Symmetries and Anomalies on the Lattice

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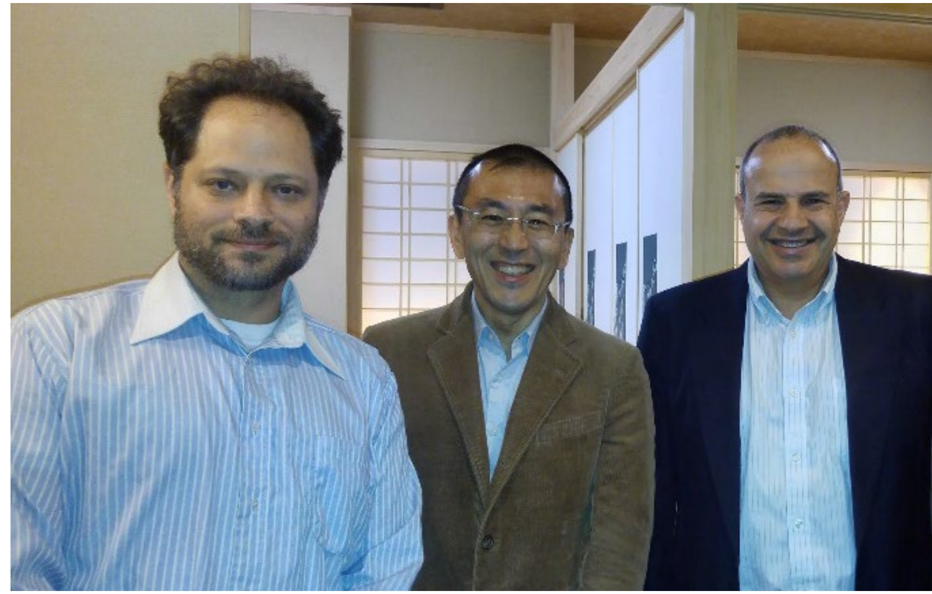
IAS

Happy Birthday, Hirosi

Thanks for the wonderful friendship

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Pranay Gorantla, Ho Tat Lam, NS, and Shu-Heng Shao, 2103.01257

Meng Cheng and NS, to appear

What are anomalies?

A global symmetry has an ‘t Hooft anomaly when it cannot be coupled to classical gauge fields.

In some (not all) textbooks:

- Anomalies are associated with fermions
- Anomalies are associated with divergences (need an infinite number of degrees of freedom)
- Anomalies signal an inconsistency of the theory. The theory makes sense only on the boundary of a bulk theory with a Symmetry Protected Topological Phase (SPT).

The simplest anomaly

Consider a two-level system – a qubit – with vanishing Hamiltonian.

Operators are 2×2 matrices.

The global symmetry acts by conjugation. It is $SO(3)$.

It is realized projectively on the Hilbert space, which is in an $SU(2)$ doublet [Wigner (1931)].

Therefore, we cannot couple it to classical $SO(3)$ gauge fields.

No need for an infinite number of degrees of freedom – unrelated to divergences.

No need for a bulk.

More interesting example in a free 1+1d field theory

$$S = \frac{\beta}{2}\int dtdx\left(\partial_\mu\phi\right)^2$$

- Global symmetries $\partial^\mu J_\mu = 0$
  - $U(1)^m$ charge, momentum $J_\mu^m = \beta\partial_\mu\phi$
  - $U(1)^w$ vorticity, winding $J_\mu^w = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu\phi$
  - Anomaly:
    - The charges commute, but $[j^w, j^m] \neq 0$.
    - Defects carry non-quantized charges (more below).
- Exact self-duality (T-duality) maps $\beta \leftrightarrow \frac{1}{(2\pi)^2\beta}$ and exchanges momentum and winding.
  - In string theory it is common to use $R = \sqrt{2\pi\beta}$ $\quad(R \leftrightarrow \frac{1}{R})$

Lattice formulation

[…; Jose, Kadanoff, Kirkpatrick, Nelson (1977); …]

Euclidean-time lattice, Lagrangian formulation (or equivalently, classical statistical mechanics):

$$S = -\beta \sum_{links} \cos(\Delta\phi)$$

Continuous Lorentzian time, Hamiltonian formulation:

$$H = \frac{1}{2\beta} \sum_{sites} p^2 - \frac{\beta}{2} \sum_{links} \cos(\Delta\phi)$$

Only the momentum symmetry $\phi(x,\tau) \to \phi(x,\tau) + \alpha$

No winding symmetry and hence, no anomaly.

For large β, it flows to the continuum theory.

