Towards global S-matrix constraints on dark matter physics

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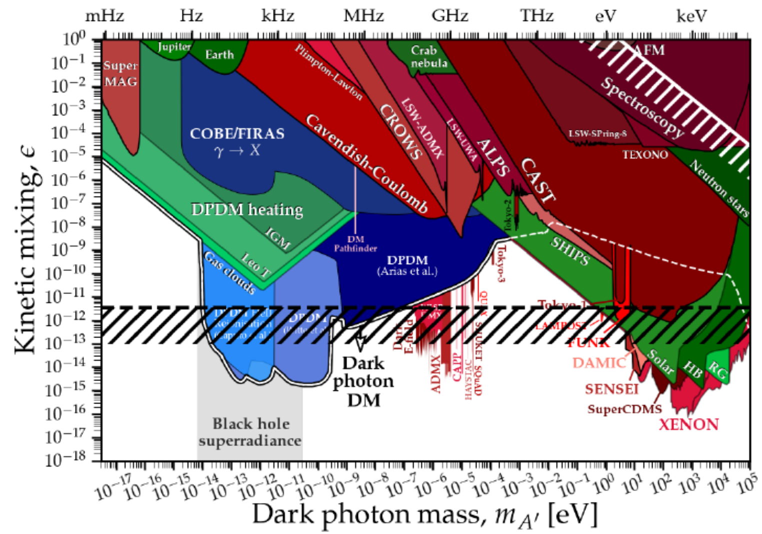
What do we aim for?

- ❑ **Consistency with UV completion?**

Fundamental principles such as unitarity and causality are already enough to find strong constraints on low-energy physics. A. Adams et al. 2006 and many.

Noumi-san's talk:
Dark sector may not be very dark?
Mass may not be very light?

There are still theoretical subtleties but the predictions are testable!



From Noumi, Sato, and Tokuda, 2022.

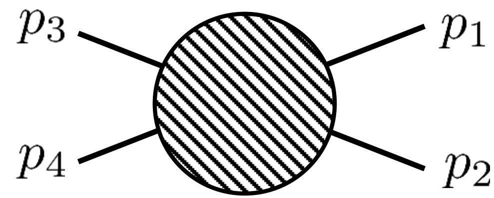
- ❑ **We need to**

✓ resolve the theoretical subtleties and get rigorous&stronger bounds,
✓ understand the bounds systematically and their pheno. consequences,
✓ …

Towards global constraints

- The bounds are well established
  for 2-to-2 scattering of the lightest state in the gapped system.

$$\mathcal{M}(s,t) =$$



$$s = -(p_1 + p_2)^2$$
$$t = -(p_1 - p_3)^2$$
$$u = -(p_1 - p_4)^2$$

Nice properties (analytic structure, high energy behaviour) are known.
Unitarity gives a simple positivity constraint $\mathrm{Im}\mathcal{M}|_{t=0} > 0.$

⇒ Positivity bounds on coupling constants of EFT. A. Adams et al. 2006 and many.

- However, our world is more complicated!

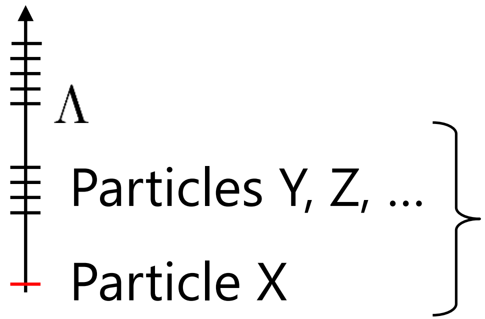
There are massless particles (photon, graviton)
→ Graviton may give non-trivial constraints a la swampland.

Cf. Noumi-san's talk.

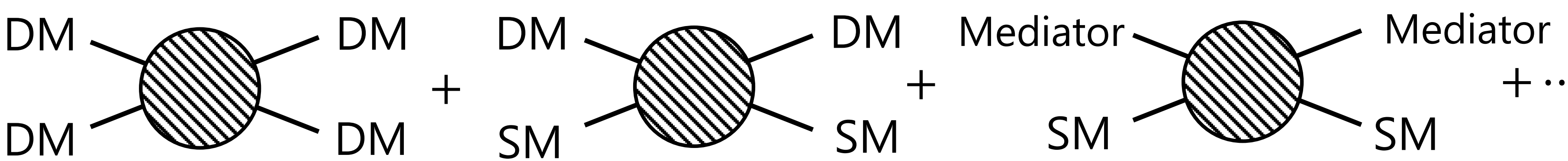
There are many massive particles.

