2024.01.18

Strong/weak CY/CY correspondence and the grade restriction rule

Kentaro Hori

Based on

KH, David Tong 2006

Manfred Herbst, KH, David Page 2008

KH 2011

KH, Mauricio Romo 2013

Richard Eager, KH, Johanna Knapp, Mauricio Romo 2024

A class of 2d (2,2) supersymmetric gauge theories called

gauged linear sigma models provide explanation of,

or gave birth to

Calabi-Yau/Landau-Ginzburg correspondence

McKay correspondence

Calabi-Yau/Calabi-Yau correspondence

$G = U(1)$

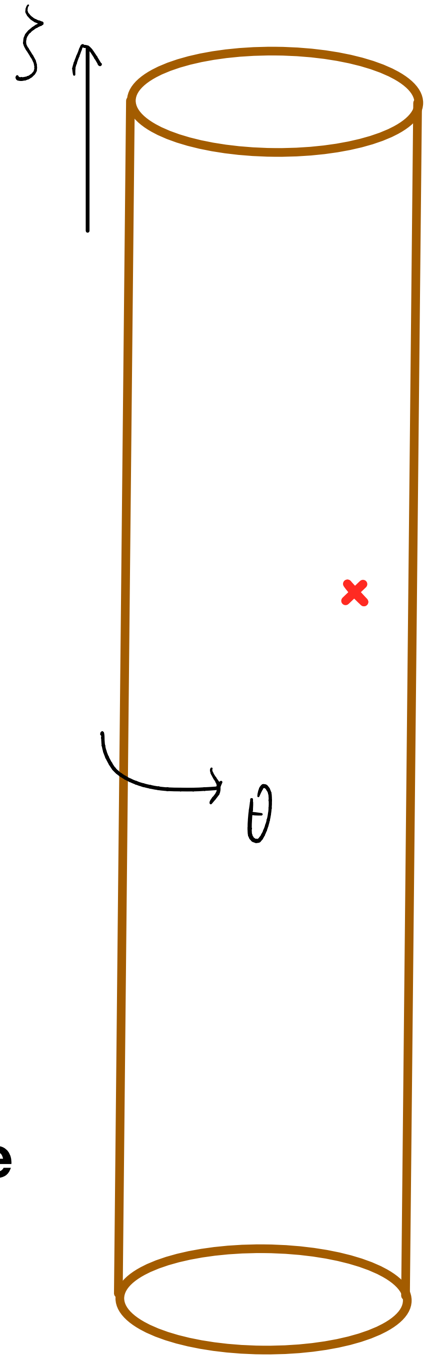
$V = \mathbb{C}(1)^{\oplus 5} \oplus \mathbb{C}(-5)$

$\quad x_1, \dots, x_5 \qquad p$

$W = p\, f(x_1, \dots, x_5)$

↑

quintic polynomial



quintic Calabi-Yau 3-fold

$X_f = \{ f = 0 \} \subset \mathbb{CP}^4$

∃ Coulomb branch

Landau-Ginzburg orbifold

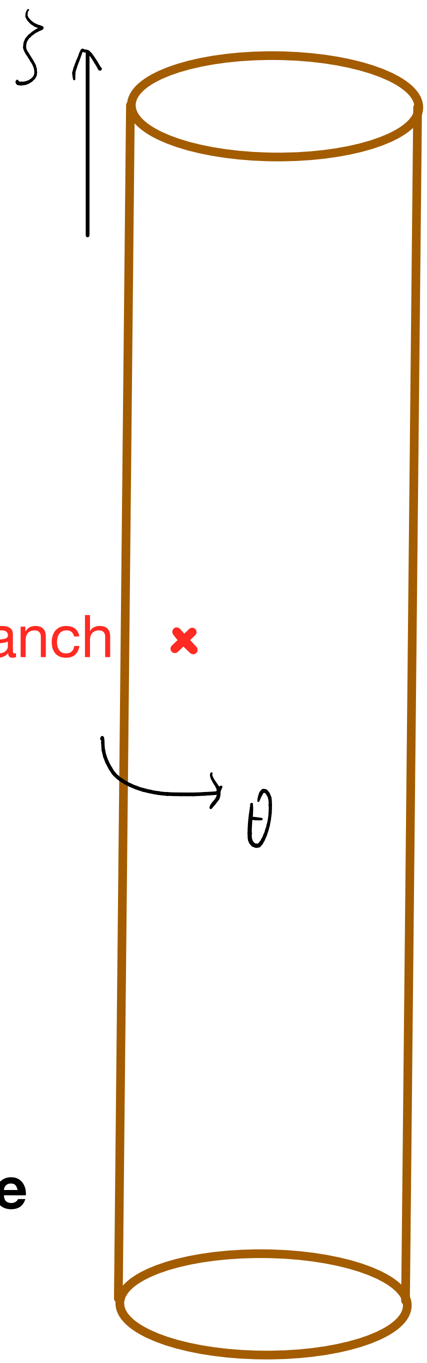
$(\mathbb{C}^5, f(x))/\mathbb{Z}_5$

CY/LG correspondence

Witten 1993

$G = U(1)$

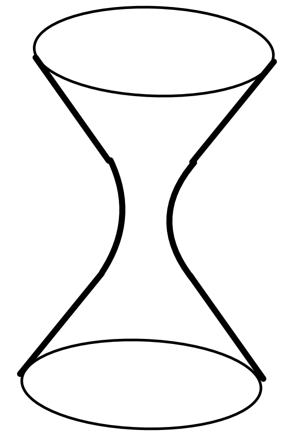
$V = \mathbb{C}(1)^{\oplus 2} \oplus \mathbb{C}(-2)$



∃ Coulomb branch ×

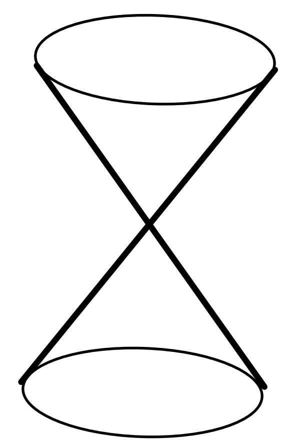
Eguchi-Hanson space

$\mathcal{O}(-2)$
$\downarrow$
$\mathbb{CP}^1$



Orbifold

$\mathbb{C}^2/\{\pm 1\}$

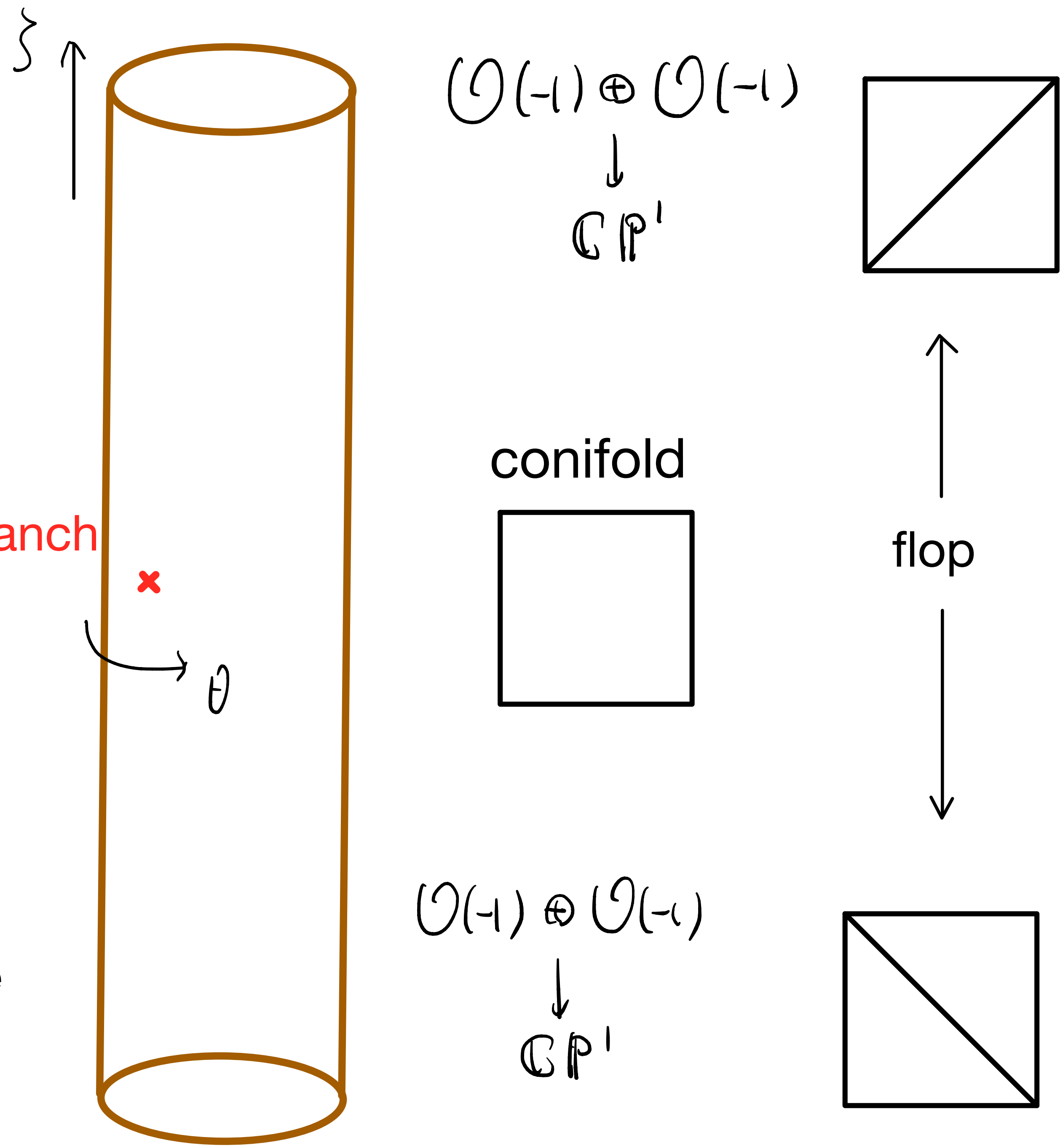


McKay correspondence

$G = U(1)$

$V = \mathbb{C}(1)^{\oplus 2} \oplus \mathbb{C}(-1)^{\oplus 2}$

$\exists$ Coulomb branch



CY/CY correspondence

Witten 1993

In these models, the phases are **weakly coupled**:

the gauge symmetry is completely broken or broken to a finite subgroup, and the classical analysis is enough to read off the low energy behavior.

There are interesting models with **strongly coupled phases**:

a continuous gauge symmetry remains unbroken, and you need a quantum analysis to find the low energy behavior.

Rødland model

H Tong 2006

$$G = U(2)$$

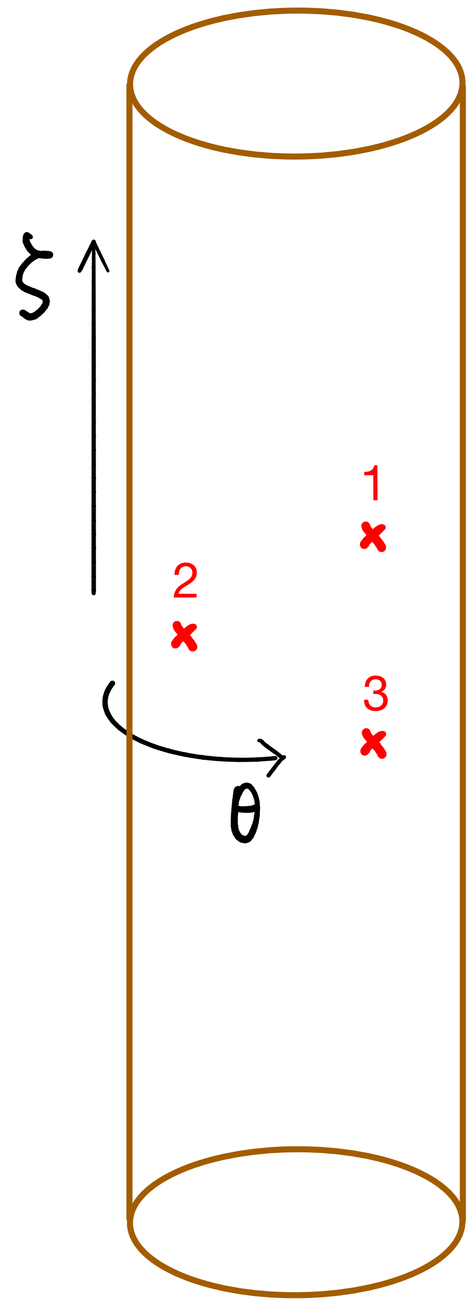
$$V = (\det{}^{-1})^{\oplus 7} \oplus (\mathbb{C}^2)^{\oplus 7} \ni (p^1, \cdots, p^7, X_1, \cdots, X_7)$$

$$W = \sum_{i,j,k} A^{ij}_k \, p^k [X_i X_j]$$

$$[X_i X_j] := X_i^1 X_j^2 - X_i^2 X_j^2 ,$$

write $W = \sum_{ij} A^{ij}(p) [X_i X_j] = \sum_k p^k A_k(X)$

mass matrix



$U(2) \longrightarrow \{1\}$

"weakly coupled phase"

$\exists$ Coulomb branch

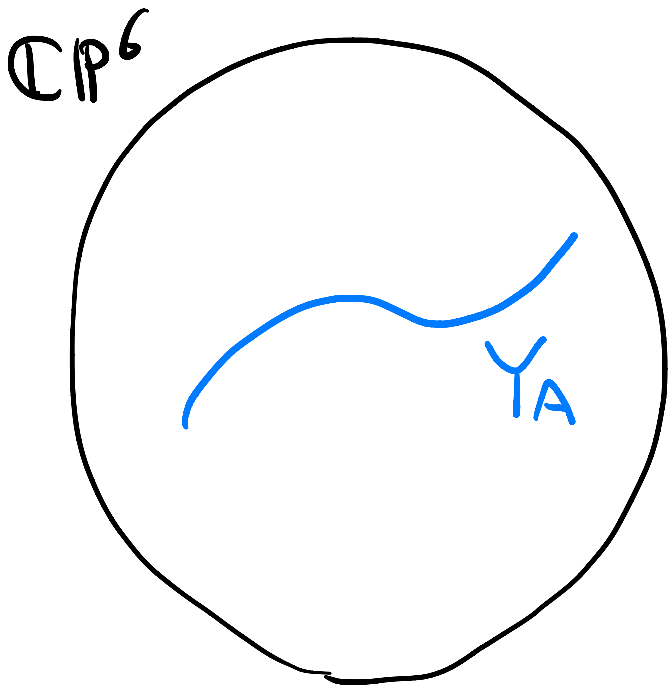
at $\begin{cases} \zeta_a = 7 \log\left(2\cos\left(\frac{\pi a}{7}\right)\right) \\ \theta_a \equiv \pi a \end{cases}$ $a = 1,\ 2,\ 3.$

