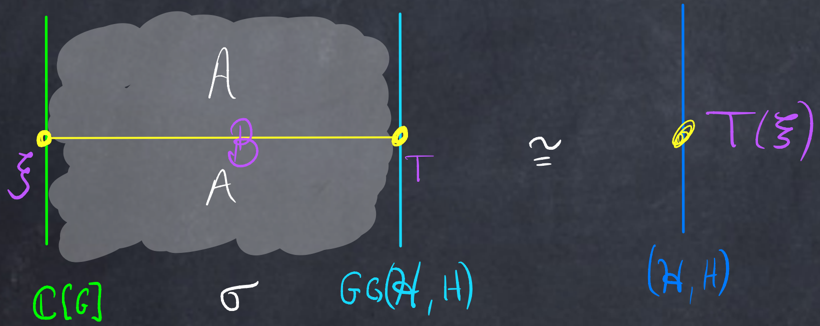


The general point defect

A general point defect in F can be realized by a line defect in $(\rho, \sigma, \widetilde{F})$.

Label the defect beginning with the highest dimensional strata and work down in dimension

B	(A, A)-bimodule
ξ	vector in B
T	(A, A)-bimodule map $B \longrightarrow \mathrm{End}(\mathcal{H})$

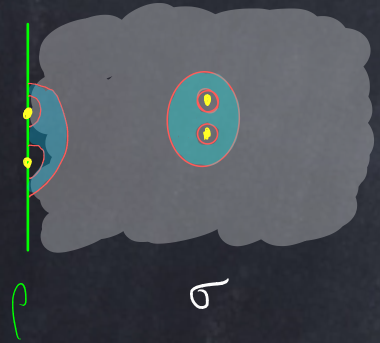


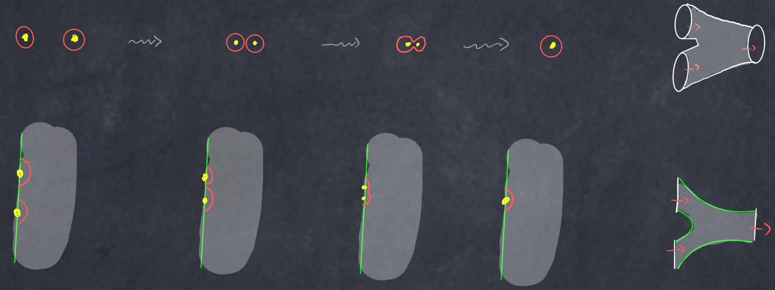
Composition law on defects

Compute using the links of the defects—2 incoming and 1 outgoing

σ-defects: pair of pants

ρ-defects: pair of chaps





Commutation relations among defects

The sandwich realization makes clear that

- ρ-defects (symmetries) commute with $\widetilde{F}$-defects
- σ-defects (central symmetries) commute with both ρ-defects and with $\widetilde{F}$-defects

Commutation relations among defects

The sandwich realization makes clear that

- ρ-defects (symmetries) commute with $\widetilde{F}$-defects
- σ-defects (central symmetries) commute with both ρ-defects and with $\widetilde{F}$-defects

However, ρ-defects do not necessarily commute with each other

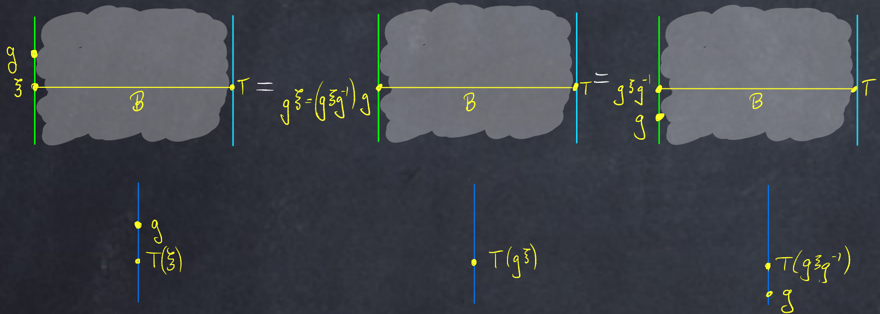
Commutation relations among defects

The sandwich realization makes clear that

- ρ-defects (symmetries) commute with $\widetilde{F}$-defects
- σ-defects (central symmetries) commute with both ρ-defects and with $\widetilde{F}$-defects

However, ρ-defects do not necessarily commute with each other

Nor do they commute with the general defect



Quotients: augmentations

Definition: An *augmentation* of an algebra A is an algebra homomorphism $\epsilon\colon A \to \mathbb{C}$.