Towards global constraints

- The bounds are well established
  for 2-to-2 scattering of ~~the lightest state~~ in the gapped system.
  → **What are the bounds arising from all possible 2-to-2 scatterings?**



Below the cutoff, we can study
$XX \to XX, XY \to XY, YY \to YY, YZ \to YZ, \ldots$



Of course, this is not a new idea but must be crucial, e.g.
Four-fermi interaction → W boson is required
W boson scattering → Higgs is required.

Towards global constraints

- The bounds are well established
  for 2-to-2 scattering of ~~the lightest state~~ in the gapped system.
  → **What are the bounds arising from all possible 2-to-2 scatterings?**

- Technically cumbersome but straightforward?

M: the heavy particle A
μ: the lightest particle φ

$2\mu^2 < M^2 < 4\mu^2$: S-matrix has new singularities (anomalous thresholds).
$4\mu^2 < M^2$: A can decay to $\varphi\varphi$.
→ A does not appear in the asymptotic state. Veltman 1963.

Then, what is the S-matrix of A?
If exists, what is the consequence of unitarity? …

Towards global constraints

- Physically, there must be unitarity constraints! $\mathrm{Im}\mathcal{M}|_{t=0} > 0$???

  But, when is this naïve intuition correct and when is it wrong?

- **Current status:**

  There indeed exist unitarity constraints and "optical theorem" even for unstable-particle amplitude.

$$\mathrm{Im}\mathcal{M}|_{t\simeq[\mathrm{Im}M^2]^2/s} > 0$$

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(Optimistic) expectation:

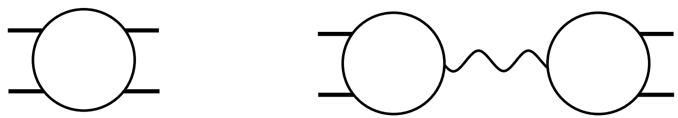
Positivity bounds on unstable particles might be similar to the stable case if there is no t-channel exchange of a spin-J ($J \geq 2$) particle.

→ Graviton is spin-2! May be crucial to swampland (and more?).

Unstable-particle amplitudes

- The unstable particles do not appear in the asymptotic states.
  Its existence is only seen in internal states (resonance in physical process).

$$\mathcal{M}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{M}_{\mathcal{A}\to\varphi\varphi}\frac{1}{s-M^2}\mathcal{M}_{\varphi\varphi\to\mathcal{A}}$$



$$M^2 \in \mathbb{C}$$

$$M^2 = M_R^2 - iM_R\Gamma$$

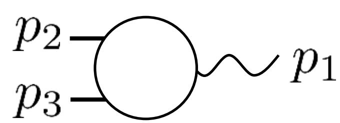
(physical) mass width

The unstable particles have a complex mass and the residue is factorized.

The amplitude for $\mathcal{A}\to\varphi\varphi$ can be defined by the residue.

See e.g. *The analytic S-matrix*, R. J. Eden et al, 1966

$$\mathcal{M}_{\mathcal{A}\to\varphi\varphi} =$$



The “on-shell” conditions are understood as

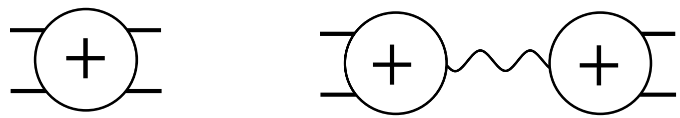
$$p_2^2 = p_3^2 = -\mu^2\,,\quad p_1^2 = -M^2 \in \mathbb{C}$$

p_1 is decaying mode.

