Hirosifest@Caltech

- A few features in common with Hirosi: topological strings, Cumrun, bipolar, administration
- In awe of Hirosi: Presentations in real time, ability to absorb and summarize results of others

- NB, HO, C. Vafa: On the worldsheet derivation of large-N dualities for the superstring (hep-th/0310118)

Studied open/closed dualities through F-terms computed using a $\hat{c} = 5$ topological string

In this talk, I will discuss how another $\hat{c} = 5$ topological string relates the RNS and pure spinor formalisms

D=5 Holomorphic Chern-Simons and the Pure Spinor Superstring

(to appear soon on arXiv)

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1) Review of RNS and Pure Spinor formalisms
2) D=5 Holomorphic Chern-Simons as an N=1 and N=2 string theory
3) B-RNS-GSS superstring with worldsheet and spacetime susy
4) Twisting the N=1 B-RNS-GSS formalism into the N=2 pure spinor formalism
5) Comments on multiloop amplitude prescriptions

Review of superstring formalisms

- RNS formalism has manifest N=1 worldsheet susy but D=10 spacetime susy is hidden
- Up to 2-loop 4-pt NS amplitudes, but need to sum over spin structures
- Ramond vertex operators and RR backgrounds are complicated
- Spacetime susy is manifest in Green-Schwarz formalism, but covariant quantization is complicated
- For compactifications to lower dimensions, can use hybrid formalism with manifest D=4 or D=6 spacetime susy
- To covariantly quantize with D=10 spacetime susy, use pure spinor formalism with twisted N=2 worldsheet susy
- Up to 2-loop 5-pt and 3-loop 4-pt computations with no sum over spin structures
- Can quantize in $AdS_5 \times S^5$ background with manifest PSU(2,2|4) invariance

Review of Pure Spinor formalism

- Worldsheet variables: $x^m, (\theta^\alpha, p_\alpha)$, $(\lambda^\alpha, w_\alpha)$, $(\bar\lambda_\alpha, \bar w^\alpha), (r_\alpha, s^\alpha)$ $\alpha = 1 \text{ to } 16$

  Green-Schwarz-Siegel variables + pure spinor ghosts + non-minimal variables

  Pure spinor constraints: $\lambda\gamma^m\lambda = 0$, $\bar\lambda\gamma^m\bar\lambda = \bar\lambda\gamma^m r = 0$

- Formalism has twisted $\hat c$=3 N=2 worldsheet susy

$$T = \frac{1}{2}\partial x^m \partial x_m + p_\alpha \partial\theta^\alpha + w_\alpha\partial\lambda^\alpha + \bar w^\alpha \partial\bar\lambda_\alpha + s^\alpha\partial r_\alpha$$

$$G^+ = j_{BRST} = \lambda^\alpha d_\alpha + \bar w^\alpha r_\alpha$$

$$G^- = B_{pure} = s^\alpha\partial\bar\lambda_\alpha + w_\alpha\partial\theta^\alpha + \left(\lambda\bar\lambda\right)^{-1}\Pi^m\left(\bar\lambda\gamma_m d\right) + \left(\lambda\bar\lambda\right)^{-2}(\bar\lambda\gamma^m r)\,(d\gamma_m d) + \ldots$$

$$J = j_{ghost} = w_\alpha\lambda^\alpha + r_\alpha s^\alpha$$

$d_\alpha = p_\alpha - \frac{1}{2}\partial x^m (\gamma_m\theta)_\alpha + \frac{1}{8}(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha$, $\Pi^m = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta$ are GSS definitions

Pure spinor vertex operators and scattering amplitudes

- Physical states are vertex operators in the cohomology of the BRST operator

$$Q_{pure} = \int dz\, G^+ = \int dz(\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha)$$

