Some Observations about Black Hole Partition Functions

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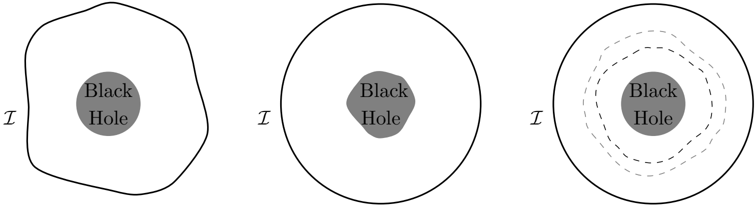
Black hole entropy as a phase space volume

A general BTZ black hole metric has a soft mode that visible from outside

$$ds^2 = dr^2 + (e^{2r} + e^{-2r}T_{uu}T_{vv})dudv + T_{uu}du^2 + T_{vv}dv^2$$

$$T_{uu} = \{U, u\}, \qquad T_{vv} = \{V, v\}$$



$$M + J = \oint T_{uu}du = \oint \{U, u\}du$$

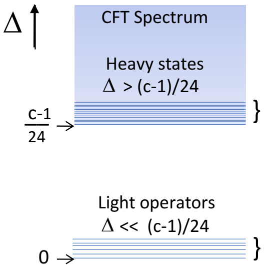
$$M - J = \oint T_{vv}dv = \oint \{V, v\}dv$$

Black hole entropy as a phase space volume

Consider the volume of a thin slice of phase space around (M, J)

$$\mathrm{Vol}(M, J)\,\delta M \delta J$$

Conjecture: in the $c \to \infty$, $M \pm J \to 0$ limit with $S_{BH}(M, J)$ finite



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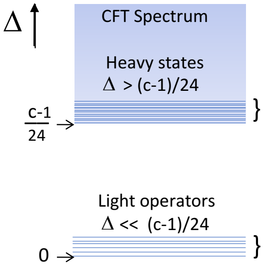
$$\boxed{\mathrm{Vol}(M, J) = e^{S_{BH}(M,J)}}$$



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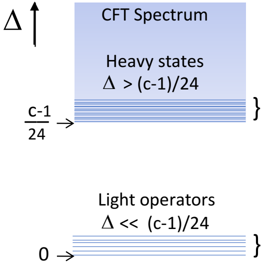
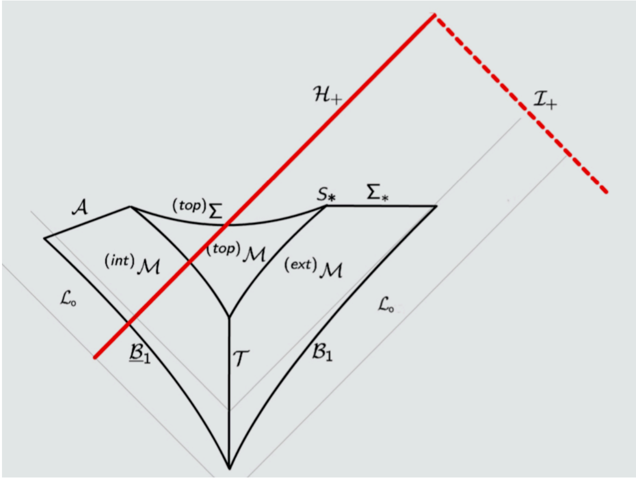


The 'proof' makes use of the identities

$$\int [dU][dV] e^{-\beta_+(M+J)-\beta_-(M-J)} = Z_{Schw}(\beta_+)\, Z_{Schw}(\beta_-)$$

$$Z_{Schw}(\beta) = \lim_{c\to\infty} Z_{\mathrm{CFT}}(\beta_c) \quad \text{with} \quad \beta_c = \frac{24\pi^2}{c\beta}$$

Most ingredients of the story generalize to 4D



Partition function of $T\overline{T}$ deformed CFT

Zamolodchikov and Smirnov showed that the energy spectrum of the deformed CFT defined by turning on the $T\overline{T}$ coupling

$$S_{T\overline{T}} \;=\; S_{\rm CFT} - \int \lambda T\overline{T}$$

is exactly given by

$$\mathcal{E}_n(\lambda) \;=\; \frac{1}{\lambda}\big(-1 + \sqrt{1 + 2E_n\lambda + j_n^2\lambda^2}\big).$$

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The corresponding $T\overline{T}$ deformed CFT partition function

$$Z_1(\lambda, \sigma) \;=\; \sum_n \exp(2\pi i(\sigma_1 j_n + i\sigma_2 \mathcal{E}_n(\lambda)))$$

with $\sigma = \sigma_1 + i\sigma_2 =$ shape of the torus, is modular invariant

$$Z_1(\lambda, \sigma) \;=\; 2\rho_2 \int \frac{d^2\tau}{\tau_2^2}\, e^{-\frac{\pi\rho_2}{\sigma_2\tau_2}|\sigma-\tau|^2}\, Z_{\mathrm{CFT}}(\tau) \qquad\qquad \rho_2 \equiv \frac{\sigma_2}{\lambda}$$