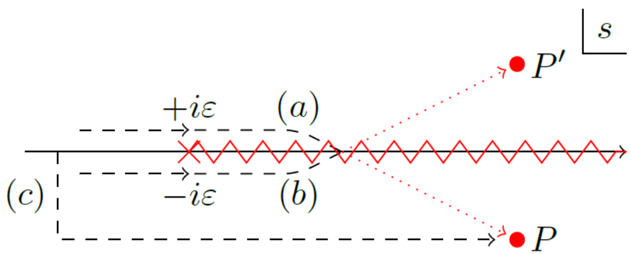
Unstable-particle amplitudes

- There is also a growing solution (complex conjugate).

$$\mathcal{M}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{M}_{\mathcal{A}\to\varphi\varphi} \frac{1}{s-M^2} \mathcal{M}_{\varphi\varphi\to\mathcal{A}}$$ at P



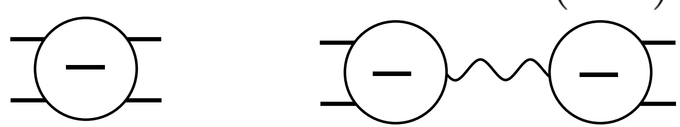
Decaying mode



Continued from $+i\varepsilon$

Continued from $-i\varepsilon$

$$\mathcal{M}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{M}_{\mathcal{A}\to\varphi\varphi} \frac{1}{s-(M^2)^*} \mathcal{M}_{\varphi\varphi\to\mathcal{A}}$$ at P′



Growing mode

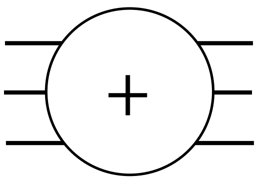
Decaying ($+i\varepsilon$) and growing ($-i\varepsilon$) are denoted by $+$ and $-$.

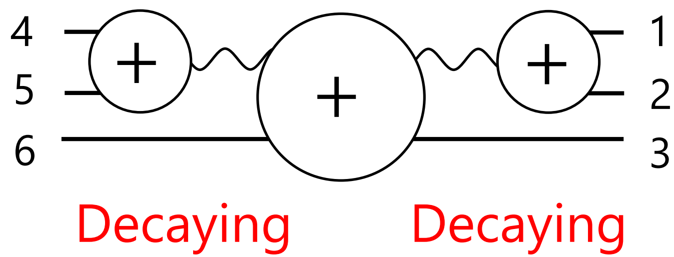
Unstable-particle amplitudes

- The 2-to-2 amplitudes are similarly defined.

$$s_{ij} = -(p_i + p_j)^2$$

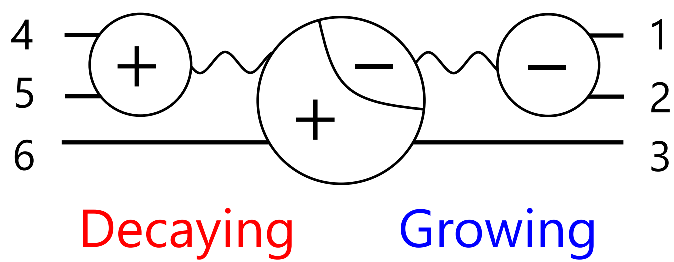
$$\mathcal{M}_{\varphi\varphi\varphi\to\varphi\varphi\varphi} \sim \mathcal{M}_{\mathcal{A}\to\varphi\varphi} \frac{1}{s_{45} - M^2} \mathcal{M}_{\varphi\mathcal{A}\to\varphi\mathcal{A}} \frac{1}{s_{12} - M^2} \mathcal{M}_{\varphi\varphi\to\mathcal{A}}$$





or

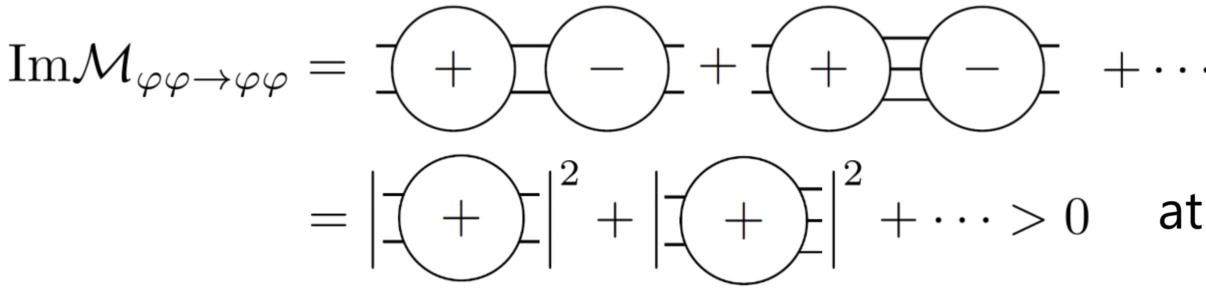
$$\sim \mathcal{M}_{\mathcal{A}\to\varphi\varphi} \frac{1}{s_{45} - M^2} \mathcal{M}_{\varphi\mathcal{A}\to\varphi\mathcal{A}} \frac{1}{s_{12} - (M^2)^*} \mathcal{M}_{\varphi\varphi\to\mathcal{A}}$$