$SU(2)$ unbroken

"strongly coupled phase"

$\zeta \ll 0$: $SU(2)$ gauge theory fibered over $\mathbb{CP}^6 \ni p$:

seven fundamentals with mass matrix $A^{ij}(p) \xrightarrow{\text{rank}} \{4, 6\}$



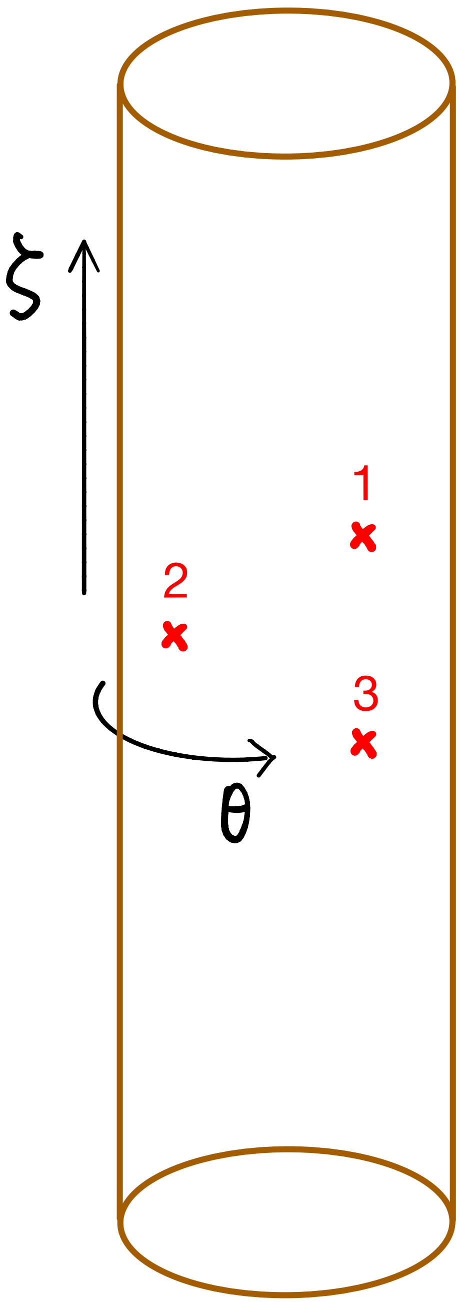
rank $A(p) = 6$: ~~SUSY~~

rank $A(p) = 4$: 3 massless $\square$

$\xrightarrow{\text{I.R.}}$ free theory of composites

$[X_1 X_2]$, $[X_2 X_3]$, $[X_3 X_2]$

$\xrightarrow{\text{I.R.}}$ σ -model with target Y_A **Pfaffian variety**



$$X_A = \{ A_1(x) = \cdots = A_7(x) = 0 \} \subset G(2,7)$$

Grassmannian

$\exists$ Coulomb branch

at $\begin{cases} \zeta_a = 7 \log\left(2\cos\left(\frac{\pi a}{7}\right)\right) \\ \theta_a \equiv \pi a \end{cases}$ $\quad a = 1, 2, 3.$

$$Y_A = \{ \mathrm{rk}\, A(p) = 4 \} \subset \mathbb{CP}^6$$

Pfaffian

This is an example of a

strong/weak Calabi-Yau/Calabi-Yau correspondence

known also as **Pfaffian/Grassmannian correspondence**

There are other examples:

Hosono-Takagi model

H 2011

$$G = \frac{O(2) \times U(1)}{\{(\pm \mathbb{1}_2, \pm 1)\}}$$

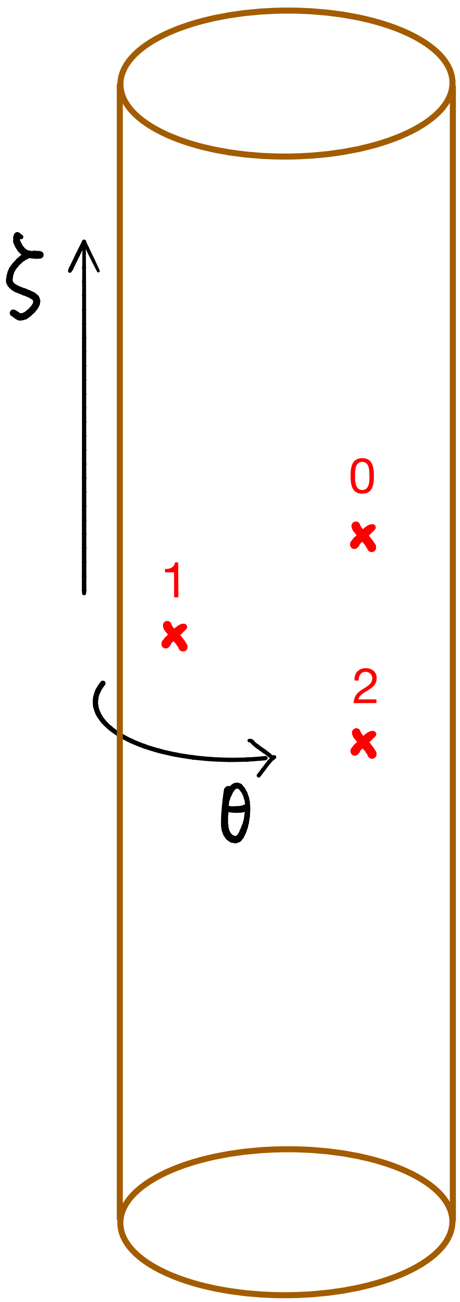
$$V = \mathbb{C}_+(-2)^{\oplus 5} \oplus \mathbb{C}^2(1)^{\oplus 5} \ni (p^1, \cdots, p^5, x_1, \cdots, x_5)$$

$$W = \sum_{i,j,k} S_k^{ij} p^k (x_i x_j)$$

$$(x_i x_j) := x_i^1 x_j^1 + x_i^2 x_j^2$$

write $W = \sum_{ij} S^{ij}(p)(x_i x_j) = \sum_k p^k S_k(x)$

mass matrix



$G \to \{1\}$

"weakly coupled phase"

$\exists$ Coulomb branch

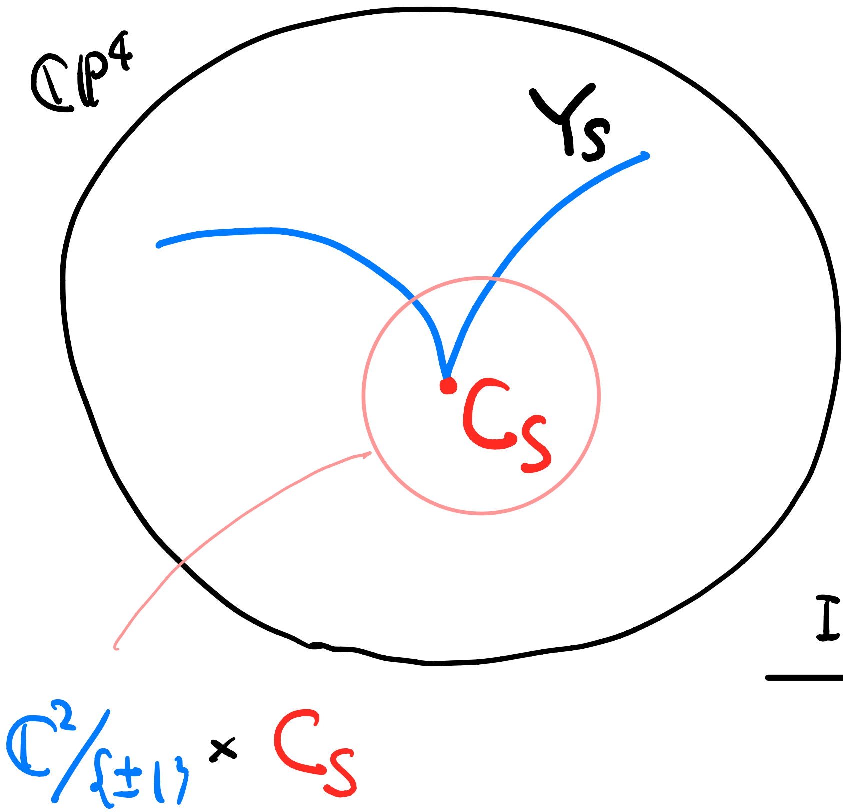
at $\begin{cases} \zeta_a = 5 \log\left(2\cos\left(\frac{\pi a}{5}\right)\right) \\ \theta_a \equiv \pi a + \pi \end{cases}$ $a = 0, 1, 2.$

$O(2)$ unbroken

"strongly coupled phase"

$\zeta \ll 0$: $O(2)$ gauge theory fibered over $\mathbb{CP}^4 \ni p$:

five fundamentals with mass matrix $S^{ij}(p) \xrightarrow{\text{rank}} \{3, 4, 5\}$



$\operatorname{rank} S(p) = 5$: ~~SUSY~~

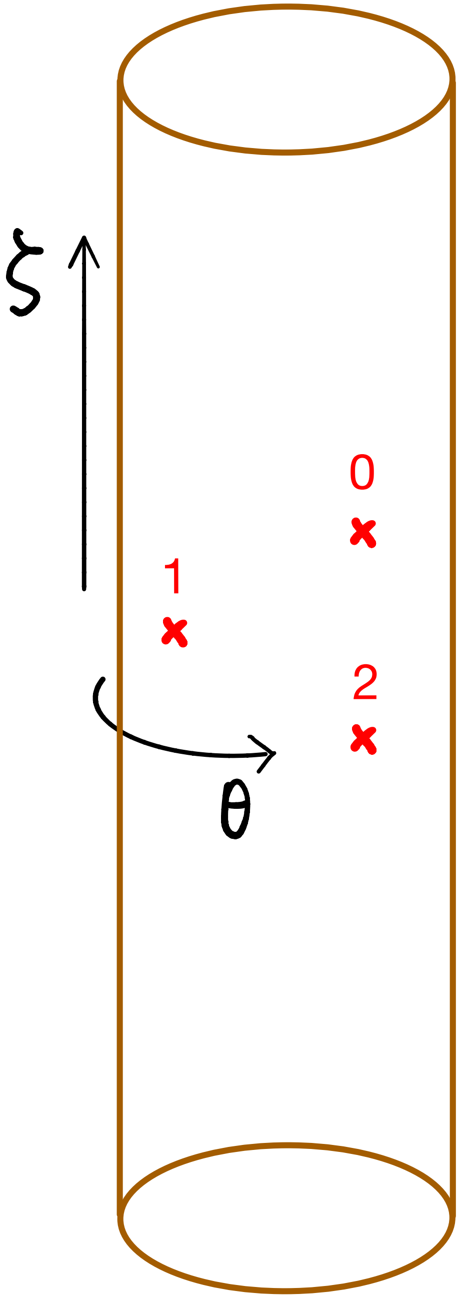
$\operatorname{rank} S(p) = 4$: two vacua

$\operatorname{rank} S(p) = 3$: single vacuum

$\xrightarrow{\text{I.R.}}$ σ-model with target

ramified double cover $\tilde{Y}_S \longrightarrow Y_S \supset C_S$

$\tilde{Y}_S \longrightarrow C_S$



$$X_S = \{ S_1(x) = \cdots = S_5(x) = 0 \} \subset \frac{\mathbb{CP}^4 \times \mathbb{CP}^4}{\text{exchange}}$$

Reye congruence

$\exists$ Coulomb branch

at
$$\begin{cases} \zeta_a = 5 \log\left(2 \cos\left(\frac{\pi a}{5}\right)\right) \\ \theta_a \equiv \pi a + \pi \end{cases} \qquad a = 0, 1, 2.$$

$$\tilde{Y}_S \longrightarrow Y_S = \{ \mathrm{rk}\, S(p) \leq 4 \}$$

$$\cup$$

$$\tilde{Y}_S \searrow C_S = \{ \mathrm{rk}\, S(p) = 3 \}$$

There are dual models where

strong and **weak** are exchanged.