- Super-YM vertex operator: $V = \lambda^\alpha A_\alpha(x,\theta)$ where $A_\alpha(x,\theta)$ is spinor gauge superfield
- Integrated operator: $\mathrm{U} = \int dz\, [\partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + (w\gamma^{mn}\lambda)F_{mn}]$
- n-point g-loop amplitude prescription:

$$A = \int d^{3g-3+n}\tau \, \langle\, \mathfrak{N} \prod_{r=1}^{n} V(z_r) \prod_{s=1}^{3g-3+n} \int dy_s\, \mu(y_s)\, G^-(y_s)\, \rangle$$

- No sum over spin structures since all variables have integer conformal weight
- Integration over zero modes naively produces 0/0 which can be regularized using $\mathfrak{N}$

How is this topological N=2 amplitude prescription related to the N=1 worldsheet supersymmetric prescription of the RNS formalism?

Clue: D=5 holomorphic Chern-Simons theory

D=5 Holomorphic Chern-Simons theory

- D=5 holomorphic Chern-Simons theory describes onshell D=10 super-YM states which are in the cohomology of the D=10 spacetime susy generator q_+
- Under the subgroup U(1)xSU(5) of SO(10), q_+ carries U(1) charge $+\frac{5}{2}$
- Since $\{q_+, q_a\}$=P_a where $P_m = (P_a, P^a)$ for a=1 to 5, states in the cohomology of $q_+$are sYM states with P_a=0 and can be described by the SU(5)-covariant superfield

$$\mathrm{V}\big(x_a, \Gamma^b\big) = \chi(x) + A_a(x)\Gamma^a + \xi_{ab}(\mathrm{x})\Gamma^a\Gamma^b + F^{ab}(x)(\Gamma^3)_{ab} + \partial^a\xi^+(x)(\Gamma^4)_a$$

$\xi^\alpha = (\xi^a, \xi_{ab}, \xi^+)$ is the gluino where ξ^a=$\partial^a\chi$, $\mathrm{A_m} = (\mathrm{A^a}, \mathrm{A_a})$ is the gluon

- 3-point tree amplitude is $\int d^5\Gamma\, d^5x\, V_1(x,\Gamma)V_2(x,\Gamma)V_3(x,\Gamma)$
- Multiloop scattering amplitudes in D=5 holomorphic Chern-Simons can be computed using either the usual RNS formalism where the external states are restricted to super-YM states with P_a=0, or using a topological theory with the twisted $\hat{c}$=5 N=2 superconformal generators

$$T = \partial x^a \partial x_a + \overline{\Gamma}_a \partial \Gamma^a, \quad G^+ = \partial x_a \Gamma^a, \quad G^- = \partial x^a \overline{\Gamma}_a, \quad J = \Gamma^a \overline{\Gamma}_a$$

Twisting N=1 into N=2 for D=5 holomorphic CS theory

- To relate the N=1 and N=2 amplitude prescriptions, start with the RNS formalism and define

$$\Gamma^a = \gamma(\psi^a + i\psi^{a+5}),\ \overline{\Gamma}_a = \gamma^{-1}(\psi^a - i\psi^{a+5}), \quad \tilde{\gamma} = \gamma^2,\ \tilde{\beta} = \gamma^{-1}\beta + \gamma^{-2}\psi^a\,\psi^{a+5}$$

$$\Rightarrow Q_{RNS} = \int dz\,[\,c\,T + G^+ + \tilde{\gamma}(b + G^-) + \mathrm{c}(\tilde{\beta}\partial\tilde{\gamma} + \partial(\tilde{\beta}\tilde{\gamma}) + \mathrm{b}\partial\mathrm{c})]$$

- Defining $\tilde{\gamma} = \tilde{\eta}\, e^{\tilde{\phi}},\ \tilde{\beta} = \partial\tilde{\xi}e^{-\tilde{\phi}},\ \partial H = J = \psi^a\psi^{a+5} \Rightarrow \tilde{\eta} = e^{-\frac{1}{2}(\phi+H)} = e^{-\frac{\phi}{2}}\Sigma_+$
- The RNS spacetime susy generator is $q_\alpha = \int dz\, e^{-\frac{\phi}{2}}\Sigma_\alpha$, so $q_+ = \int dz\,\tilde{\eta}$

  and D=5 hCS states annihilated by q_+ are in the “small” tilded Hilbert space
- Finally, perform the similarity transformation with $R = \int dz\,(cG^- + c\partial c\tilde{\beta})$ so

$$e^{-R}Q_{RNS}e^{R} = \int dz\,(G^+ + \tilde{\gamma}b), \qquad e^{-R}b\,e^{R} = b - G^- - \partial c\tilde{\beta} - \partial(c\tilde{\beta})$$