Unitarity of unstable-particle amplitudes

- Stable case:

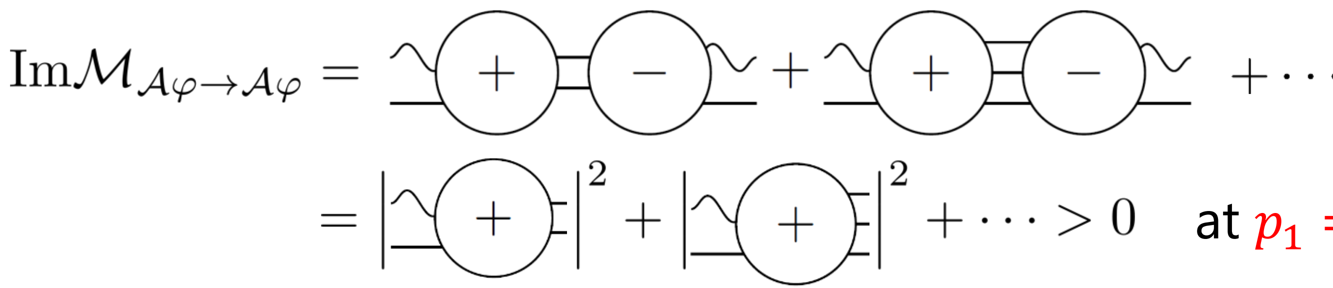
+: amplitude, -: its Hermitian conjugate.



at $p_1 = p_3$ $(t = 0)$.

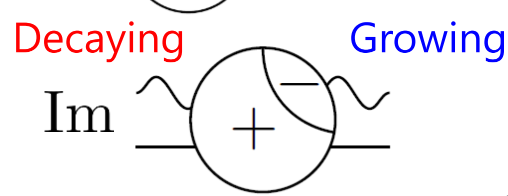
- Unstable case: we want to keep (Amplitude) × (Conjugate) > 0 on RHS.

Expected unitarity constraint



at $p_1 = p_3^*$.

The positivity may hold for



Let's prove it!

Unitarity of unstable-particle amplitudes

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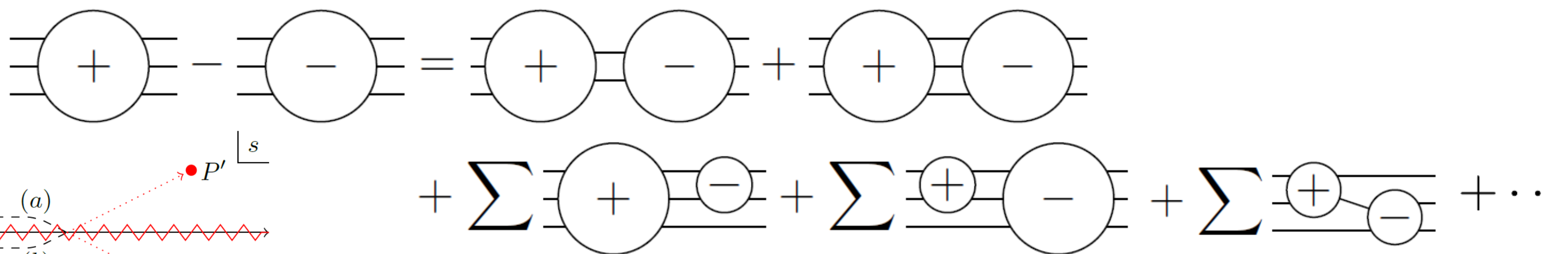
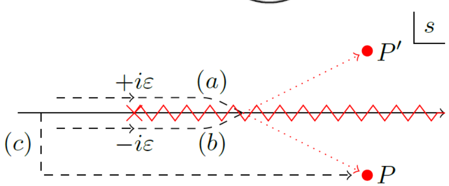
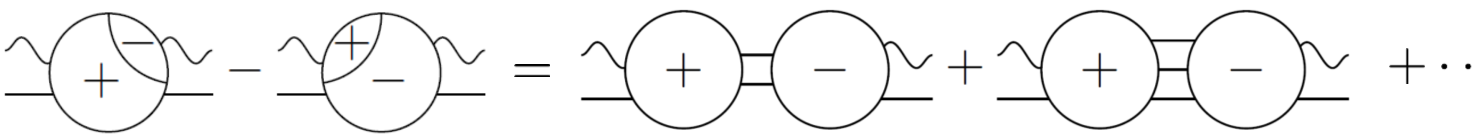
Assumptions:

1. Lorentz invariance, 2. Unitarity, 3. Analyticity of higher pt. amplitudes.

⇒ Unitarity equations for $\mathcal{M}_{\mathcal{A}\varphi\to\mathcal{A}\varphi}$ indeed arise from unitarity of higher pt.



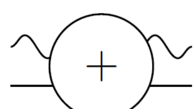


⇒ 

$\mathrm{Im}\mathcal{M}_{\mathcal{A}\varphi\to\mathcal{A}\varphi}(s,t) > 0$ holds in $(\mathrm{Im}M^2)^2/s < t < t_*$

The positivity may be extended up to the nearest singularity t_* in the t-plane.

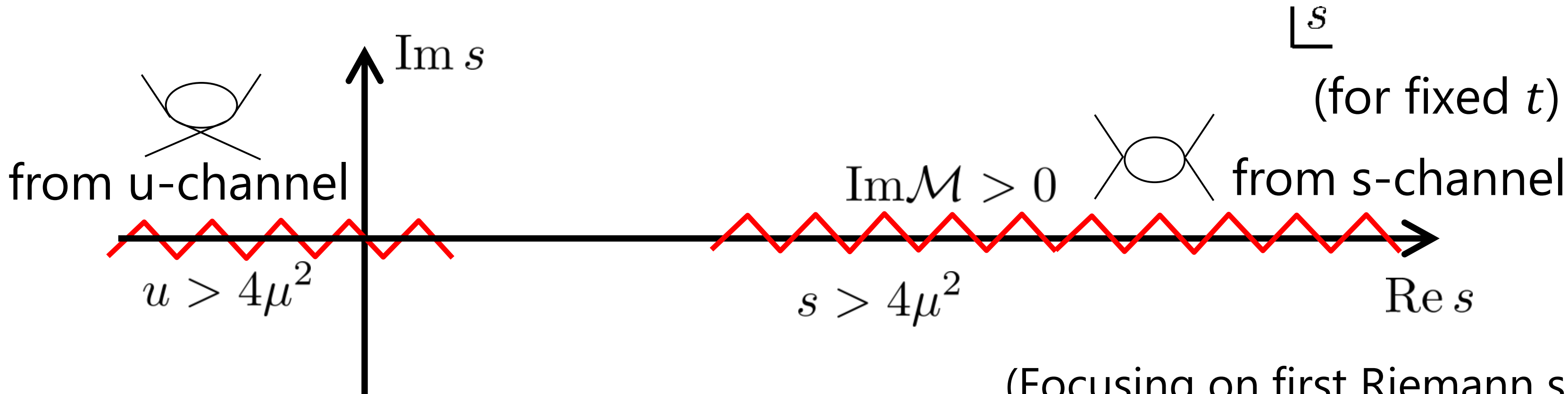
e.g. $t_{\rm tri} \simeq 4\mu^2(\mathrm{Im}M^2)^2/(\mathrm{Re}M^2)^2$ (narrow width)

Unitarity equation for  can be also derived although complicated.

Analyticity of unstable-particle amplitudes

Analyticity is important:
singularities determine properties of complex functions → positivity bounds

- Analyticity of $\varphi\varphi \to \varphi\varphi$ (in s-plane).
  All singularities are normal thresholds (= easily predicted by unitarity)



(Focusing on first Riemann sheet)