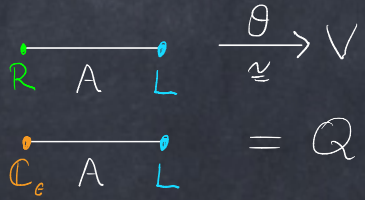
Use ϵ to give a right A-module structure to $\mathbb{C}$: $\lambda \cdot a = \lambda\epsilon(a)$, $\lambda \in \mathbb{C}$

Example: $A = \mathbb{C}[G]$:

$$\epsilon\colon \mathbb{C}[G] \longrightarrow \mathbb{C}$$
$$\sum_{g\in G} \lambda_g g \longmapsto \sum_{g\in G} \lambda_g$$

The "quotient" of a left A-module L is the vector space

$$Q = \mathbb{C} \otimes_A L = \mathbb{C} \otimes_\epsilon L$$



Quotients: augmentations

Definition: An *augmentation* of an algebra A is an algebra homomorphism $\epsilon \colon A \to \mathbb{C}$.

Use ϵ to give a right A-module structure to $\mathbb{C}$: $\lambda \cdot a = \lambda\epsilon(a)$, $\lambda \in \mathbb{C}$

Example: $A = \mathbb{C}[G]$:

$$\epsilon \colon \mathbb{C}[G] \longrightarrow \mathbb{C}$$
$$\sum_{g \in G} \lambda_g g \longmapsto \sum_{g \in G} \lambda_g$$

The "quotient" of a left A-module L is the vector space

$$Q = \mathbb{C} \otimes_A L = \mathbb{C} \otimes_\epsilon L$$

Example: $A = \mathbb{C}[G]$, S a finite G-set, $L = \mathbb{C}\langle S \rangle$: then $Q = \mathbb{C} \otimes_A \mathbb{C}\langle S \rangle \cong \mathbb{C}\langle S/G \rangle$

Quotients and quotient defects

We use the yoga of fully local topological field theory: let $\mathcal{C}'$ be a symmetric monoidal n-category and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$, the $(n+1)$-category whose objects are algebras in $\mathcal{C}'$

Definition: An *augmentation* $\epsilon_A : A \to 1$ of an algebra $A \in \mathrm{Alg}(\mathcal{C}')$ is an algebra homomorphism from A to the tensor unit $1 \in \mathcal{C}$

Definition: Let $\mathcal{F}$ be a collection of $(n+1)$-dimensional fields, and suppose $\sigma : \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is a topological field theory. A right boundary theory ϵ for σ is an *augmentation* of σ if $\epsilon(\mathrm{pt})$ is an augmentation of $\sigma(\mathrm{pt})$

Augmentations are also called *Neumann boundary theories*

Quotients and quotient defects

We use the yoga of fully local topological field theory: let $\mathcal{C}'$ be a symmetric monoidal n-category and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$, the $(n+1)$-category whose objects are algebras in $\mathcal{C}'$

Definition: An *augmentation* $\epsilon_A : A \to 1$ of an algebra $A \in \mathrm{Alg}(\mathcal{C}')$ is an algebra homomorphism from A to the tensor unit $1 \in \mathcal{C}$

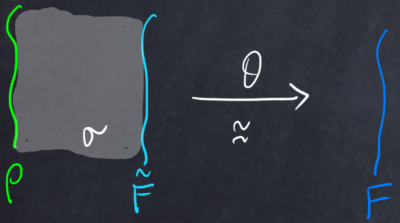
Definition: Let $\mathcal{F}$ be a collection of $(n+1)$-dimensional fields, and suppose $\sigma\colon \mathrm{Bord}_{n+1}(\mathcal{F}) \to \mathcal{C}$ is a topological field theory. A right boundary theory ϵ for σ is an *augmentation* of σ if $\epsilon(\mathrm{pt})$ is an augmentation of $\sigma(\mathrm{pt})$

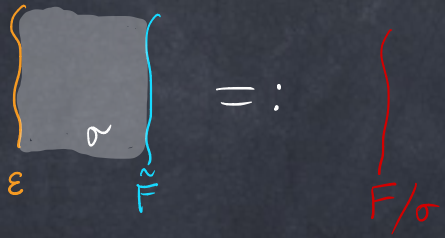
Augmentations are also called *Neumann boundary theories*