Constructed from 2d Seiberg duality.   H 2011

Dual Rødland

H 2011

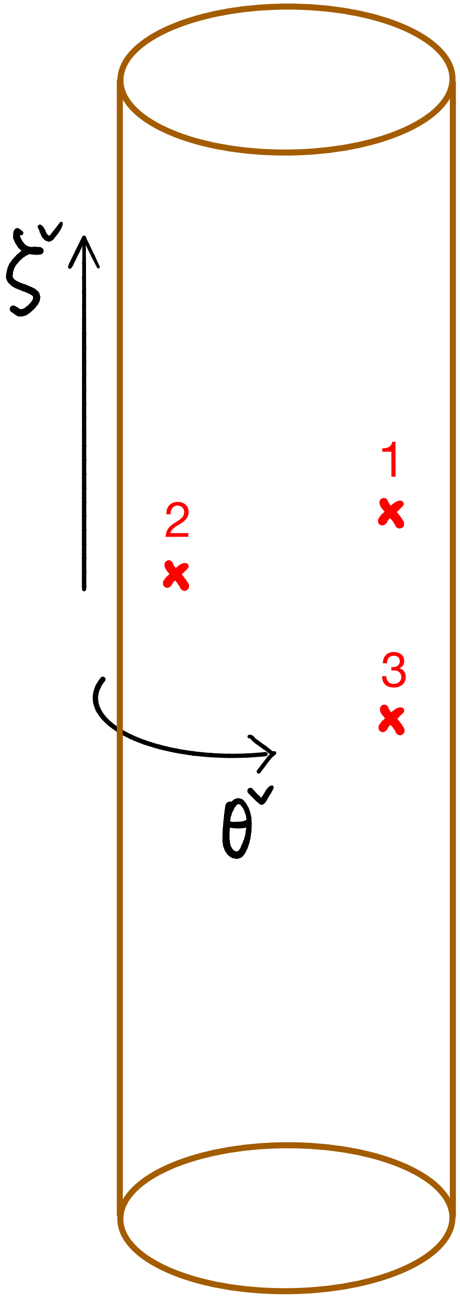
$$G^\vee = \frac{USp(4) \times U(1)}{\{(\pm \mathbb{1}_4, \pm 1)\}}$$

$$V^\vee = \mathbb{C}(-2)^{\oplus 7} \oplus \mathbb{C}^4(-1)^{\oplus 7} \oplus \mathbb{C}(2)^{\oplus \binom{7}{2}} \ni (p^k ; \tilde{X}^i ; a_{ij})$$

$$W^\vee = \sum_{i,j,k} A^{ij}_k \, p^k a_{ij} + \sum_{i,j} a_{ij} [\tilde{X}^i \tilde{X}^j]$$

$$[\tilde{X}^i \tilde{X}^j] = J^{ab} \tilde{X}^i_a \tilde{X}^j_b$$

$$a_{ij} = - a_{ji}$$



$USp(4)$ unbroken

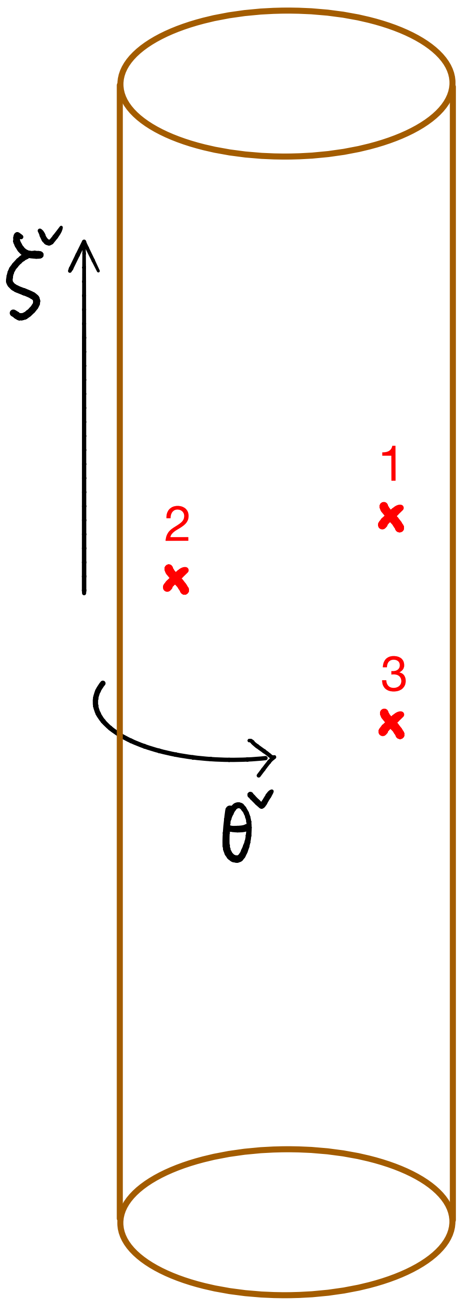
"strongly coupled phase"

$\exists$ Coulomb branch

at $\begin{cases} \zeta^\vee_a = 7 \log\left(2\cos\left(\frac{\pi a}{7}\right)\right) \\ \theta^\vee_a \equiv \pi a \end{cases}$ $a = 1, 2, 3.$

$G^\vee \to \{1\}$ (superpotential needed)

"weakly coupled phase"



$$\left\{ \mathrm{rk}\, a = 2,\ \sum_{ij} A_k^{ij} a_{ij} = 0 \right\} \subset \mathbb{CP}^{20}$$

$$a_{ij} = [x_i x_j] \quad \xrightarrow{\text{Plücker}} \quad \cong X_A$$

$\exists$ Coulomb branch

at $\begin{cases} \zeta^\vee_a = 7 \log\left(2\cos\left(\frac{\pi a}{7}\right)\right) \\ \theta^\vee_a \equiv \pi a \end{cases}$ $\quad a = 1, 2, 3.$

$$\left\{ A_{ij}(p) + [\tilde{x}^i \tilde{x}^j] = 0 \right\}^S \Big/ G^\vee_{\mathbb{C}} \cong Y_A$$

$$\Downarrow$$

$$\mathrm{rk}\, A(p) = 4 \longrightarrow Y_A$$

Dual Hosono-Takagi

H 2011

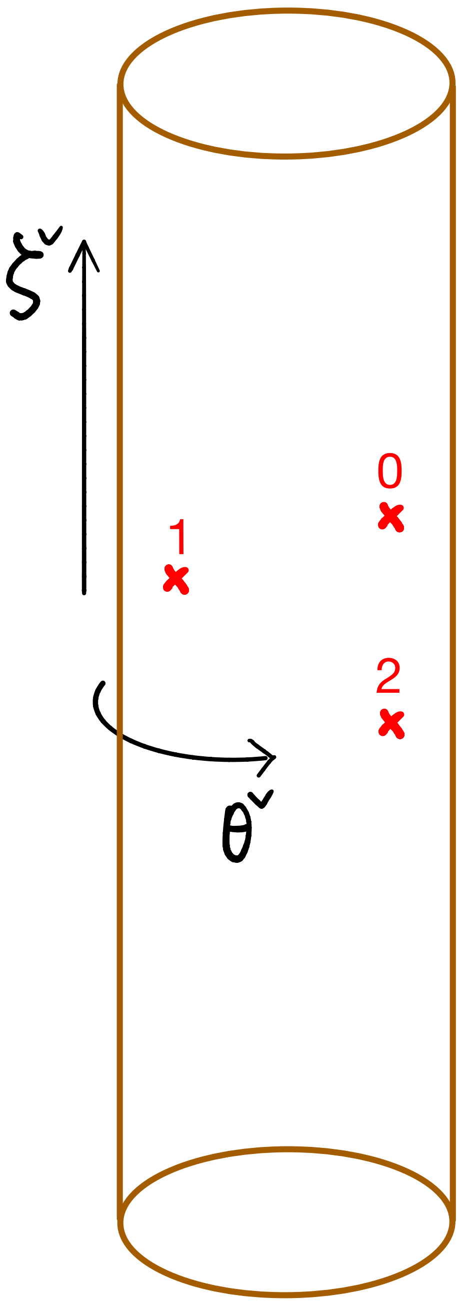
$$G^\vee = \frac{SO(4) \times U(1)}{\{(\pm \mathbb{1}_4, \pm 1)\}}$$

$$V^\vee = \mathbb{C}(-2)^{\oplus 5} \oplus \mathbb{C}^4(-1)^{\oplus 5} \oplus \mathbb{C}(2)^{\oplus 15} \ni (p^k; \tilde{x}^i; s_{ij})$$

$$W^\vee = \sum_{i,j,k} S^{ij}_k p^k s_{ij} + \sum_{ij} s_{ij} (\tilde{x}^i \tilde{x}^j)$$

$$[\tilde{x}^i \tilde{x}^j] = \delta^{ab} \tilde{x}^i_a \tilde{x}^j_b$$

$$s_{ij} = s_{ji}$$



$SO(4)$ unbroken

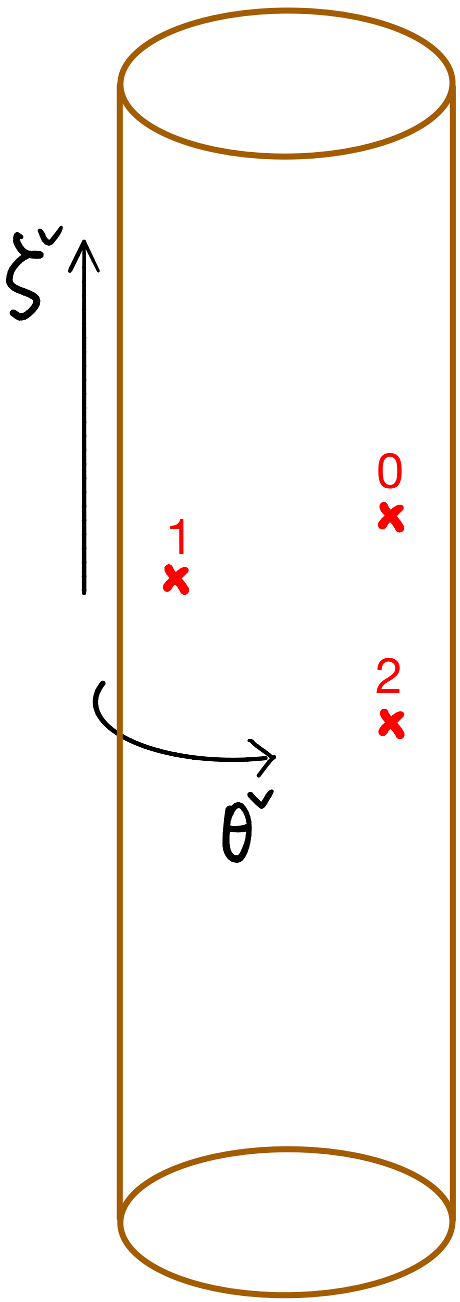
"strongly coupled phase"

$\exists$ Coulomb branch

at
$$\begin{cases} \zeta^\vee_a = 5\log\left(\cos\left(\frac{\pi a}{5}\right)\right) \\ \theta^\vee_a \equiv \pi a + \pi \end{cases} \qquad a = 0, 1, 2.$$

$G^\vee \to \{1\}$ (superpotential needed)

"weakly coupled phase"



$$\left\{ \operatorname{rank} S = 2,\ \sum_{ij} S_k^{ij} s_{ij} = 0 \right\} \subset \mathbb{CP}^{14}$$

$s_{ij} = (x_i x_j)$

$$\cong X_S$$

$\exists$ Coulomb branch

at $\begin{cases} \zeta^\vee_a = 5 \log\left(\cos\left(\frac{\pi a}{5}\right)\right) \\ \theta^\vee_a \equiv \pi a + \pi \end{cases}$ $a = 0, 1, 2.$

$$\left\{ S^{ij}(p) + (\tilde{x}^i \tilde{x}^j) = 0 \right\}^s / G_G^\vee \cong \tilde{Y}_S$$

$$\downarrow \qquad\qquad \downarrow$$

$$\left\{ \operatorname{rk} S(p) \leq 4 \right\} = Y_S$$

The correspondences have mathematical consequences.

For example:

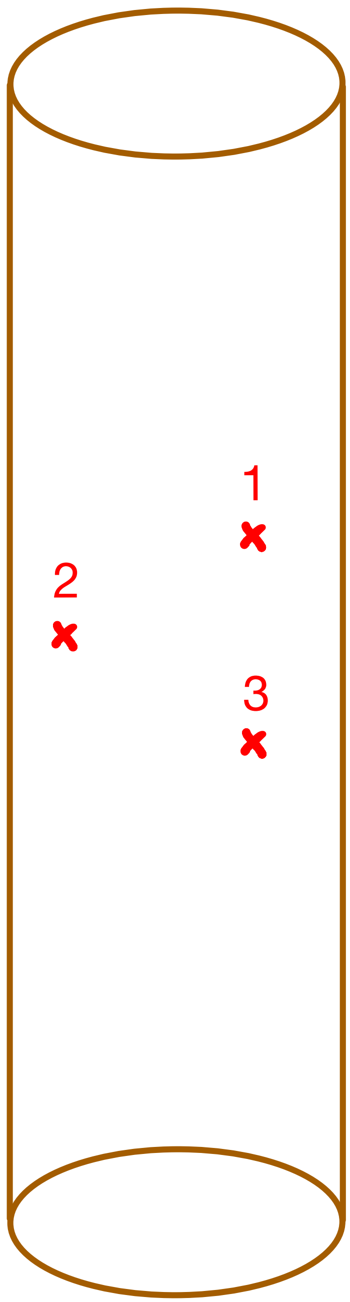
Gromov-Witten (or FJRW) invariants are related by

analytic continuation.

Categories of B-branes are equivalent.

