Unraveling Turbulence: Modern Viewpoints On An Unsolved Problem

HirosiFest (October 2022)

A Simple Model

Hirosi is a Quantum Particle

Outline

Leonardo da Vinci (1452-1519)

Turbulence

Turba is a Latin word for crowd. Turbulence originally refers to the disorderly motion of a crowd. Scientifically it refers to a complex and unpredictable motion of a fluid.

Turbulence

- Fluid turbulence: "You know it when you see it".
- Emergent complex structure from simple rules (Newton's Second Law).

Mathematical Framework

- The incompressible Navier-Stokes (NS) equations (1822) provide a mathematical formulation of the fluid flow evolution at low Mach number:

$$\partial_t v^i + v^j \partial_j v^i = -\partial^i p + \nu \partial_{jj} v^i + F^i, \qquad \partial_i v^i = 0 \qquad (1)$$

- $v^i, i = 1, ..., d$ is the fluid velocity and p is the fluid pressure, ν is the kinematic viscosity and F^i is an external random force.

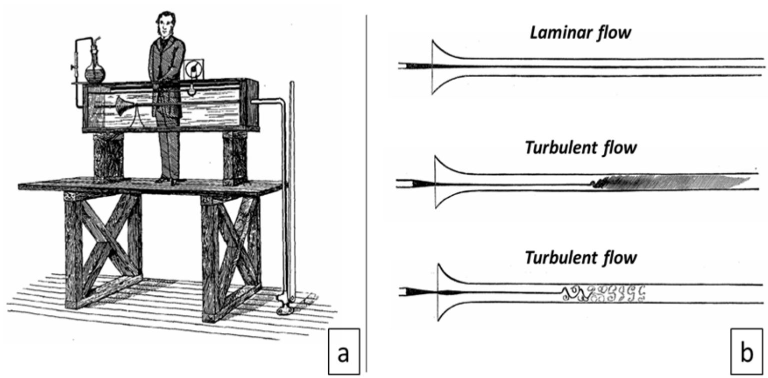
Reynolds Number

- Reynolds number (1883):

$$\mathcal{R}_e = \frac{lv}{\nu} \tag{2}$$

  where l is a characteristic length scale, v is the velocity difference at that scale, and ν is the kinematic viscosity.
- When the Reynolds number is of order 10^3 or more, one observes numerically and experimentally a turbulent structure of the flow.

Transition to Turbulence



(Source: Wikipedia)

Turbulence in Nature

Most flows in nature are turbulent: the kinematic viscosity of water is $\nu \simeq 10^{-6} \frac{m^2}{sec}$ and that of air is $\nu \simeq 1.5 \times 10^{-5} \frac{m^2}{sec}$. A medium size river has a Reynolds number $\mathcal{R}_e \sim 10^7$

Statistical Turbulence

- One defines the inertial range to be the range of distance scales $l \ll r \ll L$, where the scales l and L are determined by the viscosity and forcing, respectively.
- Numerical and experimental data show that the statistical average properties exhibit a universal structure shared by all turbulent flows, independently of the details of the flow excitations.

Fluid Observables

- Define the longitudinal velocity difference between points separated by a fixed distance $r = |\vec{r}|$

$$\delta v(r) = \left(\vec{v}(\vec{r}, t) - \vec{v}(0, t)\right) \cdot \frac{\vec{r}}{r} \tag{3}$$

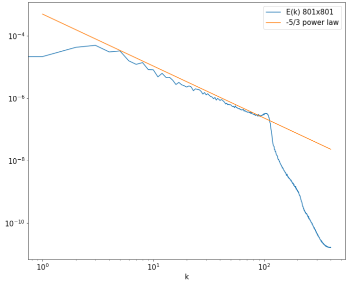
- The structure functions exhibit in the inertial range a scaling

$$S_n(r) = \langle (\delta v(r))^n \rangle \sim r^{\xi_n} \tag{4}$$

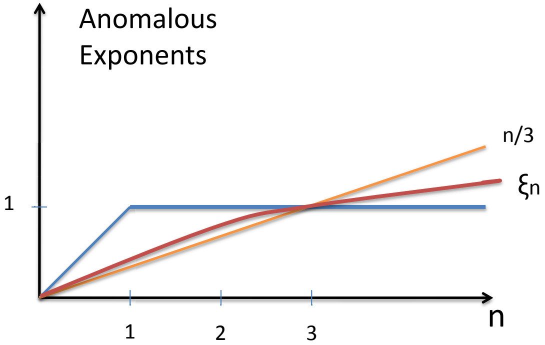
K41 Theory

$$S_n(r) \equiv \left\langle \left((\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{r}}{r} \right)^n \right\rangle \sim r^{\frac{n}{3}} \tag{5}$$

$$E(k) \sim k^{-5/3} \tag{6}$$



Anomalous Scaling



Field Theory

- We derive under certain assumptions an exact formula for the anomalous scalings ξ_n

$$\xi_n - \frac{n}{3} = \mathcal{G}^2(d)\xi_n(1 - \xi_n) \tag{7}$$

- $\mathcal{G}(d)$ is a numerical real parameter that depends on the number of space dimensions.