- Cohomology of $\int dz\,(\partial x_a\Gamma^a + \tilde{\gamma}b)$ can only depend on (x_a, Γ^a) zero modes ⇒ hCS states
- If RNS picture-changing operators are put on top of b ghosts, path integral over (b,c) and $(\tilde{\beta}, \tilde{\gamma})$ ghosts cancel and the RNS prescription reproduces the topological N=2 prescription

$$A = \int d^{3g-3+n}\tau\,\langle \textstyle\prod_{r=1}^{n} V(z_r)\ \prod_{s=1}^{3g-3+n} \int dy_s\,\mu(y_s)\,G^-(y_s)\,\rangle$$

B-RNS-GSS formalism for the superstring

- Has both N=1 worldsheet susy and D=10 spacetime susy
- Worldsheet matter variables: (x^m, ψ^m), $(\theta^\alpha, \Lambda^\alpha)$, $(\Omega_\alpha, p_\alpha)$, $(\overline{\Lambda}_\alpha, R_\alpha)$, $(S^\alpha, \overline{\Omega}^\alpha)$

$$S = \int d^2z \, [\ \tfrac{1}{2}\left(\partial x^m \overline{\partial} x_m + \psi^m \overline{\partial} \psi_m\right) \quad + p_\alpha \overline{\partial}\theta^\alpha + \Omega_\alpha \overline{\partial}\Lambda^\alpha \quad + \overline{\Omega}^\alpha \overline{\partial}\overline{\Lambda}_\alpha + S^\alpha \overline{\partial} R_\alpha]$$

$$G \quad = \quad \psi^m \partial x_m \quad + \Lambda^\alpha p_\alpha + \Omega_\alpha \partial\theta^\alpha \quad + \overline{\Omega}^\alpha R_\alpha + S^\alpha \partial \overline{\Lambda}_\alpha$$

- After performing similarity transformation $\mathcal{O} \to e^A\, \mathcal{O} e^{-A}$ with $A = \int dz\, \psi^m (\Lambda \gamma_m \theta)$, formalism has manifest D=10 spacetime supersymmetry with

$$T = \tfrac{1}{2}(\Pi^m \Pi_m + \psi^m \partial \psi_m) \ + d_\alpha \overline{\partial}\theta^\alpha + \tfrac{1}{2}(\Omega_\alpha \partial \Lambda^\alpha - \Lambda^\alpha \partial \Omega_\alpha + \overline{\Omega}^\alpha \partial \overline{\Lambda}_\alpha - \overline{\Lambda}_\alpha \partial \overline{\Omega}^\alpha + S^\alpha \partial R_\alpha + R_\alpha \partial S^\alpha)$$

$$G = \ \psi^m \Pi_m + \Lambda^\alpha d_\alpha + \Omega_\alpha \partial \theta^\alpha + \overline{\Omega}^\alpha R_\alpha + S^\alpha \partial \overline{\Lambda}_\alpha$$

- BRST operator: $Q = \int dz\, [cT + \gamma G + \gamma^2 b + c\left(b\, \partial c + \beta \partial \gamma + \partial(\beta\gamma)\right)]$
- Super-YM vertex: $V = \int dz\, G[\Lambda^\alpha A_\alpha(x,\theta) + \psi^m A_m(x,\theta) + \Omega_\alpha W^\alpha(x,\theta)]$

$$= \int dz\, [\partial\theta^\alpha A_\alpha + \ \Pi^m A_m + \ d_\alpha W^\alpha + (\psi^m \psi^n + \Omega \gamma^{mn} \Lambda) F_{mn} + \Omega_\alpha \psi^m \partial_m W^\alpha]$$