⋮

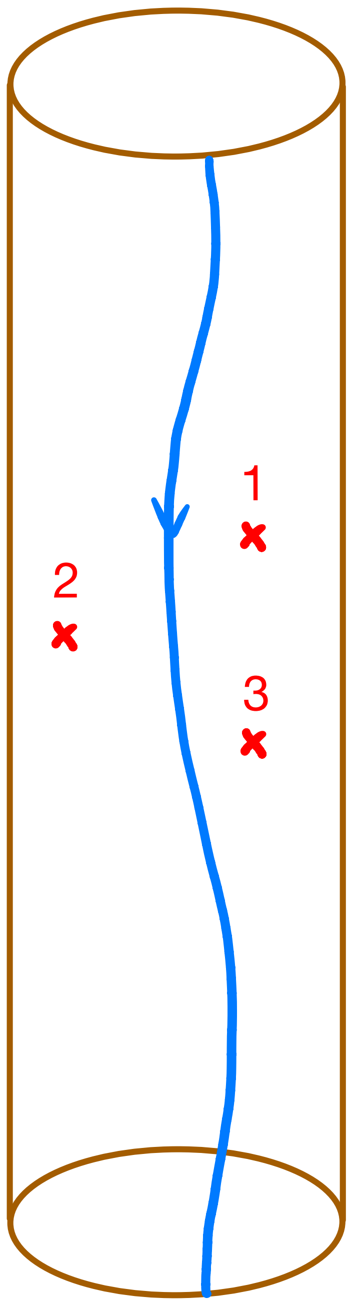
Categories of B-branes



$$D^b(X_A)$$

$$D^b(Y_A)$$

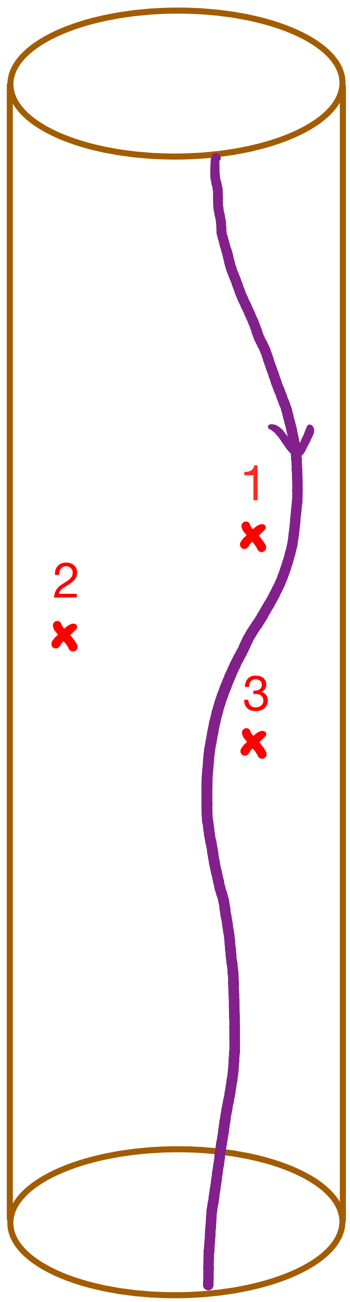
A path



An equivalence of categories

$$D^b(X_A) \xrightarrow{\ \cong\ } D^b(Y_A)$$

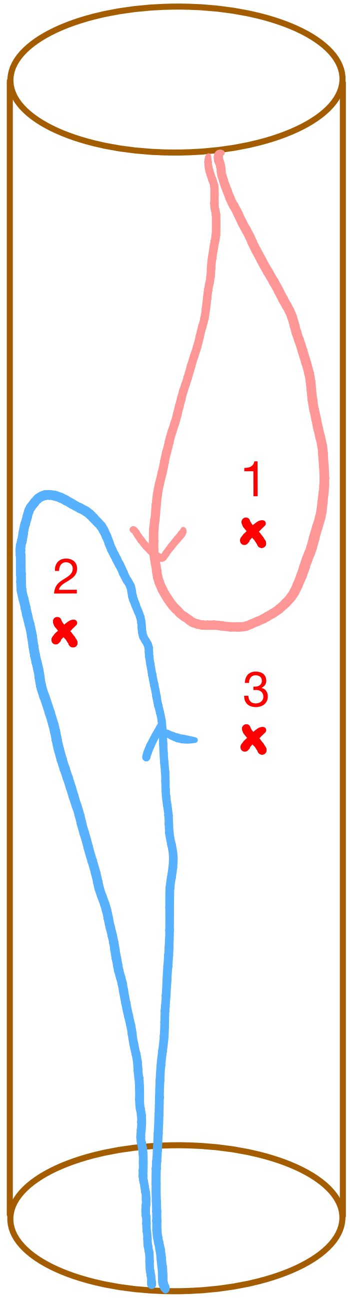
Another path



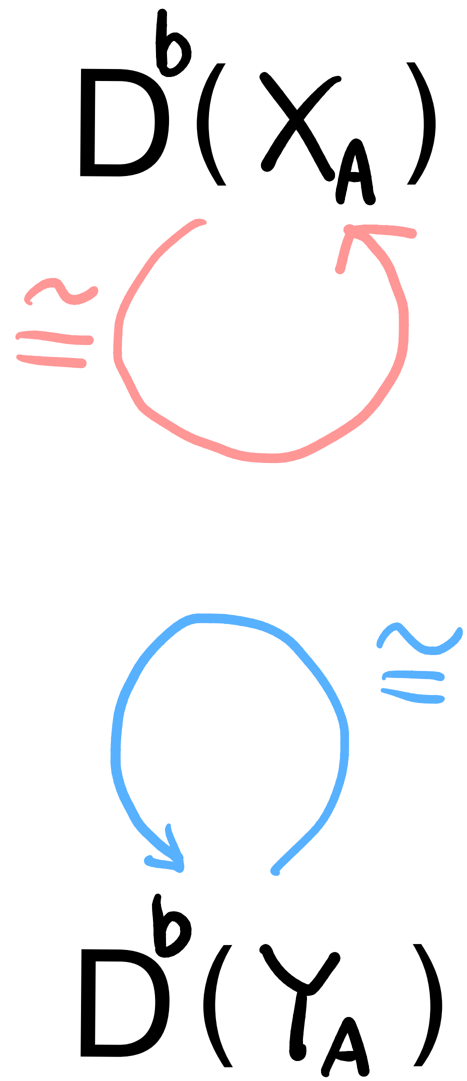
Another equivalence of categories

$$D^b(X_A) \xrightarrow{\cong} D^b(Y_A)$$

Loops



Autoequivalences of categories



We would like to find what they are,

using GLSMs.

In GLSM with gauge group G, matter V, superpotential W,

a classical B-brane is specified by

(M, Q) ···· a G *-equivariant matrix factorization of* (V, W)

$M = M^{ev} \oplus M^{od}$ a $\mathbb{Z}_2$ graded representation of G

$Q : V \longrightarrow \mathrm{End}^{od}(M)$ a G-equivariant polynomial function

s.t. $Q^2 = W \, id_M$

+ an additional grading compatible with $\mathbb{Z}_2$.

They form a category $D_G(V, W)$.

However, a classical B-brane may induce instability in quantum theory, especially when a continuous gauge symmetry remains unbroken.

E.g. Around phase boundaries

Inside strongly coupled phases

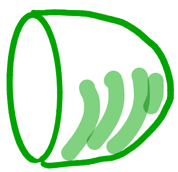
Theories with simple gauge groups

We would like to find a condition on $(\mathcal{M}, Q)$ to define a well behaved B-brane in the quantum theory.

A proposal

Convergence of **hemisphere partition function**

HR, Sugishita-Terashima, Honda-Okuda 2013

$\Big\langle$  $\Big\rangle$

$$= \int_{\gamma} d^{\ell}\sigma \prod_{\alpha>0} \langle \alpha, \sigma \rangle \sinh(\pi \langle \alpha, \sigma \rangle) \times \prod_i \Gamma\left(i\langle Q_i, \sigma \rangle + \frac{R_i}{2} \right) \times e^{i\langle t, \sigma \rangle} \cdot \mathrm{tr}_M \left(e^{\pi i R_M} e^{2\pi\sigma} \right)$$

$$V|_{T \times U(1)_V} = \bigoplus_i \mathbb{C}(Q_i, R_i)$$ weight decomposition

This imposes a severe constraint on representations of G

that can be included in M.

For theories with Abelian gauge groups, this reproduces

the **grade restriction rule** for D-brane transport across

phase boundaries. HHP 2008

We shall continue to call the constraint

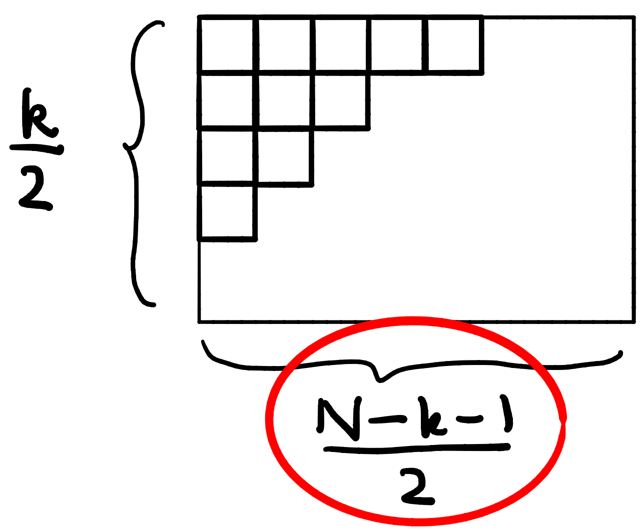
grade restriction rule.

Example G simple, V symmetric: $\{\pm Q_i\}_{i>0} \cup \{0\text{'s}\}$

The representation of highest weight λ can be included in M when $\langle \lambda+\rho, \sigma\rangle < \frac{1}{2}\sum_{i>0} \left|\langle Q_i, \sigma\rangle\right| \quad \forall\, 0 \neq \sigma \in \mathfrak{t}$.

E.g. super-QCD $G = USp(k)$, $V = (\mathbb{C}^k)^{\oplus N}$ (N odd):

$S_Y(\mathbb{C}^k)^{\otimes |Y|}_0$ for Y in



Young symmetrizer J-traceless

Consistent with 2d Seiberg duality H2011.

Let us apply this to Rødland model, Hosono-Takagi model,

and their duals. Eager-H-Knapp-Romo 2014-2024

These models have gauge groups of the form $G = \dfrac{H \times U(1)}{\{(\pm 1_H, \pm 1)\}}$

with $H = USp(k), O(k), SO(k)$

e.g. $U(2) = \dfrac{USp(2) \times U(1)}{\{(\pm 1_2, \pm 1)\}}$.

Irreps of such groups is of the form $Y(i)$

- Y Young diagram specifying an irrep of $H \subset G$
- $U(1) \subset G$ acts by $\lambda \mapsto \lambda^{|Y|+2i}$

Grade restriction rule in strongly coupled phases:

Rodland $\zeta \ll 0$: $\mathbb{C}(i)$, $\square(i)$, $\square\square(i)$

Dual Rodland $\zeta^\vee \gg 0$ & $\zeta^\vee \ll 0$: $\mathbb{C}(i)$, $\square(i)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}(i)$

Hosono-Takagi $\zeta \ll 0$: $\mathbb{C}_\pm(i)$, $\square(i)$, $\square\square(i)$

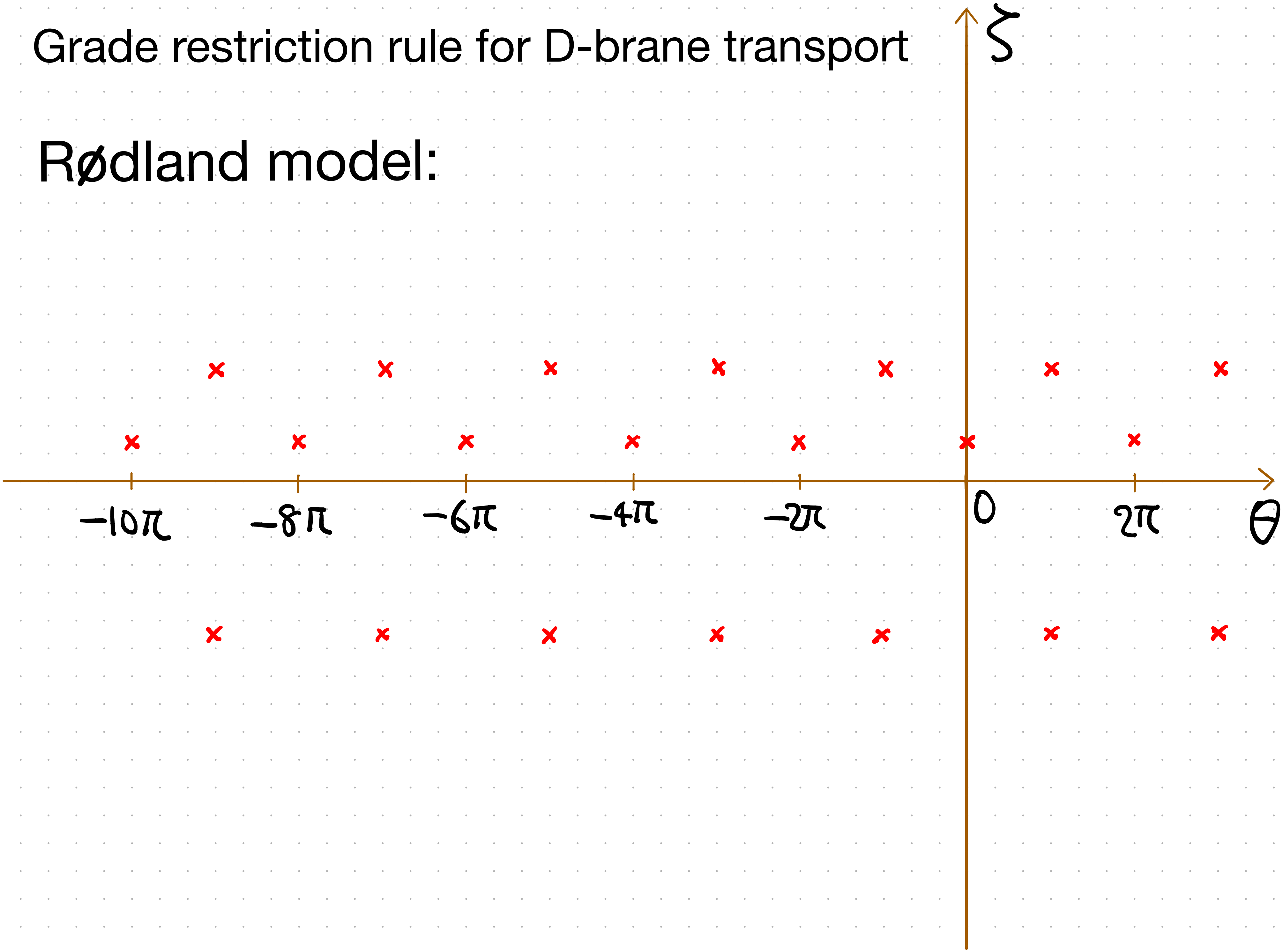
Dual Hosono-Takagi $\zeta^\vee \gg 0$ & $\zeta^\vee \ll 0$: $\mathbb{C}(i)$, $\square(i)$, $\begin{smallmatrix}\square\\\square\end{smallmatrix}_\pm(i)$