Augmentations do not always exist

Definition: Suppose given finite symmetry data (σ, ρ) and a (σ, ρ)-module structure $(\widetilde{F}, \theta)$ on a quantum field theory F. Suppose ϵ is an augmentation of σ. Then the *quotient* of F by the symmetry σ is

$$F \underset{\epsilon}{/} \sigma = \epsilon \otimes_\sigma \widetilde{F}$$





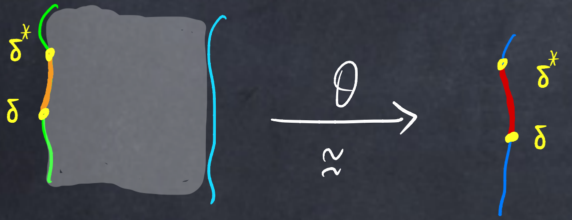
Dirichlet-to-Neumann and Neumann-to-Dirichlet domain walls

The categories of domain walls $\rho \to \epsilon$ and $\epsilon \to \rho$ are each free of rank one; let

$$\delta : \rho \longrightarrow \epsilon$$
$$\delta^* : \epsilon \longrightarrow \rho$$

be generators. Transporting via θ we obtain domain walls

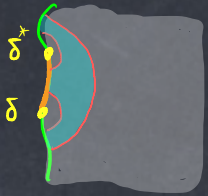
$$\delta : F \longrightarrow F/\sigma$$
$$\delta^* : F/\sigma \longrightarrow F$$



Dirichlet-to-Neumann and Neumann-to-Dirichlet domain walls

The categories of domain walls $\rho \to \epsilon$ and $\epsilon \to \rho$ are each free of rank one; let

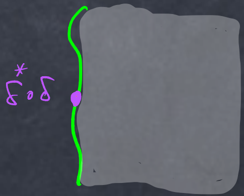
$$\delta : \rho \longrightarrow \epsilon$$
$$\delta^* : \epsilon \longrightarrow \rho$$



be generators. Transporting via θ we obtain domain walls

$$\delta : F \longrightarrow F/\sigma$$
$$\delta^* : F/\sigma \longrightarrow F$$

We will soon compute the self-domain wall

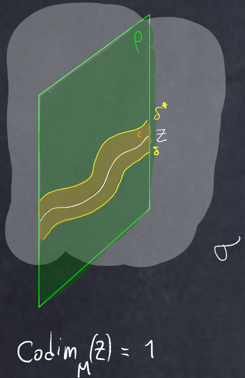


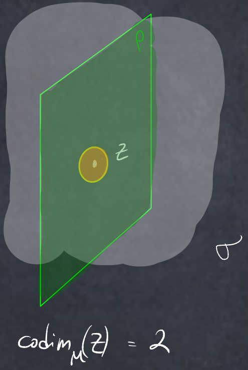
$$\delta^* \circ \delta : F \longrightarrow F$$

Quotient defects (after Roumpedakis–Seifnashri–Shao arXiv:2204.02407)

Passing from F to F/σ on a manifold M places the topological defect ϵ on all of M

There is also a *quotient defect* $\epsilon(Z)$—it is a ρ-defect—supported on a submanifold $Z \subset M$, defined using a tubular neighborhood ν of $Z \subset M$. It is *topological*, as are all ρ-defects





Quotient defects (after Roumpedakis–Seifnashri–Shao arXiv:2204.02407)

Passing from F to F/σ on a manifold M places the topological defect ϵ on all of M

There is also a *quotient defect* $\epsilon(Z)$—it is a ρ-defect—supported on a submanifold $Z \subset M$, defined using a tubular neighborhood ν of $Z \subset M$. It is *topological*, as are all ρ-defects

If $\mathrm{codim}_M(Z) = 1$, then

$$\epsilon(Z) = \delta^* \circ \delta$$

Computation for finite homotopy theories

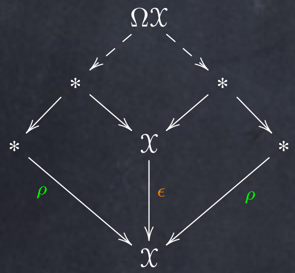
Finite homotopy theories are a special class of topological field theories, introduced in 1988 by Kontsevich, picked up a few years later by Quinn, developed by Turaev, . . .