- Can compute scattering amplitudes without spin fields using standard N=1 prescription, but extra variables means BRST invariance is not enough to obtain physical superstring spectrum

Twisting the B-RNS-GSS formalism

- To relate the B-RNS-GSS and pure spinor formalisms, first define U(1) generator J so that $[J, G] = G^+ - G^-$ where $[J, G^+, G^-, T]$ form an N=2 algebra

$$J = \Lambda^\alpha \Omega_\alpha + R^\alpha S_\alpha + \frac{\lambda\gamma^{mn}\bar\lambda}{\lambda\bar\lambda}\psi_m\psi_n + \frac{\bar\lambda\gamma^{mnp}r}{(\lambda\bar\lambda)^2}\psi_m\psi_n\psi_p$$

where $$\Lambda^\alpha = \lambda^\alpha + \frac{u^m(\gamma_m\bar\lambda)^\alpha}{\lambda\bar\lambda}, \bar\Lambda_\alpha = \bar\lambda_\alpha + \frac{\bar u^m(\gamma_m\lambda)_\alpha}{\lambda\bar\lambda}, \quad R_\alpha = r_\alpha + \frac{\rho^m(\gamma_m\lambda)_\alpha}{\lambda\bar\lambda}$$

Constraints: $\lambda\gamma_m\lambda = \bar\lambda\gamma_m\bar\lambda = r\gamma_m\bar\lambda = v_m(\gamma^m\bar\lambda)^\alpha = \bar u^m(\gamma_m\bar\lambda)^\alpha = \rho^m(\gamma_m\bar\lambda)^\alpha = 0$

$$u_m = \Lambda\gamma_m\Lambda, \quad \bar u_m = \bar\Lambda\gamma_m\bar\Lambda, \quad \rho_m = R\gamma_m\bar\Lambda \qquad (\hat c = 5)$$

- Twist all spin ½ variables to have integer conformal weight by defining

$$\Gamma^m = \gamma\frac{\lambda\gamma^{mn}\bar\lambda}{\lambda\bar\lambda}\psi_n, \quad \bar\Gamma_m = \gamma^{-1}\left(\frac{\lambda\gamma_{mn}\bar\lambda}{\lambda\bar\lambda}\psi^n + \frac{\bar\lambda\gamma_{mnp}r}{(\lambda\bar\lambda)^2}\psi^n\psi^p\right), \tilde\Lambda^\alpha = \gamma\Lambda^\alpha, \quad \tilde R^\alpha = \gamma R^\alpha,$$

$$\tilde\gamma = \gamma^2, \quad \tilde\beta = \gamma^{-1}\beta + \gamma^{-2}J, \quad J = \Lambda^\alpha\Omega_\alpha + R^\alpha S_\alpha + \Gamma^m\bar\Gamma_m$$

$$\tilde\gamma = \tilde\eta\, e^{\tilde\phi}, \ \tilde\beta = \partial\tilde\xi e^{-\tilde\phi}, \ \partial H = J \Rightarrow \tilde\eta = e^{-\frac{1}{2}(\phi+H)} = e^{-\frac{\phi}{2}}\left(\lambda^\alpha\Sigma_\alpha + \frac{(\bar\lambda\gamma_{mnp}r)(\lambda\gamma^{mnp}\Sigma)}{(\lambda\bar\lambda)^2}\right)$$

Non-minimal term in $\tilde\eta$ is needed for $\{\tilde\eta, Q_{B-RNS-GSS}\} = 0$