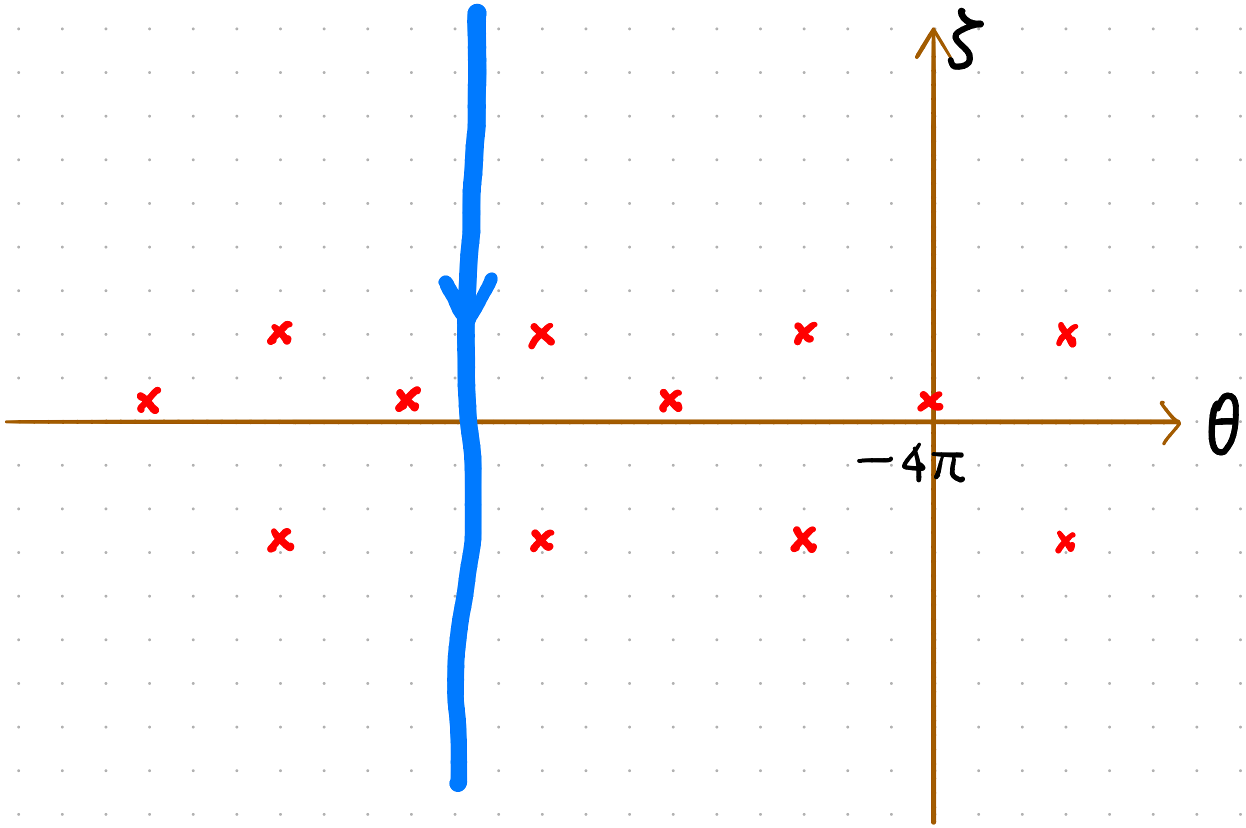
$O(2)$ representation $\mathbb{C}_\pm$: $\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix} \mapsto \pm 1$

$SO(4)$ representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}_\pm$: $\left\{\begin{array}{l}\text{self-dual}\\ \text{anti-self-dual}\end{array}\right.$ 2-form

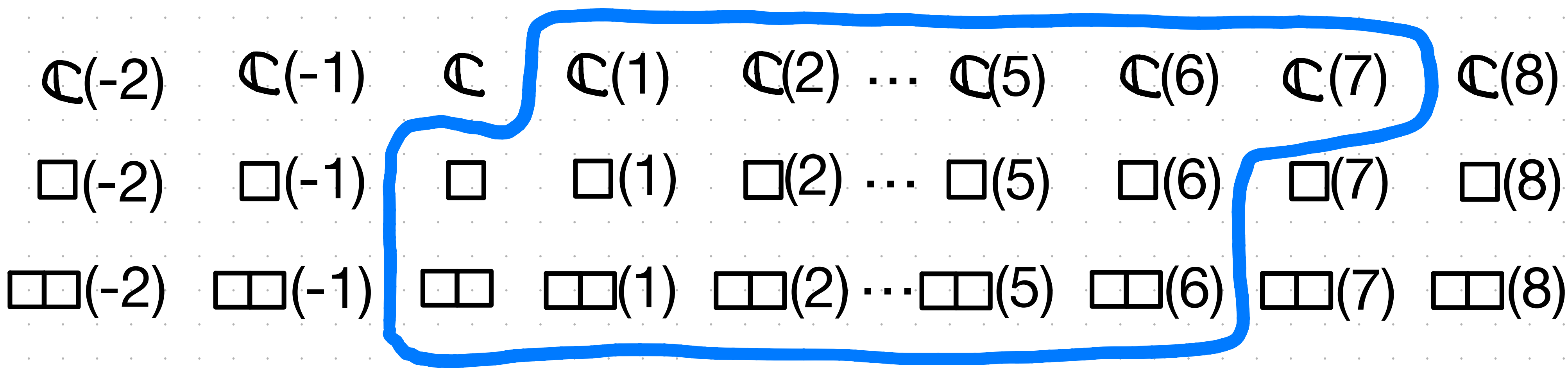
Grade restriction rule for D-brane transport

Rødland model:





ζ
θ
−4π



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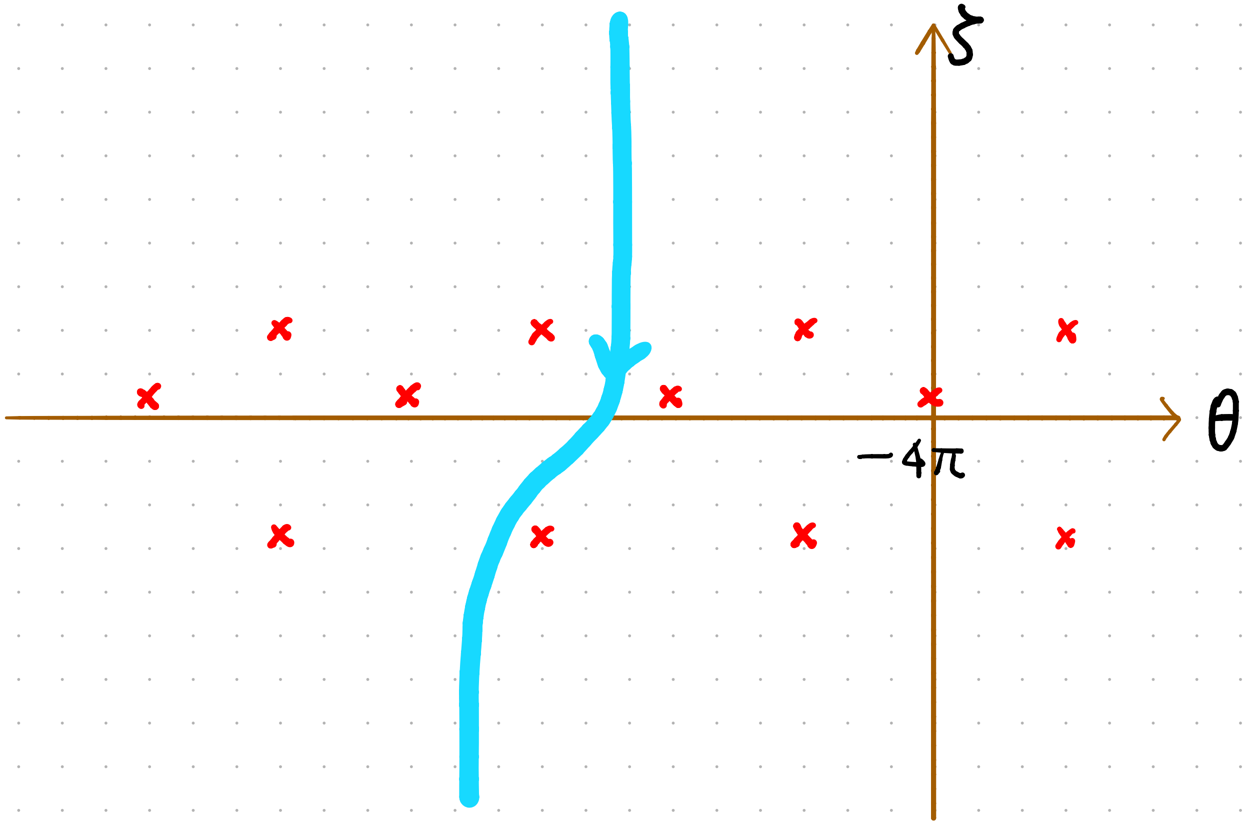
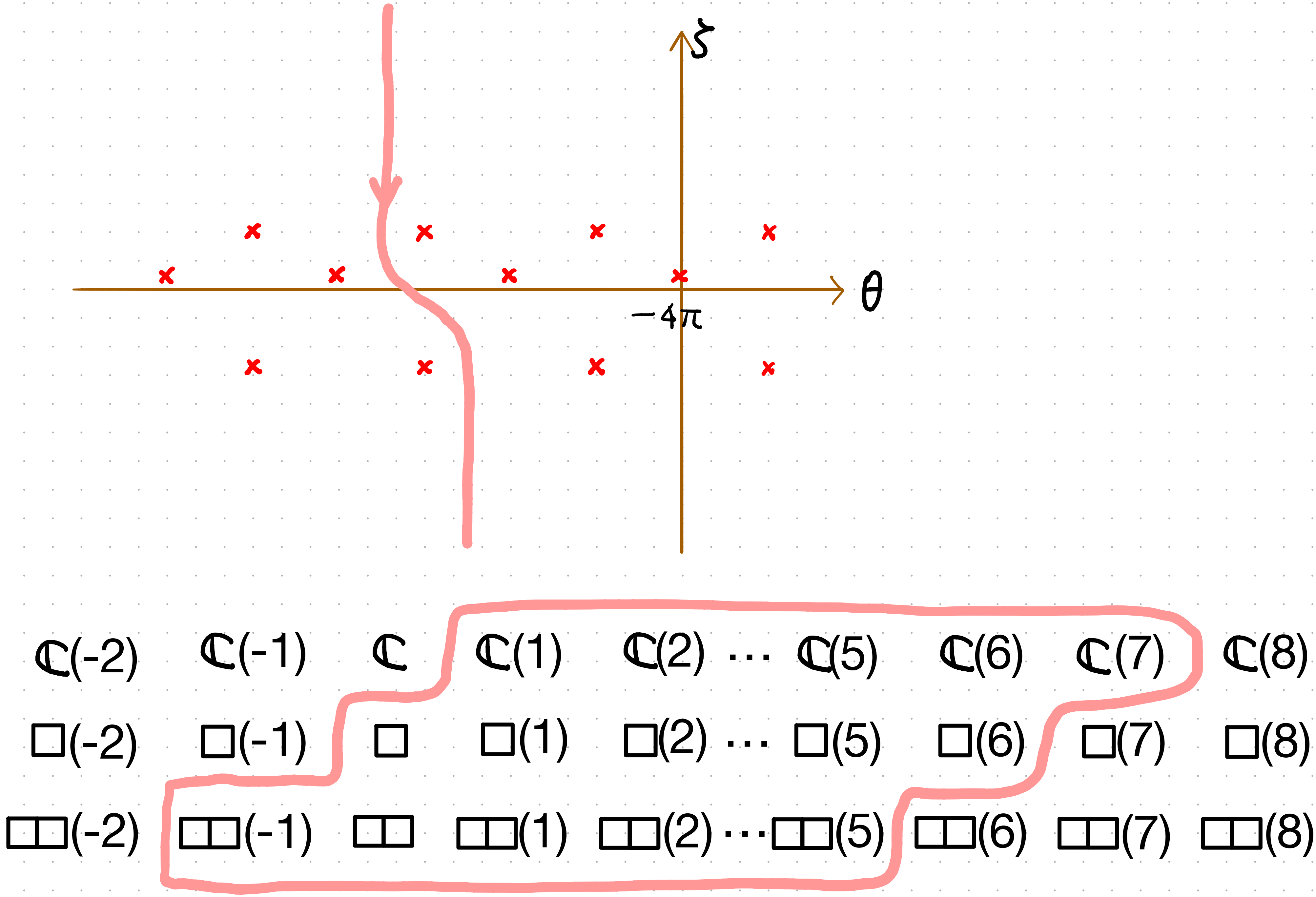


$\mathbb{C}(-2)$	$\mathbb{C}(-1)$	$\mathbb{C}$	$\mathbb{C}(1)$	$\mathbb{C}(2)$	$\cdots$	$\mathbb{C}(5)$	$\mathbb{C}(6)$	$\mathbb{C}(7)$	$\mathbb{C}(8)$
□(-2)	□(-1)	□	□(1)	□(2)	$\cdots$	□(5)	□(6)	□(7)	□(8)
□□(-2)	□□(-1)	□□	□□(1)	□□(2)	$\cdots$	□□(5)	□□(6)	□□(7)	□□(8)