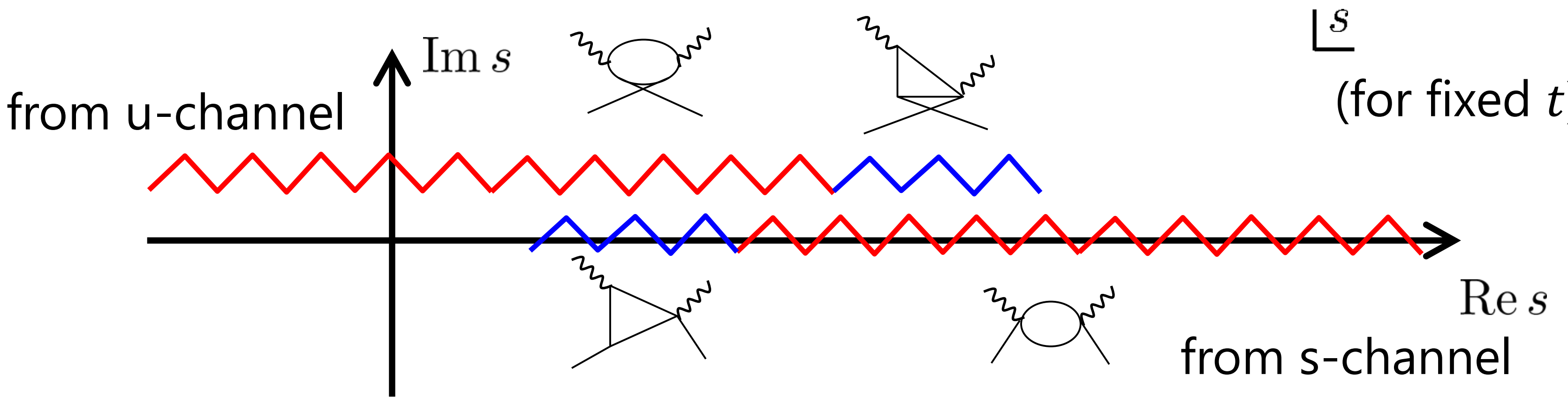
Analyticity of unstable-particle amplitudes

Analyticity is important:
singularities determine properties of complex functions → positivity bounds

- Analyticity of $\mathcal{A}\varphi \to \mathcal{A}\varphi$ (in s-plane).

  Anomalous thresholds appear (= not immediately followed from unitarity)

  But consistent with and can be derived from unitarity



$3\mu^2 < M^2 < 4\mu^2$

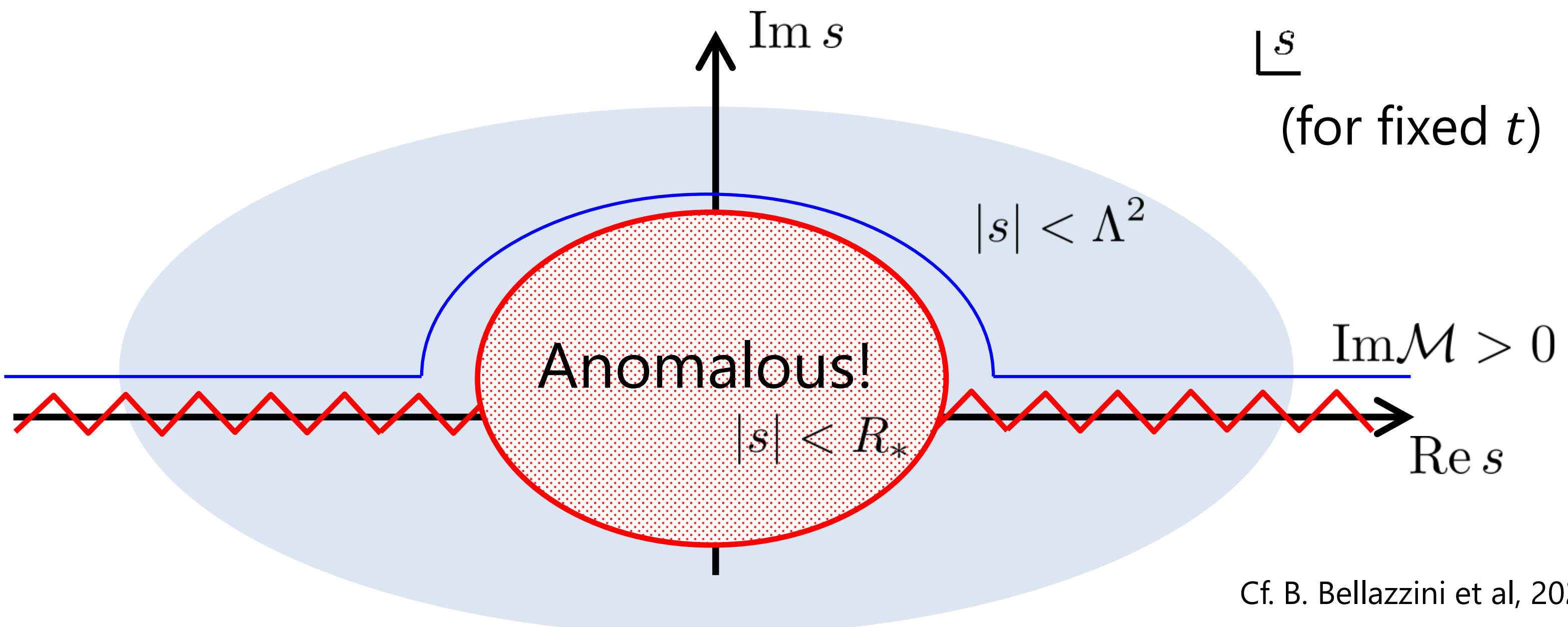
(Focusing on first Riemann sheet)