Partition function of $T\overline{T}$ deformed CFT

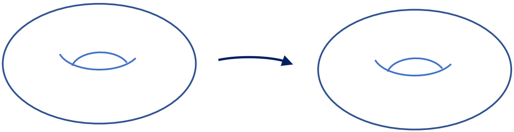
We can rewrite the deformed partition function as

$$Z_1(\lambda, \sigma) \;=\; \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} K_1(\tau; \rho_2, \sigma)\, Z_{\rm CFT}(\tau)$$

with:

$$K_1(\tau; \rho_2, \sigma) \;=\; 2\rho_2 \sum_{\gamma \in {\rm PSL}(2,\mathbb{Z})} e^{-\frac{\pi\rho_2}{\sigma_2(\gamma\tau)_2}|\sigma-\gamma\tau|^2} = \sum_{n,m} e^{iS[X_{nm}]}$$

$$X_{nm} = \frac{1}{2i\tau_2}(n - m\bar{\tau})\, z + c.c. \qquad\qquad ds^2 = \frac{\rho_2}{\sigma_2}\,|\, dx + \sigma\, dy\,|^2$$



$\hat{K}_1(\tau; \rho, \sigma) =$ restricted $\Gamma_{2,2}$ Narain sum with wrapping number 1

Partition function of a Symmetric Product CFT

$$Z(\rho_1, \sigma) \;=\; \sum_N p^N Z_N(\sigma), \qquad\qquad p \equiv e^{2\pi i \rho_1}$$

$$Z_{\rm DMVV}(p, \sigma) \;=\; \left| \prod_{d>0} \prod_{\substack{n,m\geq 0\\ j_n = md}} \frac{1}{1 - p^d e^{\frac{2\pi i}{d}(\sigma_1 j_n + i\sigma_2 E_n)}} \right|^2 .$$

Partition function of a $T\overline{T}$ deformed Symmetric Product CFT

$$Z(\rho, \sigma) \;=\; \sum_N p^N Z_N(\lambda, \sigma), \qquad\qquad e^{2\pi i \rho} \equiv p\, e^{-\frac{2\pi\sigma_2}{\lambda}}$$

$$Z(\rho, \sigma) \;=\; \left| \prod_{d>0} \prod_{\substack{n,m\in\mathbb{Z}\\ j_n = md}} \frac{1}{1 - p^d e^{\frac{2\pi i}{d}(\sigma_1 j_n + i\sigma_2 \mathcal{E}_n(\lambda/d^2))}} \right|^2$$

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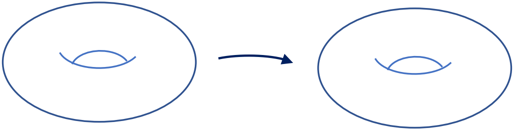
Partition function of a $T\overline{T}$ deformed Symmetric Product CFT

Now let us consider the partition function defined by:

$$\hat{Z}(\rho,\sigma) = e^{-\hat{F}(\rho,\sigma)} \qquad \hat{F}(\rho,\sigma) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \hat{K}(\tau;\rho,\sigma)\, Z_{\rm CFT}(\tau)$$

with:

$$\hat{K}(\tau;\rho,\sigma) \;=\; \rho_2 \sum_{\vec{n},\vec{m}\in\mathbb{Z}^2} e^{\frac{i\pi}{2\tau_2\sigma_2}(\rho|n_2+n_1\sigma-\tau(m_1+m_2\sigma)|^2)+c.c.}$$



$\hat{K}(\tau;\rho,\sigma) =$ Full $\Gamma_{2,2}$ Narain partition sum

S-duality invariant $T\overline{T}$ deformed partition function

$$\hat{Z}(\rho,\sigma) = e^{-F_0(\rho_2,\sigma)} \left| \prod_{d>0} \prod_{\substack{n,m\in\mathbb{Z}\\ j_n = md}} \frac{1}{1 - p^d e^{\frac{2\pi i}{d}(\sigma_1 j_n + i\sigma_2 \mathcal{E}_n(\lambda/d^2))}} \right|^2$$

$$F_0(\rho_2,\sigma) = A\rho_2 + \sum_{n\in\mathcal{S}} \log\det(\Delta + \rho_2 E_n)$$

$\hat{Z}(\rho,\sigma)$ has many remarkable properties:

- Mirror symmetry: $\hat{Z}(\rho,\sigma) = \hat{Z}(\sigma,\rho)$
- S-duality symmetry: $\hat{Z}(\rho,\sigma) = \hat{Z}(\tilde{\rho},\sigma)$, $\tilde{\rho} = \frac{a\rho+b}{c\rho+d}$,
- Spectral symmetry: $\Delta_\rho \hat{F}(\rho,\sigma) = \Delta_\sigma \hat{F}(\rho,\sigma)$
- Hecke symmetry: $T^\rho_j \hat{F}(\rho,\sigma) = T^\sigma_j \hat{F}(\rho,\sigma)$
- U-duality: $\mathrm{O}(2,2;\mathbb{Z}) \simeq \mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^2$

お誕生日おめでとう！

Hirosi !