Relation to pure spinor amplitudes

- Physical superstring states should be annihilated by both $\int dz\, \tilde{\eta}$ and $Q_{B-RNS-GSS}$
- In terms of the twisted variables,

$$Q_{B-RNS-GSS} = \int dz\, [\, c\, T + G^+ + \tilde{\gamma}(b + G^-) + \mathrm{c}(\tilde{\beta}\partial\tilde{\gamma} + \partial(\tilde{\beta}\tilde{\gamma}) + \mathrm{b}\partial c)]$$

$$= e^R \int dz\, (G^+ + \tilde{\gamma} b)\; e^{-R}$$

$$R = \int dz\, \left(cG^- + c\,\partial c\tilde{\beta}\right), \quad G^+ = \lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha + \mathrm{u}^{\mathrm{m}}\overline{\Gamma}_m + \overline{v}^m \rho_m, \quad \mathrm{G}^- = \mathrm{B}_{\mathrm{pure}} + \boldsymbol{\mathcal{O}}(\overline{\Gamma})$$

- States in cohomology of $\int dz\, (G^+ + \tilde{\gamma} b)$ are pure spinor states in cohomology of $\int dz\, (\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha)$ and independent of $(u^m, v_m, \Gamma^m, \overline{\Gamma}_m, \overline{u}_m, \overline{v}^m, \rho_m, \tau^m, b, c, \tilde{\beta}, \tilde{\gamma})$
- Super-YM vertex operator of B-RNS-GSS is equal to super-YM pure spinor vertex operator up to BRST-trivial terms depending on the non-minimal variables
- As in holomorphic Chern-Simons amplitudes, N=1 prescription can be related to N=2 topological prescription by inserting picture-raising operators on top of b ghosts so that the (b,c) and $(\tilde{\beta}, \tilde{\gamma})$ correlation functions cancel.
- But superstring multiloop amplitude prescription has subtleties not present in hCS prescription

Comments on multiloop amplitude prescriptions

- In pure spinor formalism, picture-changing operators (PCO´s) have no singularites with each other, but BRST-trivial terms can contribute if they are proportional to $\left(\lambda\bar{\lambda}\right)^{-11}$

  How is this consistent with equivalence of the RNS and pure spinor amplitude prescriptions?

- N=1 vertex operators and N=1 PCO´s are related to N=2 vertex operators and N=2 PCO´s by the similarity transformation $A = \int dz\, \eta\tilde{\xi} = \int dz\, e^{-2\tilde{\phi}+H}\tilde{\xi}$

  $$\mathrm{e^{A}V_{N=1}e^{-A} = V_{N=2}}, \qquad \mathrm{e^{A}}\xi\mathrm{e^{-A}} = \xi + \tilde{\xi} \;\Rightarrow\; \mathrm{e^{A}Q}(\xi)e^{-A} = Q(\xi) + Q(\tilde{\xi})$$

- Since $\xi = e^{2\tilde{\phi}-H}$ is in the small tilded Hilbert space (i.e. $\{\int dz\, \tilde{\eta}, \xi\} = 0$), $Q(\xi)$ can be ignored in the N=2 prescription and the N=1 PCO $Q(\xi)$ is mapped to the N=2 PCO $Q(\tilde{\xi})$

- But for B-RNS-GSS amplitudes, ξ contains inverse power of $\lambda\bar{\lambda}$. So if the pure spinor B ghosts contribute a sufficient inverse power of $\lambda\bar{\lambda}$, the term $Q(\xi)$ may contribute in the N=2 PCO

- For the computation of supersymmetric ``F-terms", these inverse powers of $\lambda\bar{\lambda}$ are not present and the B-RNS-GSS and pure spinor multiloop amplitude prescriptions coincide

- But for ``D-terms", $Q(\xi)$ can contribute and modify the pure spinor multiloop prescription

Conclusions

- Relation of RNS with D=5 holomorphic Chern-Simons is equivalent to relation of B-RNS-GSS with pure spinor formalism
- Twisting N=1 → N=2 requires defining $J = \partial H$ which implies $\tilde{\eta} = e^{-\frac{1}{2}(\phi+H)}$
- Physical states must be annihilated by Q and η_0 and $\tilde{\eta}_0$
- Multiloop amplitude prescription for D-terms requires careful treatment of $Q(\xi)$ term in twisted N=1 PCO $Q\left(\xi + \tilde{\xi}\right)$

Happy Birthday Hirosi!