C. Eling, Y.O. JHEP **1509** (2015) 150
Y.O. JHEP **1711** (2017) 040, Eur.Phys.J. **C78** (2018) no.8, 655, arXiv:1809.10003 (Jacob Bekenstein: The Conservative Revolutionary)

Experimental Data: Three Space Dimensions

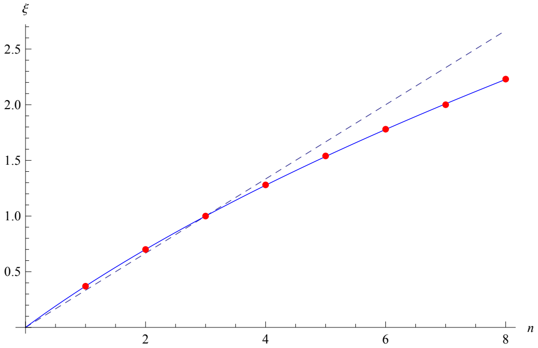


Figure: The dashed line represents Kolmogorov scaling. The best fit value of the free parameter $\mathcal{G}^2$ is about 0.161. The error on the data is about ± 1 percent (Benzi et.al. 1995).

Numerical Data: Three Space Dimensions

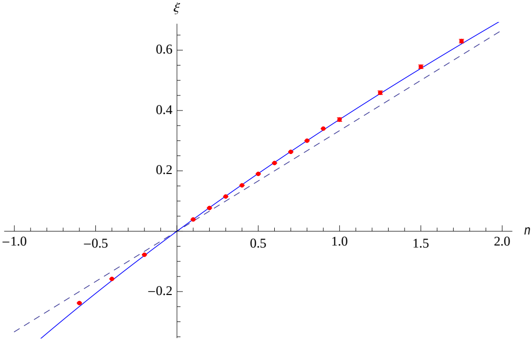


Figure: Fit to numerical data of numerical low moments (Chen et.al 2005). The dashed line represents Kolmogorov scaling. The best fit value of the free parameter $\mathcal{G}^2$ is about 0.159.

Random Geometry

- Formula (**??**) is (Knizhnik-Polyakov-Zamolodchikov)-type relation (KPZ) that arises when coupling a dynamical system to a random geometry (1988):

$$d\mu_\gamma(x) \sim e^{\gamma\phi(x)} d\mu \tag{8}$$

- The Gaussian random field $\phi(x)$ has covariance $\phi(x)\phi(y) \sim -\log|x-y|$.

Scale Symmetry Breaking

- In the absence of a viscosity term, the (inviscid) NS equations (**??**) exhibit two scale symmetries of space and time:

$$x^i \to e^{\sigma} x^i \;, \qquad t \to e^{z\sigma} t \tag{9}$$

- The local energy dissipation $\epsilon(x) = \frac{\nu}{2}\left(\partial_i v^j + \partial_j v^i\right)^2$ (alternatively the flux) breaks spontaneously the symmetries of the inviscid NS equations to $z = \frac{2}{3}$:

$$\Delta_{K41}[v^i] = \frac{1}{3} \tag{10}$$

Inertial Range Dilaton

- The dilaton $\tau(x)$ is the fluctuation:

$$\epsilon(x) = \bar{\epsilon} e^{\delta \tau(x)} \tag{11}$$

- The dilaton action reads:

$$S_D(\tau, \hat{g}) = \frac{d}{\Omega_d (d-1)!} \int_M d^d x \sqrt{\hat{g}} \left(\tau \mathcal{P}_{\hat{g}} \tau + 2Q \mathcal{Q}_{\hat{g}} \tau \right) \tag{12}$$

- T. Levy, Y.O. JHEP **1806** (2018) 119, I. Hason, arXiv:1708.08294, T. Levy, Y.O., A. Raviv-Moshe JHEP **1812** (2018) 122, JHEP **1910** (2019) 006, A. Kislev, T. Levy, Y.O. JHEP **7** (2022) 1.