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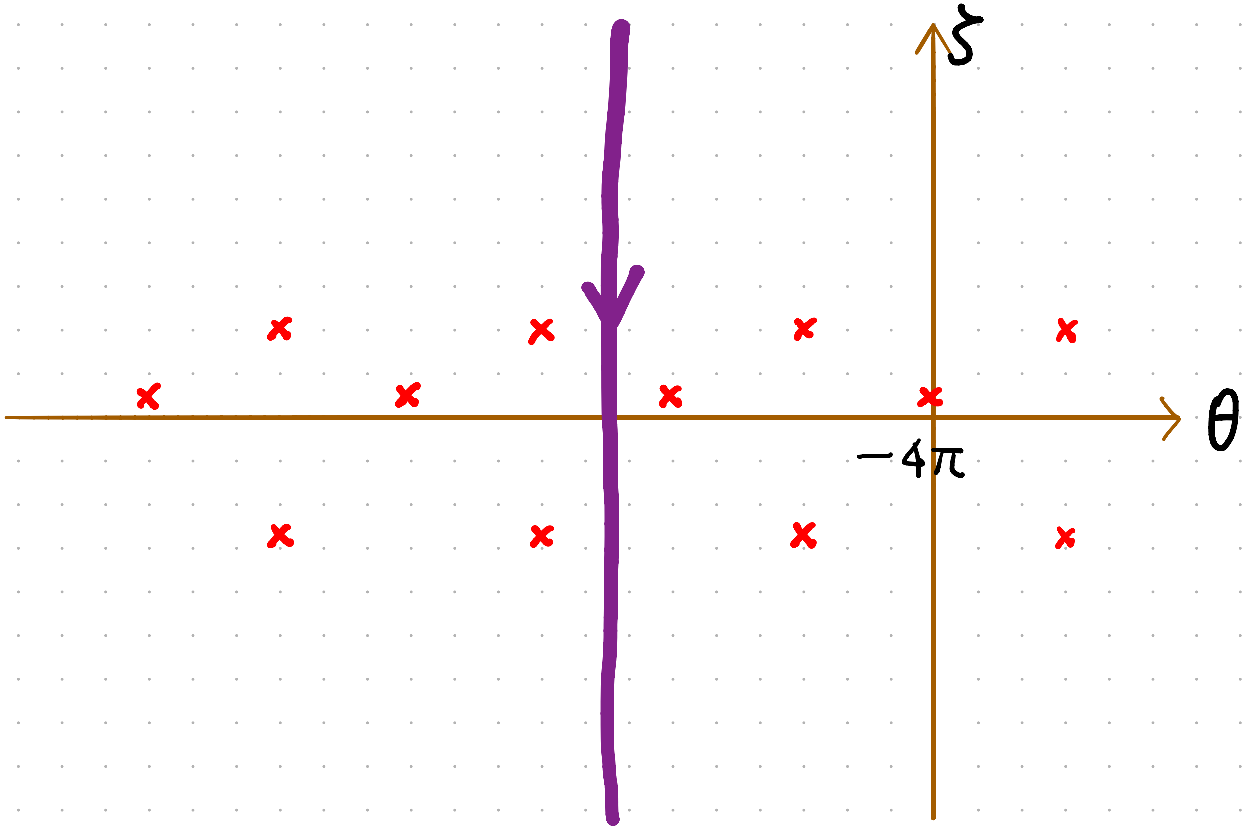
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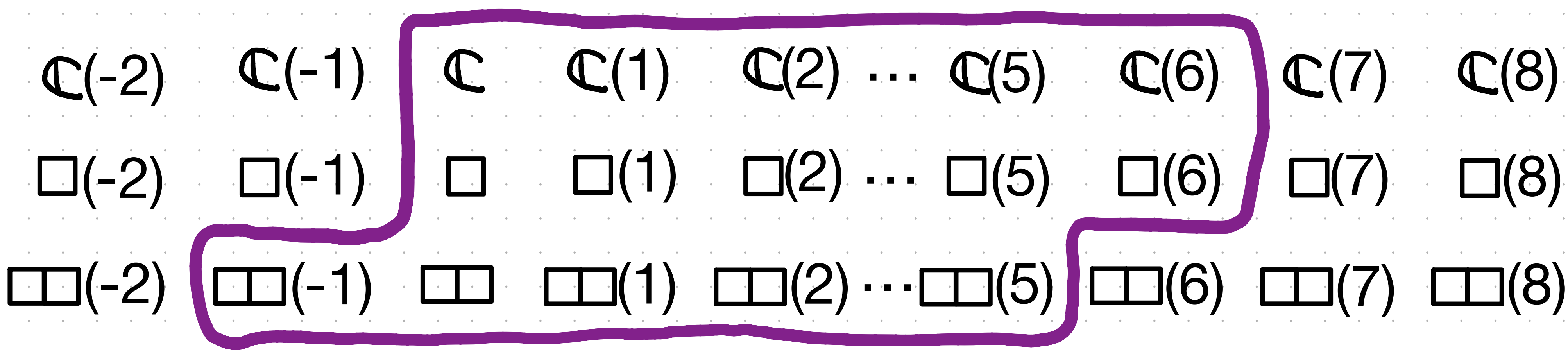
ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ⋯ ℂ(5) ℂ(6) ℂ(7) ℂ(8)
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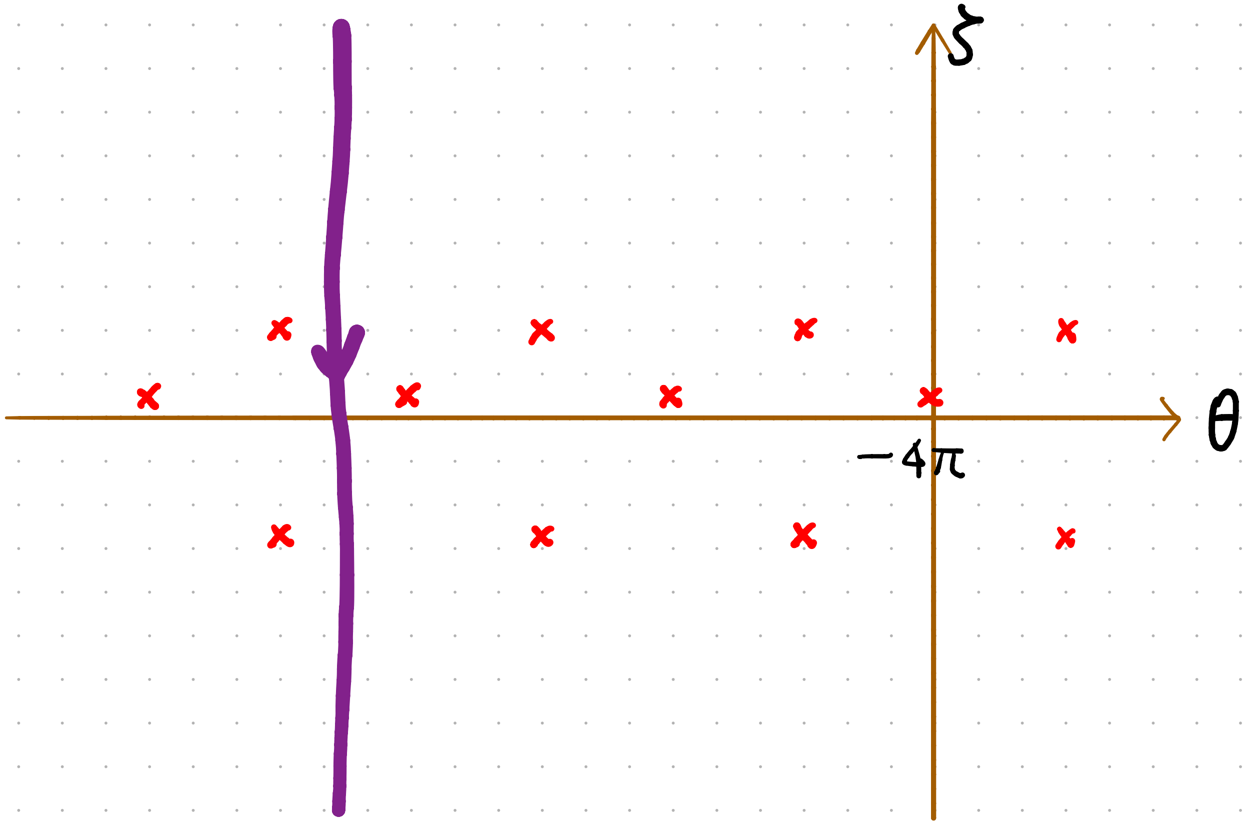
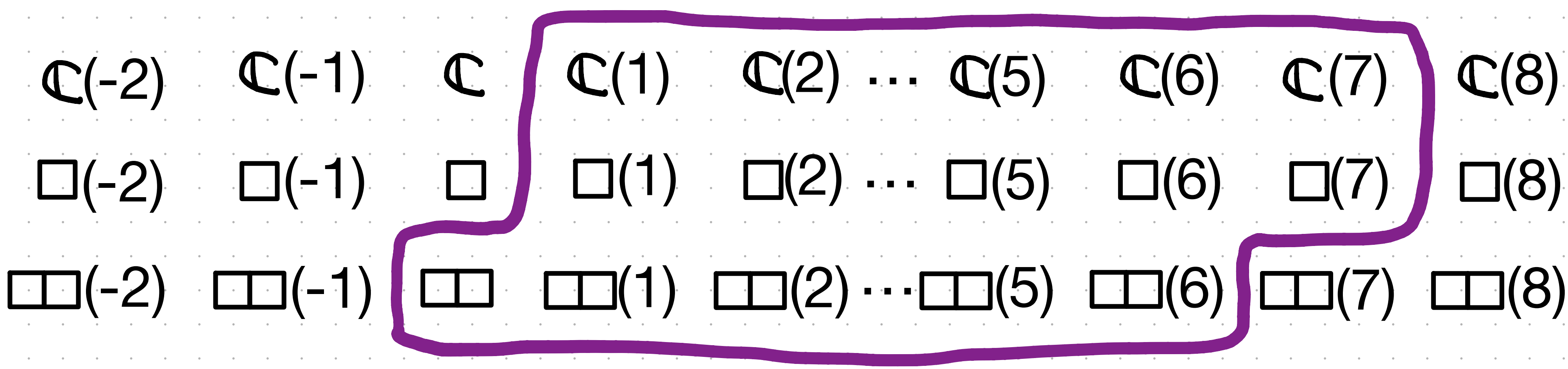
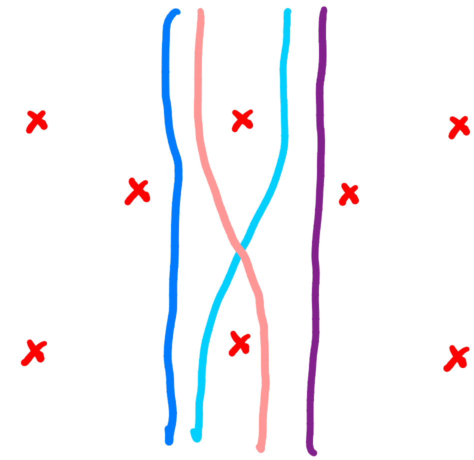

ζ
θ
−4π


ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ··· ℂ(5) ℂ(6) ℂ(7) ℂ(8)

For each homotopy class of paths (“window”) **w**

between $\zeta \gg 0$ and $\zeta \ll 0$, such as 

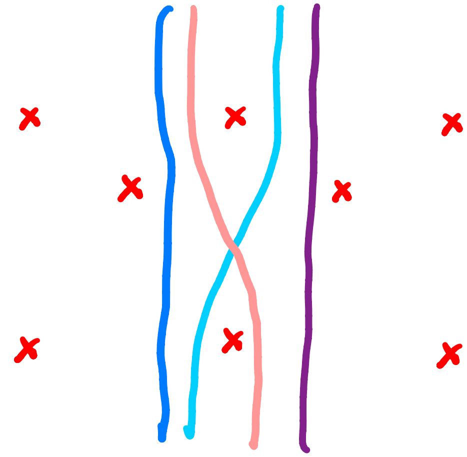
let $\mathcal{T}_{\mathbf{w}}$ be the subcategory of

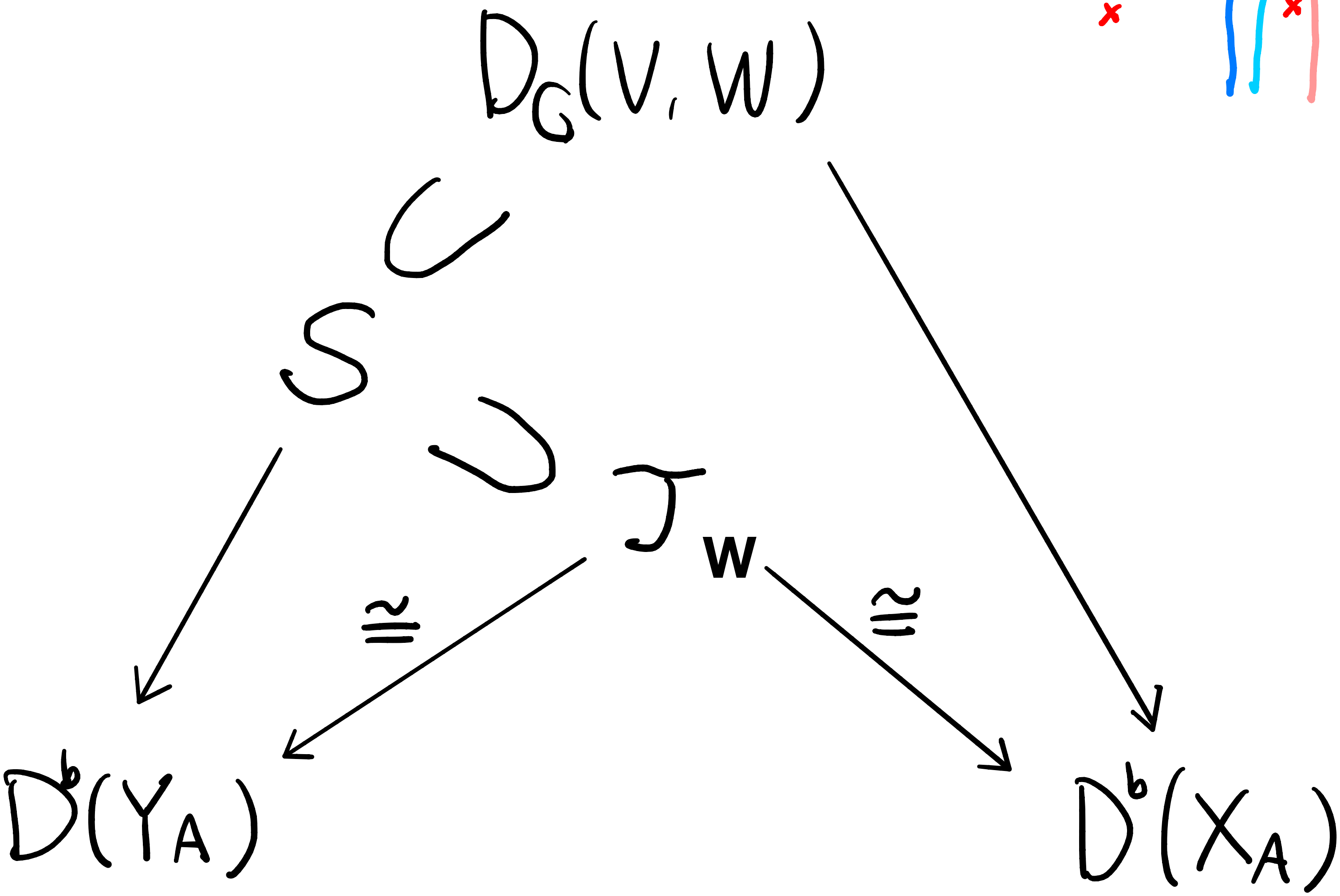
equivariant matrix factorizations made of

grade restricted representations with respect to **w**.