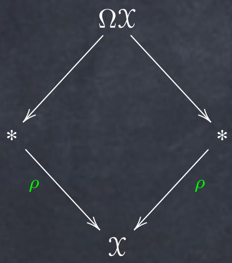
They are associated to a π-*finite* topological space $\mathcal{X}$ (possibly equipped with a “cocycle”)

They occur often in this context as $\sigma = \sigma_{\mathcal{X}}$, e.g., for $\mathcal{X} = BG$ or $\mathcal{X} = B^{p+1}A$ or extensions

$$B^2A \longrightarrow \mathcal{X} \longrightarrow BG$$

Defects—in particular quotient defects—can be made explicit and computations are easy. Here is the composition $\delta^* \circ \delta$, essentially a finite homotopy theory based on $\Omega\mathcal{X}$:





Duality defects (after Córdova–Choi–Hsin–Lam–Shao `arXiv:2111.01139`)

I conclude with an application—symmetry used to constrain dynamics via:

> *If a gapped theory F_{UV} has a (σ,ρ)-module structure, then the low energy topological field theory approximation F_{IR} should also have a (σ,ρ)-module structure*

$$\begin{array}{c} (\sigma,\rho) \circlearrowright F_{\mathrm{UV}} \\ \rightsquigarrow \text{ RG flow} \\ (\sigma,\rho) \circlearrowright F_{\mathrm{IR}} \end{array}$$

We will prove in a particular example that there does not exist a *topological* left σ-module $\tilde{\lambda}$ such that $\lambda := \rho \otimes_\sigma \tilde{\lambda}$ is invertible. Therefore, F_{UV} cannot flow to an invertible field theory, i.e., is not "trivially gapped"

Duality defect

σ	$n+1$-dimensional topological field theory
ρ	right regular σ-module
ϵ	augmentation of σ: "invertible" right σ-module
$\widetilde{F}$	left σ-module
F	n-dimensional QFT $\rho \otimes_\sigma \widetilde{F}$
F/σ	n-dimensional QFT $\epsilon \otimes_\sigma \widetilde{F}$

Suppose there is an isomorphism $\phi\colon F/\sigma \xrightarrow{\cong} F$. Recall $\delta\colon F \to F/\sigma$

Definition: The *duality defect* Δ is the self-domain wall

$$\Delta = \phi \circ \delta\colon F \longrightarrow F$$

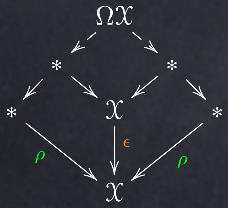
Computation: $\Delta^* \circ \Delta = (\phi\delta)^*(\phi\delta) = \delta^*\phi^*\phi\delta = \delta^* \circ \delta$ since $\phi^* = \phi^{-1}$ (ϕ is invertible)

Example

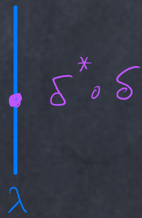
Let $n = 4$ and let σ be the 5-dimensional finite homotopy theory built from $\mathcal{X} = B^2/\!\!/\mu_2$

This models $B/\!\!/\mu_2$-symmetry ("1-form symmetry")

Recall that ρ, ϵ, δ, and δ^* and the composition $\delta^* \circ \delta$ fit into the diagram



$\delta^* \circ \delta$ is roughly 3-dimensional $/\!\!/\mu_2$-gauge theory



In an invertible (σ, ρ)-module λ, the self-domain wall $\delta^* \circ \delta$ is multiplication by 3-dimensional $/\!\!/\mu_2$-gauge theory

Now suppose F is a 4d QFT with a left (σ, ρ)-structure, and assume given an isomorphism

$$\phi \colon F/\sigma \xrightarrow{\;\cong\;} F$$

Example: F is U_1 gauge theory with coupling constant τ
F has $B\mu_2$ symmetry from $\mu_2 \subset U_1$
F/σ is U_1 gauge theory with coupling constant $\tau/4$
ϕ is S-duality which sends $\tau \mapsto -1/\tau$
Set $\tau = 2\sqrt{-1}$

We do not use details of the gauge theory beyond its $B\mu_2$ symmetry

Then the duality defect $\Delta \colon F \to F$ is a "square root" of 3d μ_2 gauge theory

Theorem: No such square root exists

Conclusion: The gauge theory F is not trivially gapped

Happy Birthday, Hirosi!