The simulation is more complicated when $\mathrm{Re}M^2 > 4\mu^2$.

Analyticity of unstable-particle amplitudes

However, the anomalous thresholds would appear only in $|s| < R_*$.

Maybe no proof but e.g. M. Correia 2022 for a recent discussion.



We only need the information in $R_* < |s| < \Lambda^2$ for positivity bounds.

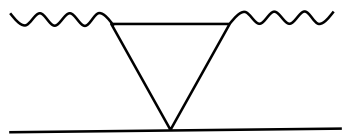
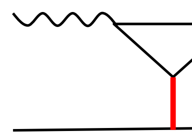
If correct, it is not so difficult to derive positivity bounds for general masses!

t-channel anomalous threshold

The s,u-channel anomalous thresholds could be handled
while we still have t-channel anomalous thresholds!

Roughly, positivity bounds are expected to give

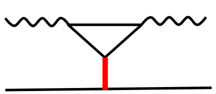
$$\frac{d^{2n}}{ds^{2n}}\left[\mathcal{M}_{\mathcal{A}\varphi\to\mathcal{A}\varphi}(s,t)-\text{subtractions}\right]_{t\simeq 0}>0,\quad n\geq 1$$

$\mathcal{M}_{\mathcal{A}\varphi\to\mathcal{A}\varphi}=$  $+$  Spin-J $\propto s^J/(m_J^2-t)$ $+\cdots$

The t-channel triangle has a singularity near $t\simeq 0$!

$t_{\rm tri}\simeq 4\mu^2({\rm Im}M^2)^2/({\rm Re}M^2)^2$ (narrow width)

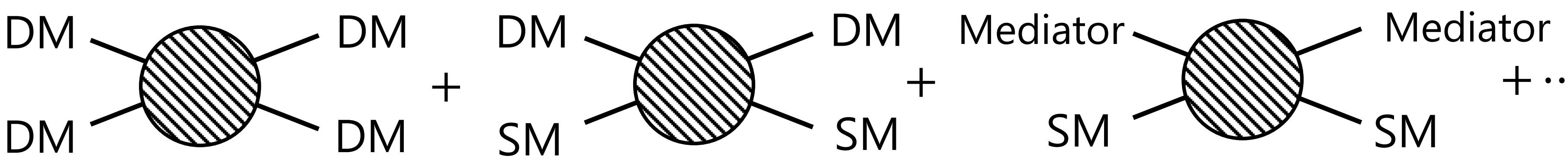
⇒ If the theory contains a spin-J ($J\geq 2$) state,

 may give a new contribution that is not present in stable case.

Relevant to gravitational EFTs (graviton) or quantum gravity (Regge states)?

Summary and Discussions

- **Global constraints**
  = constraints arising from all possible 2-to-2 scatterings



  This requires general knowledge about unstable particles.

- **There are unitarity constraints and maybe positivity bounds as well.**
  $\mathcal{A}\mathcal{A} \to \mathcal{A}\mathcal{A}$ is technically more complicated but is possible.

- **Whether stable or unstable may be important if a spin-J $(J \geq 2)$ exists.**
  Swampland and DM, Gravitational EFT, Quantum gravity, QCD?

- **They are still very preliminary and further investigations are needed.**