$\mathcal{S}$: the one at $\zeta \ll 0$.

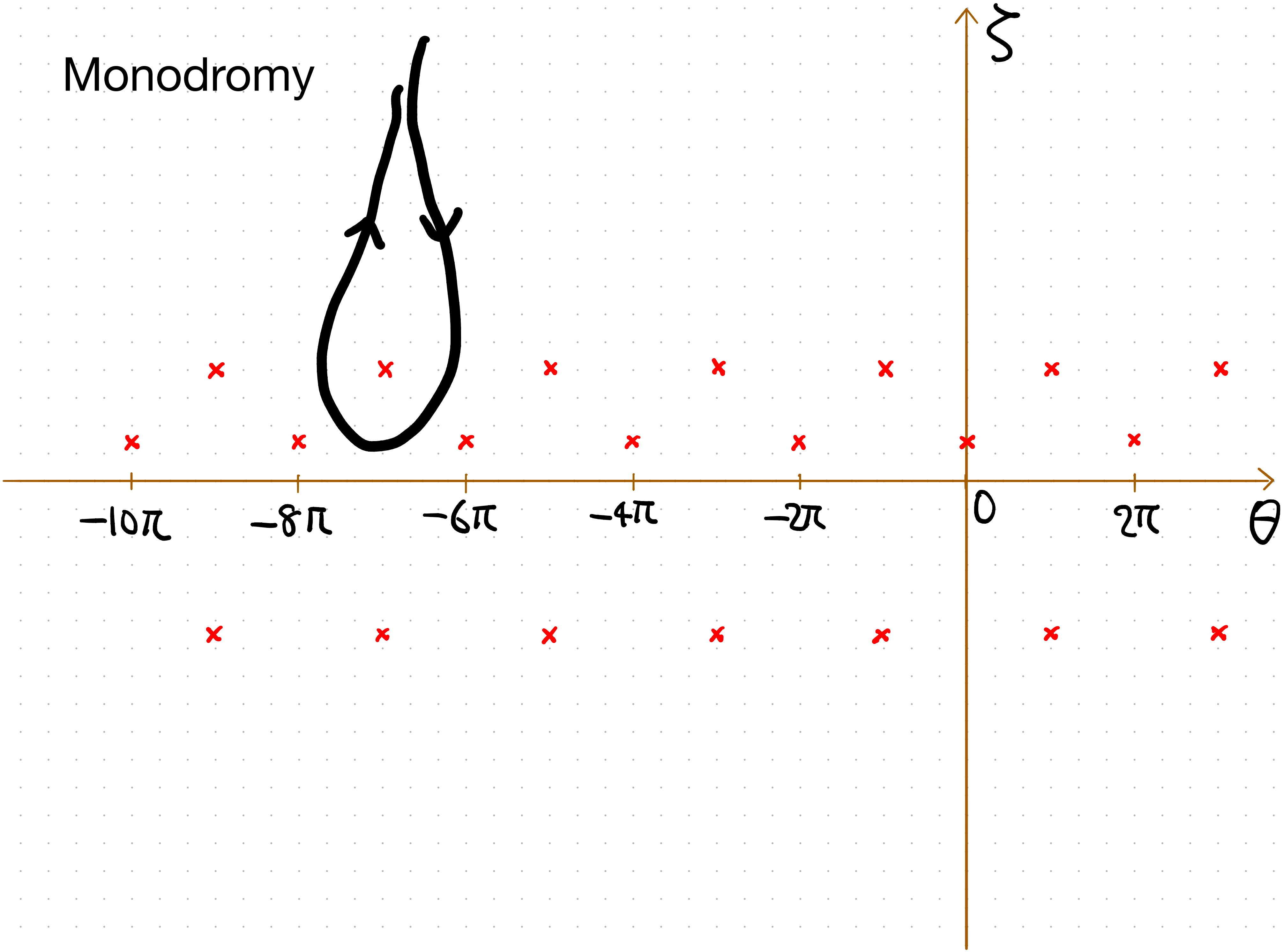
Then, we can make equivalences

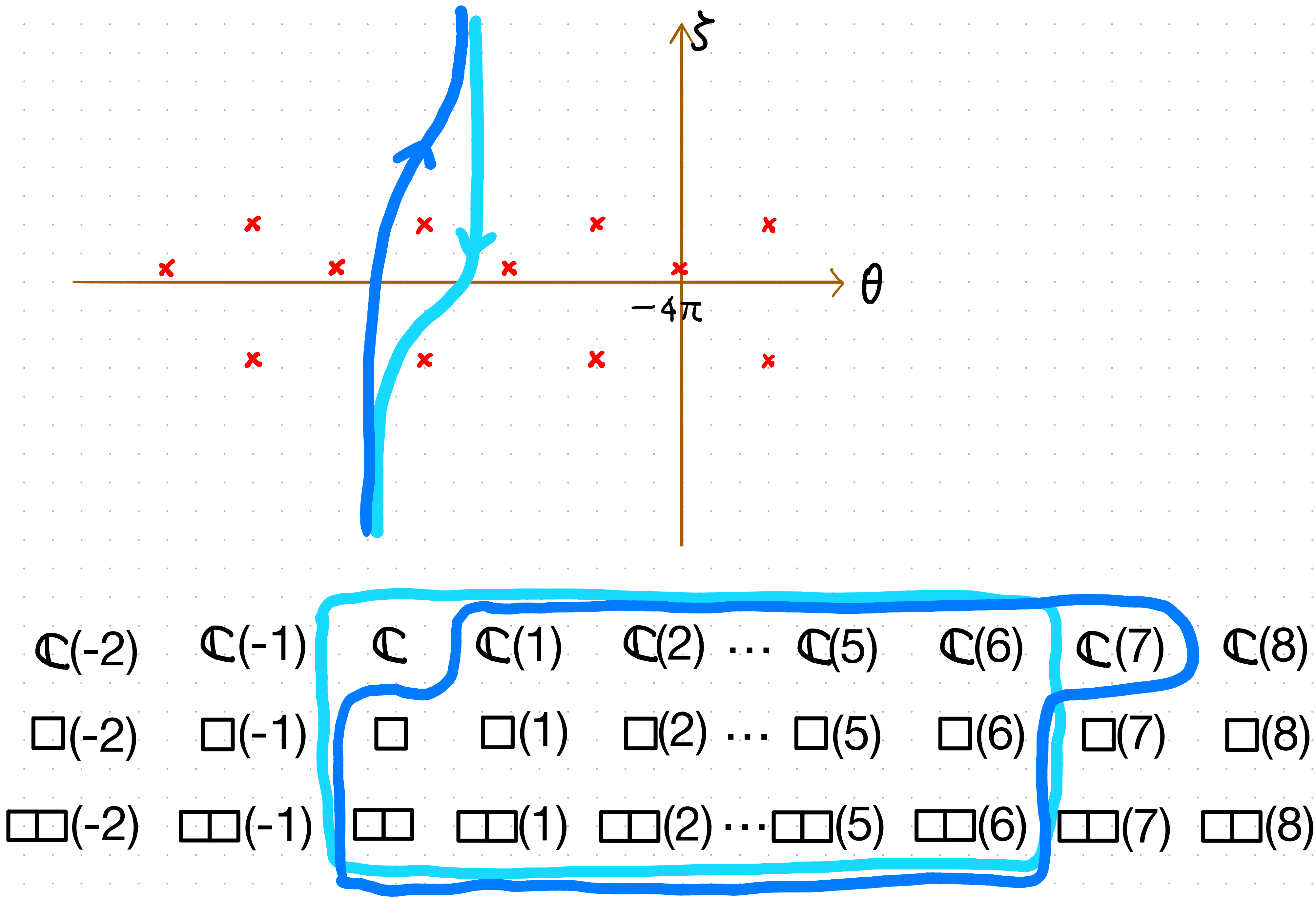


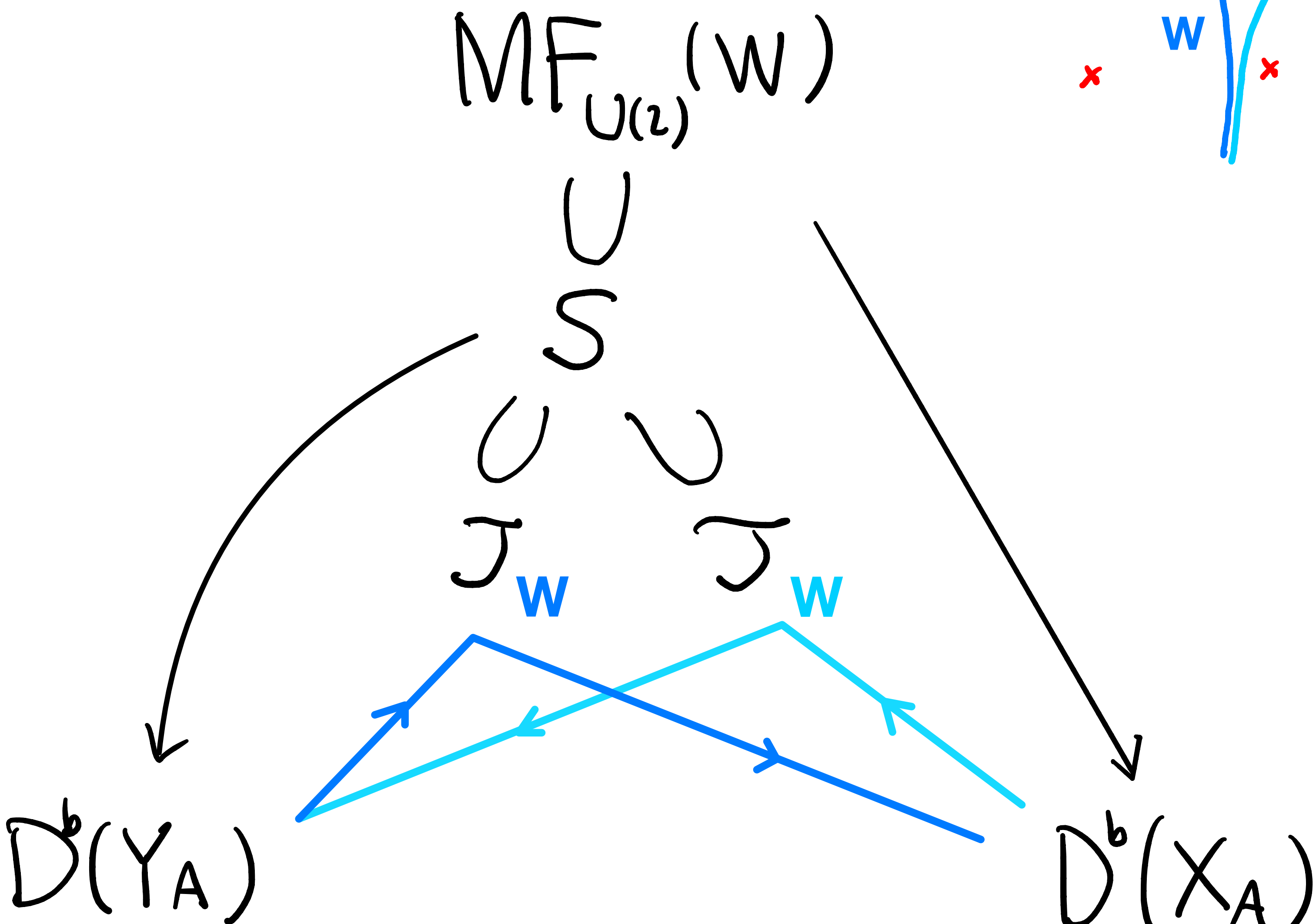


c.f. Addington-Donovan-Segal 2014

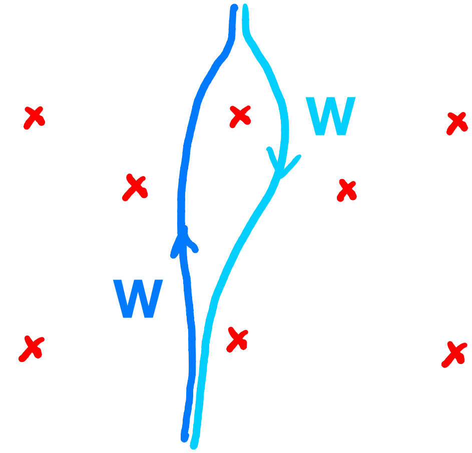
Monodromy



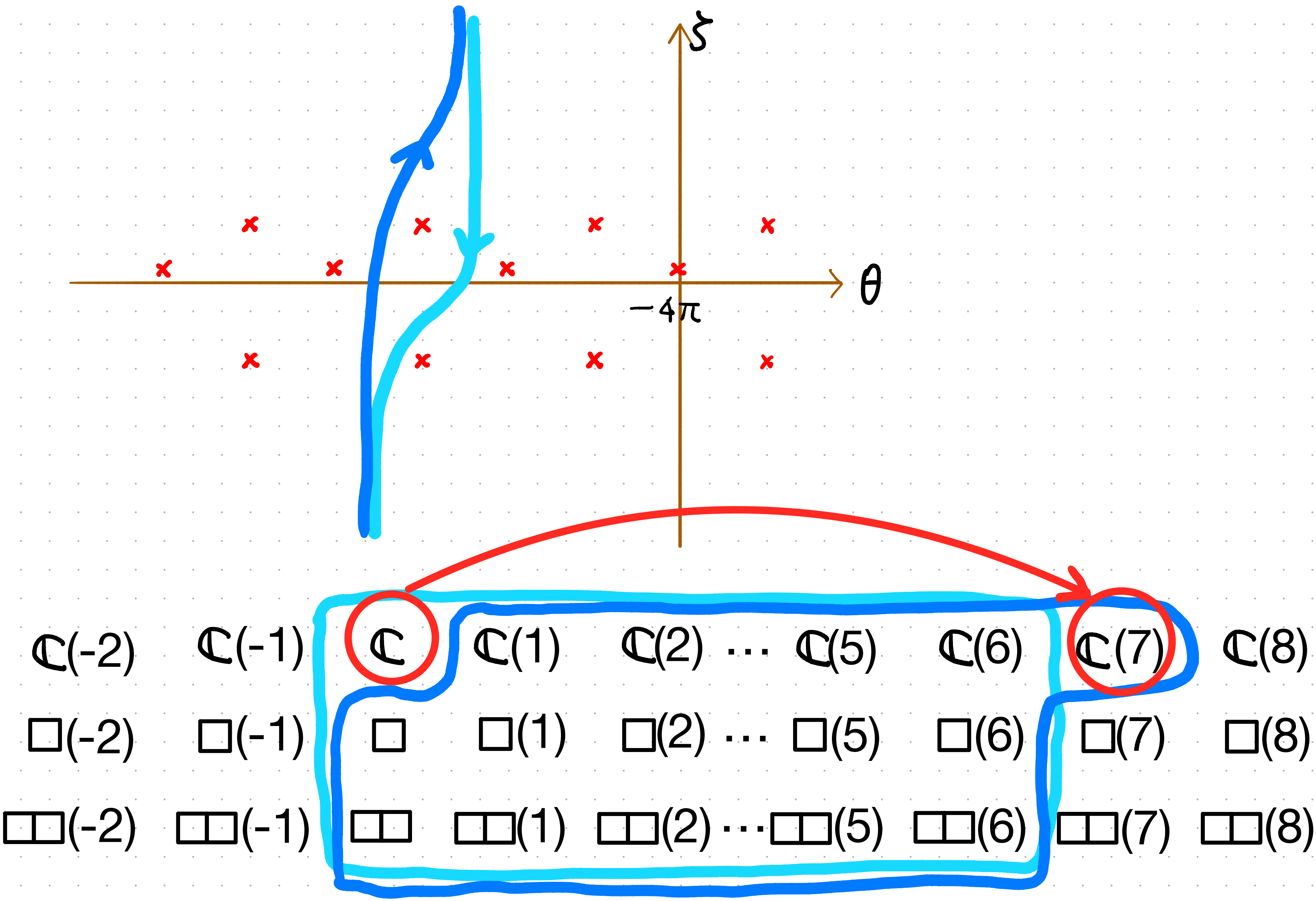

ζ
θ
−4π
ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ⋯ ℂ(5) ℂ(6) ℂ(7) ℂ(8)
□(-2) □(-1) □ □(1) □(2) ⋯ □(5) □(6) □(7) □(8)
□□(-2) □□(-1) □□ □□(1) □□(2) ⋯ □□(5) □□(6) □□(7) □□(8)



$MF_{U(2)}(W)$
$\cup$
S
$\cup$ $\cup$
J_W J_W
$D^b(Y_A)$
$D^b(X_A)$



W
W



ζ
θ
−4π
ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ··· ℂ(5) ℂ(6) ℂ(7) ℂ(8)
□(-2) □(-1) □ □(1) □(2) ··· □(5) □(6) □(7) □(8)
□□(-2) □□(-1) □□ □□(1) □□(2) ··· □□(5) □□(6) □□(7) □□(8)

Want to replace $\mathbb{C}$ by $\mathbb{C}(7)$ in the phase $\zeta \ll 0$.

— Can be done with the brane $\mathcal{K}$

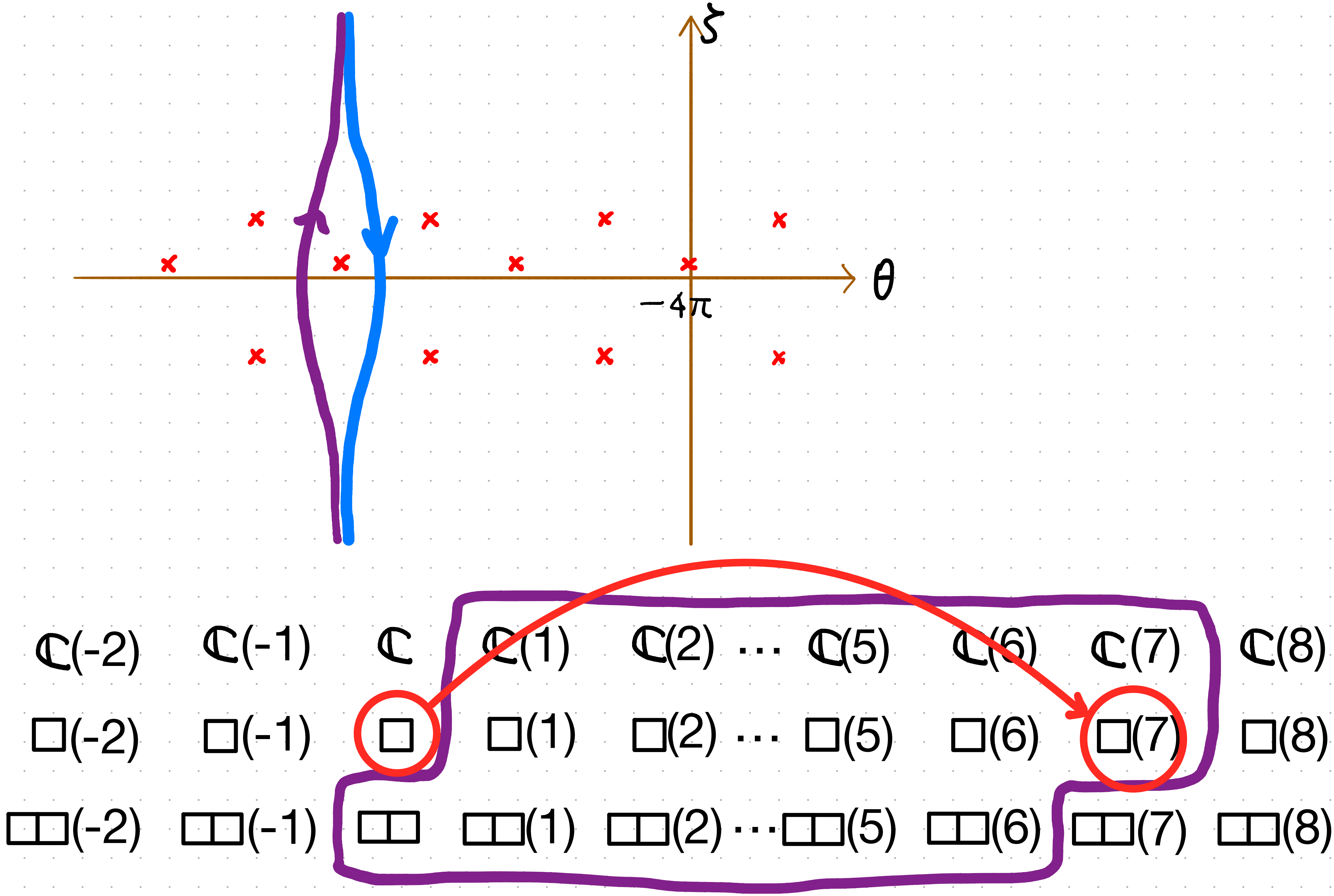
$$\mathbb{C} \underset{p}{\overset{A(x)}{\rightleftarrows}} \mathbb{C}^{7}(1) \underset{p}{\overset{A(x)}{\rightleftarrows}} \mathbb{C}^{\binom{7}{2}}(2) \underset{p}{\overset{A(x)}{\rightleftarrows}} \mathbb{C}^{\binom{7}{3}}(3) \underset{p}{\overset{A(x)}{\rightleftarrows}} \mathbb{C}^{\binom{7}{4}}(4) \underset{p}{\overset{A(x)}{\rightleftarrows}} \mathbb{C}^{\binom{7}{5}}(5) \underset{p}{\overset{A(x)}{\rightleftarrows}} \mathbb{C}^{\binom{7}{6}}(6) \underset{p}{\overset{A(x)}{\rightleftarrows}} \mathbb{C}(7)$$

0 1 2 3 4 5 6 7

$$Q = \sum_{k=1}^{7} \left(p^k \eta_k + A_k(x) \bar{\eta}^k \right) \qquad \{ \eta_k , \bar{\eta}^l \} = \delta_k^l$$