Dilaton Field Theory

- $\mathcal{P}_{\hat{g}}$ are the conformally covariant operators (GJMS 92):

$$\mathcal{P}_{\hat{g}} = (-\Delta)^{\frac{d}{2}} + \textit{lower order} \tag{13}$$

- $\mathcal{Q}_{\hat{g}}$ is the $\mathcal{Q}$-curvature scalar (Branson 91):

$$\mathcal{Q}_{\hat{g}} = \frac{1}{2(d-1)}(-\Box)^{\frac{d}{2}-1}R + \ldots \tag{14}$$

Dilaton Dressing

- The operators in the theory are K41 operators O_{K41} dressed by a dilaton factor:

$$O(x) = e^{d\alpha\tau} O_{K41}(x), \qquad \alpha = \gamma(1 - \Delta) \tag{15}$$

  where $d\Delta_{K41}$ is the undressed dimension of O_{K41}.
- We get the KPZ equation:

$$\Delta - \Delta_{K41} = \frac{\gamma^2}{2}\Delta(1 - \Delta) \tag{16}$$

Trace Anomaly

Requiring that the inertial range universal structure and in particular the anomalous scalings should not depend on the forcing scale L:

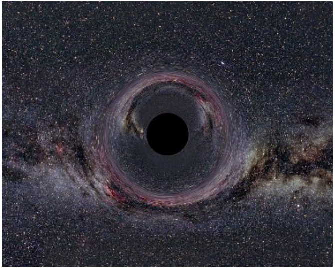
$$a_{total} = a_{dilaton} + a_{K41} = 0 \tag{17}$$

and

$$\mathcal{G}^2(d) \simeq \frac{2}{\Omega_d (d-1)! |a_{K41}(d)|} \tag{18}$$

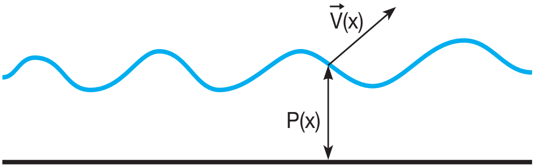
Black Hole Dynamics

Black hole horizon: fields can fall into the black hole but cannot emerge, this breaks time reversal symmetry and allows Einstein equations to describe dissipative effects.



The Fluid Variables

- The dynamics of the event horizon is described by the Navier-Stokes equations (Damour (82), Bhattacharyya, Hubeny, Minwalla and Rangamani (08), Eling, Fouxon, Y.O. (09)).
- The fluid variables in the geometrical picture :



Local Energy Dissipation

- The local energy dissipation:

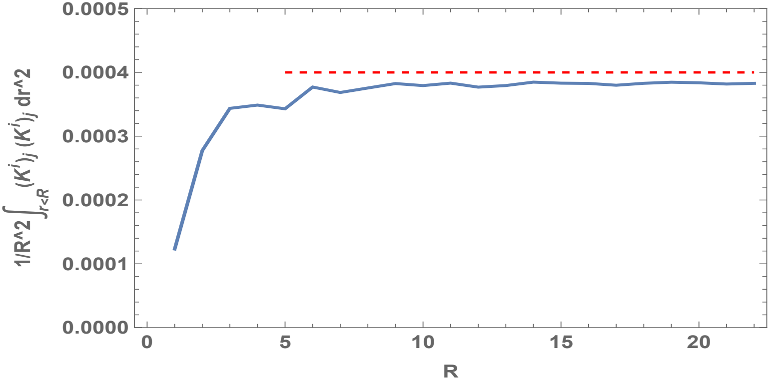
$$\epsilon(x) = \frac{\nu}{2}\left(\partial_i v^j + \partial_j v^i\right)^2 \ . \tag{19}$$

- The ensemble average of

$$\epsilon_r(x) = \frac{1}{Vol(B_d(r))}\int_{|x'-x|\leq r} d^d x' \epsilon(x') \ , \tag{20}$$

is independent of x by isotropy and of r by K41 scaling.

Horizon Calculation



With S. Waeber and A. Yarom (in peparation)

Machine Learning of Fluid Flows

- Consider a non-linear PDE:

$$\partial_t \vec{v}(\vec{x}, t) = \mathcal{L}\vec{v}(\vec{x}, t) \tag{21}$$

- A neural network evolves velocity fields, $\vec{v}(\vec{x}, t = 0)$ to a fixed time T