For smaller β, vortices proliferate and the system is gapped [Berezinskii (1971); Kosterlitz, Thouless (1973)].

Euclidean lattice formulation – suppress the vortices [...; Gross, Klebanov; Sulejmanpasic, Gattringer; Gorantla, Lam, NS, Shao]

Use the Villain formulation

$$S_{Villain} = \frac{\beta}{2} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2$$

Here $\phi \in \mathbb{R}$, $n_\mu \in \mathbb{Z}$ with the $\mathbb{Z}$ gauge symmetry (effectively making ϕ compact)

$$\phi \sim \phi + 2\pi\, m$$
$$n_\mu \sim n_\mu + \Delta_\mu m$$

Suppress the vortices by constraining $\Delta_\tau n_x - \Delta_x n_\tau = 0$.

Both momentum and winding symmetries. T-duality.

Can study the anomaly on the lattice.

We will not discuss it here.

Hamiltonian lattice formulation of the Villain model

Following [Kogut, Susskind (1975)], we represent the $\mathbb{Z}$ gauge field of the Villain model using an integer spatial gauge field n and a circle-valued electric field E on the links.

$$H_{Villain} = \frac{1}{2\beta}\sum_{sites} p^2 + \frac{\beta}{2}\sum_{links}(\Delta\phi - 2\pi n)^2 - g^2 \sum_{links} \cos E$$

$$[\phi_j, p_{j'}] = i\delta_{j,j'} \qquad [n_j, E_{j'}] = i\delta_{j,j'}$$

Gauss law at every point

$$\exp(i\Delta E - 2\pi i\, p) = 1$$

Suppress the vortices by setting $g = 0$.

Hamiltonian lattice formulation without vortices

$$H_{modified\ Villain} = \frac{1}{2\beta}\sum_{sites} p^2 + \frac{\beta}{2}\sum_{links}(\Delta\phi - 2\pi n)^2$$

$$\exp(i\Delta E - 2\pi i\, p) = 1$$

- Global symmetries
  - $U(1)^m$ momentum $\quad Q^m = \sum_{sites} J^m, \quad J^m = p$
  - $U(1)^w$ winding ($\mathbb{Z}$ Wilson line) $\quad Q^w = \sum_{links} J^w,$

    $$J^w = \frac{1}{2\pi}(\Delta\phi - 2\pi n)$$
  - The anomaly is captured by a lattice Schwinger term

    $$[J^w, J^m] \neq 0$$
- Exact T-duality: $\phi \leftrightarrow E$, momentum $\leftrightarrow$ winding, $\beta \leftrightarrow \frac{1}{(2\pi)^2\beta}$.

Hamiltonian lattice formulation – add defects

$$H_{modified\,Villain} = \frac{1}{2\beta}\sum_{sites} p^2 + \frac{\beta}{2}\sum_{links}(\Delta\phi - 2\pi n)^2$$

$$\exp(i\Delta E - 2\pi i\, p) = 1$$

Couple to background gauge fields for the global symmetry

Flat background gauge fields = Twisted boundary conditions

= Topological defects

- Momentum defect: at one link $\quad \Delta\phi - 2\pi n \to \Delta\phi - 2\pi n + \eta_m$
- Winding defect: at one site $\quad p \to p + \frac{\eta_w}{2\pi}$

The defects can be shifted by using unitary transformations – they are topological.

Hamiltonian lattice formulation – the anomaly

- A momentum defect with η_m has winding charge $\frac{\eta_m}{2\pi}$
- A winding defect with η_w has momentum charge $\frac{\eta_w}{2\pi}$
- Without defects, a $\mathbb{Z}_L$ translation symmetry (L sites with periodic boundary conditions)

$$T^L = 1$$

With defects, $[T, H] \neq 0$, but can combine T with the unitary transformation that shifts the defects, such that

$$[T(\eta_m, \eta_w), H] = 0$$

Now

$$T(\eta_m, \eta_w)^L = \underbrace{\exp(i\eta_m Q^m)\exp(i\eta_w Q^w)}_{\text{Due to the twisted boundary conditions}}\underbrace{\exp(-i\eta_m\eta_w/2\pi)}_{\text{Anomaly}}$$

Summary

- Anomalies are not specific to fermions or an infinite number of degrees of freedom.
- Anomalies do not signal an inconsistency of the theory – there is no need to add a bulk to the theory.
- Simple lattice models exhibit anomalies. This is true even for a finite lattice.
- Lattice translation plays a crucial role in identifying the anomaly.
- Many applications (did not discuss here):
  - Lieb-Schultz-Mattis type theorems
  - Luttinger theorem and filling constraints

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