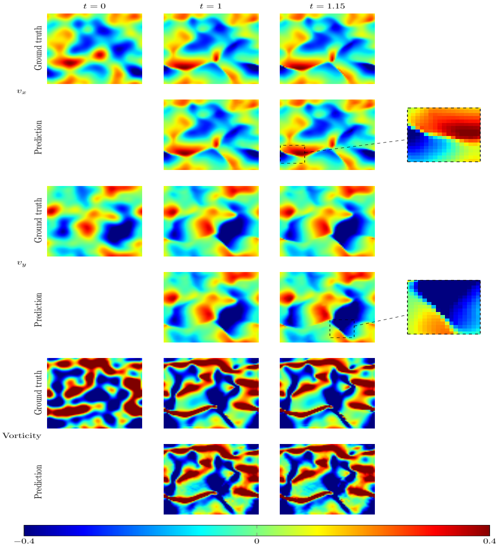
$$\Phi_T \vec{v}\left(\vec{x}, t = 0\right) = \vec{v}\left(\vec{x}, T\right) \tag{22}$$

  by learning from a set of $i = 1 \dots N$ initial conditions sampled at $t = 0$, $\vec{v}_i\left(\vec{x}, t = 0\right)$, and their corresponding time-evolved solutions of $\vec{v}_i\left(\vec{x}, t = T\right)$.

- We generalized Φ_T, to propagate solutions at intermediate times, $0 \leq t \leq T$.

With R. Rotman, A. Dekel, R. Ber, L. Wolf (arXiv:2207.14366)

Machine Learning of Fluid Flows

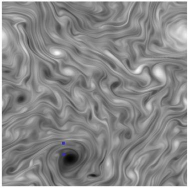
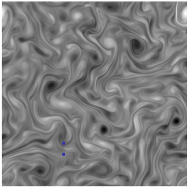
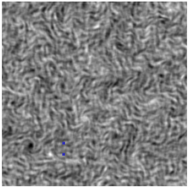
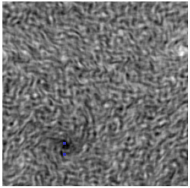


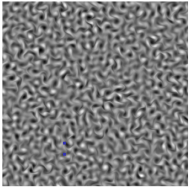
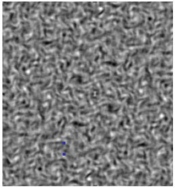
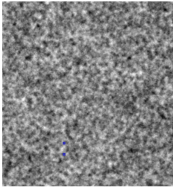
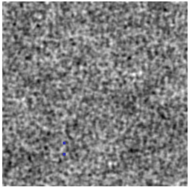
Machine Learning Complexity

- We train neural networks to distinguish turbulence fluid configurations from chaotic ones, noise and real world images.
- What is the relative complexity of the various classification tasks involving turbulence?
- How does the pattern of complexity change with depth as we go inside the neural network? How does it compare with classifying real world images?
- Can we understand what features the neural network uses to distinguish chaos from turbulence?

With R. Janik and T. Whittaker (in preparation)
R. Janik and P. Witaszczyk (effective dimension)

Images

Turbulence vs. Real World Images

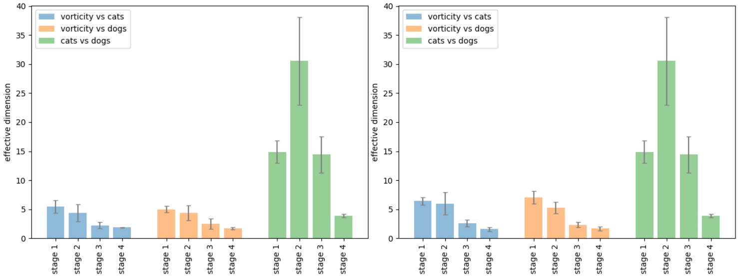


Figure: Left panel shows effective dimensions for images of weakly compressible turbulence vorticity vs. cats and dogs as well as for classifying between cats and dogs. Right panel shows the incompressible case.

Turbulence vs. Chaos

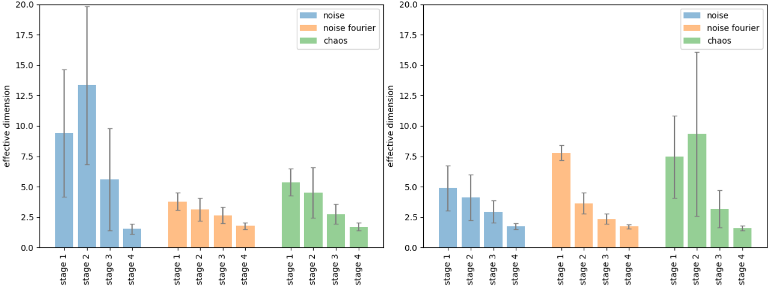


Figure: Effective dimensions for classifying weakly compressible turbulence vorticity (left) and incompressible turbulence vorticity (right).

Happy Birthday Hirosi