which is empty, $\mathcal{K} \longmapsto 0$, as

$$\{ Q, Q^\dagger \} = \| p \|^2 + \| A(x) \|^2 > 0 \text{ if } \zeta \ll 0.$$


ζ
θ
−4π
ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ⋯ ℂ(5) ℂ(6) ℂ(7) ℂ(8)
□(-2) □(-1) □ □(1) □(2) ⋯ □(5) □(6) □(7) □(8)
□□(-2) □□(-1) □□ □□(1) □□(2) ⋯ □□(5) □□(6) □□(7) □□(8)

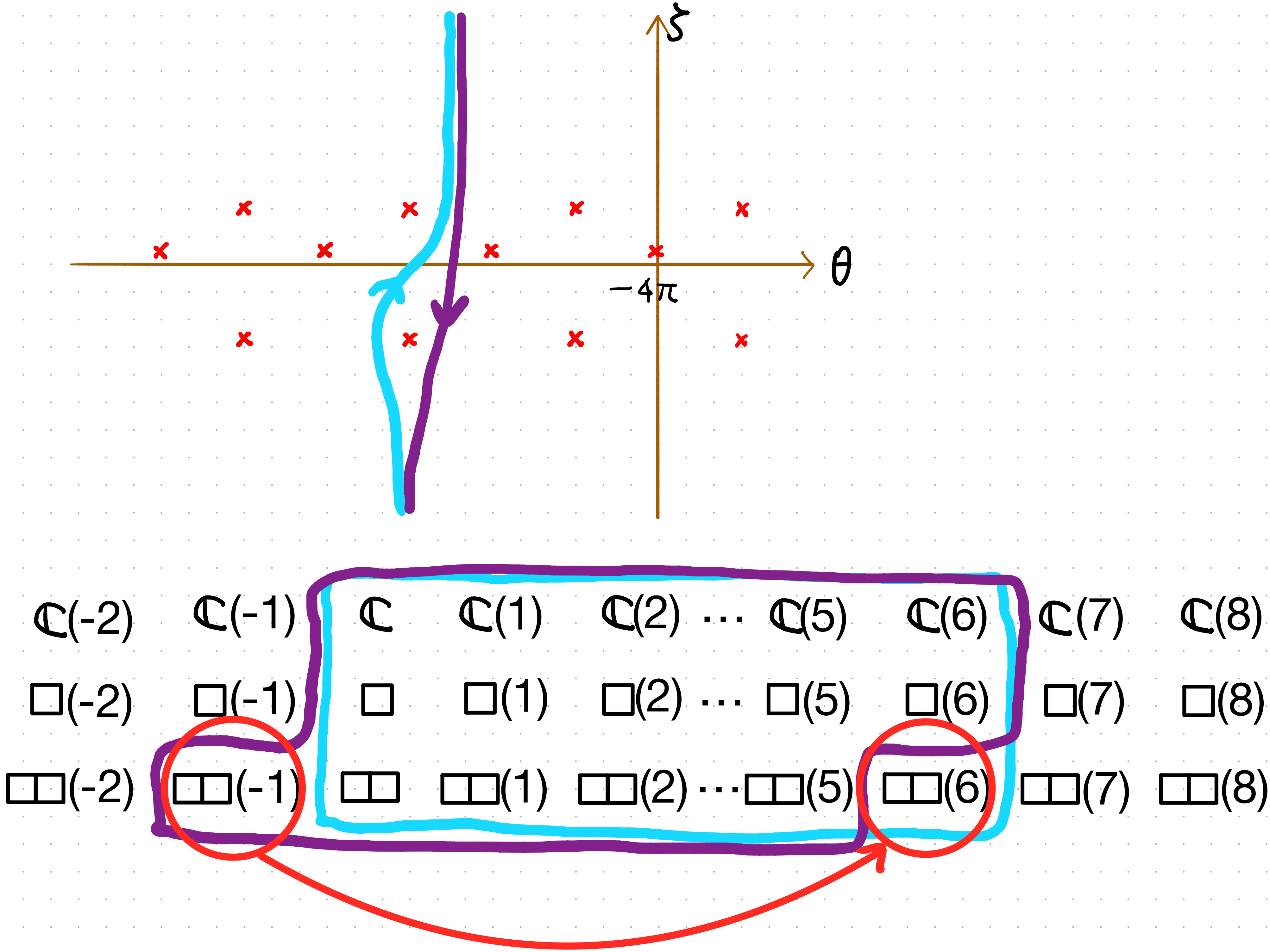
Replacement $\square \longrightarrow \square(7)$ in the phase $\zeta \ll 0$

can be done with $\square \otimes K$:

$$\square \underset{p}{\overset{A(x)}{\rightleftarrows}} \square^{\oplus 7}(1) \underset{p}{\overset{A(x)}{\rightleftarrows}} \square^{\oplus \binom{7}{2}}(2) \underset{p}{\overset{A(x)}{\rightleftarrows}} \bullet \bullet \bullet \underset{p}{\overset{A(x)}{\rightleftarrows}} \square(7)$$

0 1 2 . . . 7

which is empty there.



ζ
θ
−4π
ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ··· ℂ(5) ℂ(6) ℂ(7) ℂ(8)
□(-2) □(-1) □ □(1) □(2) ··· □(5) □(6) □(7) □(8)
□□(-2) □□(-1) □□ □□(1) □□(2) ··· □□(5) □□(6) □□(7) □□(8)

Replacement $\square\square(-1) \to \square\square(6)$ in the phase $\zeta \ll 0$

can be done with $\square\square \otimes K(-1)$:

$$\underset{0}{\square\square(-1)} \underset{p}{\overset{A(x)}{\rightleftarrows}} \underset{1}{\square\square^{\oplus 7}} \underset{p}{\overset{A(x)}{\rightleftarrows}} \underset{2}{\square\square^{\oplus \binom{7}{2}}(1)} \underset{p}{\overset{A(x)}{\rightleftarrows}} \underset{\cdots}{\cdots} \underset{p}{\overset{A(x)}{\rightleftarrows}} \underset{7}{\square\square(6)}$$

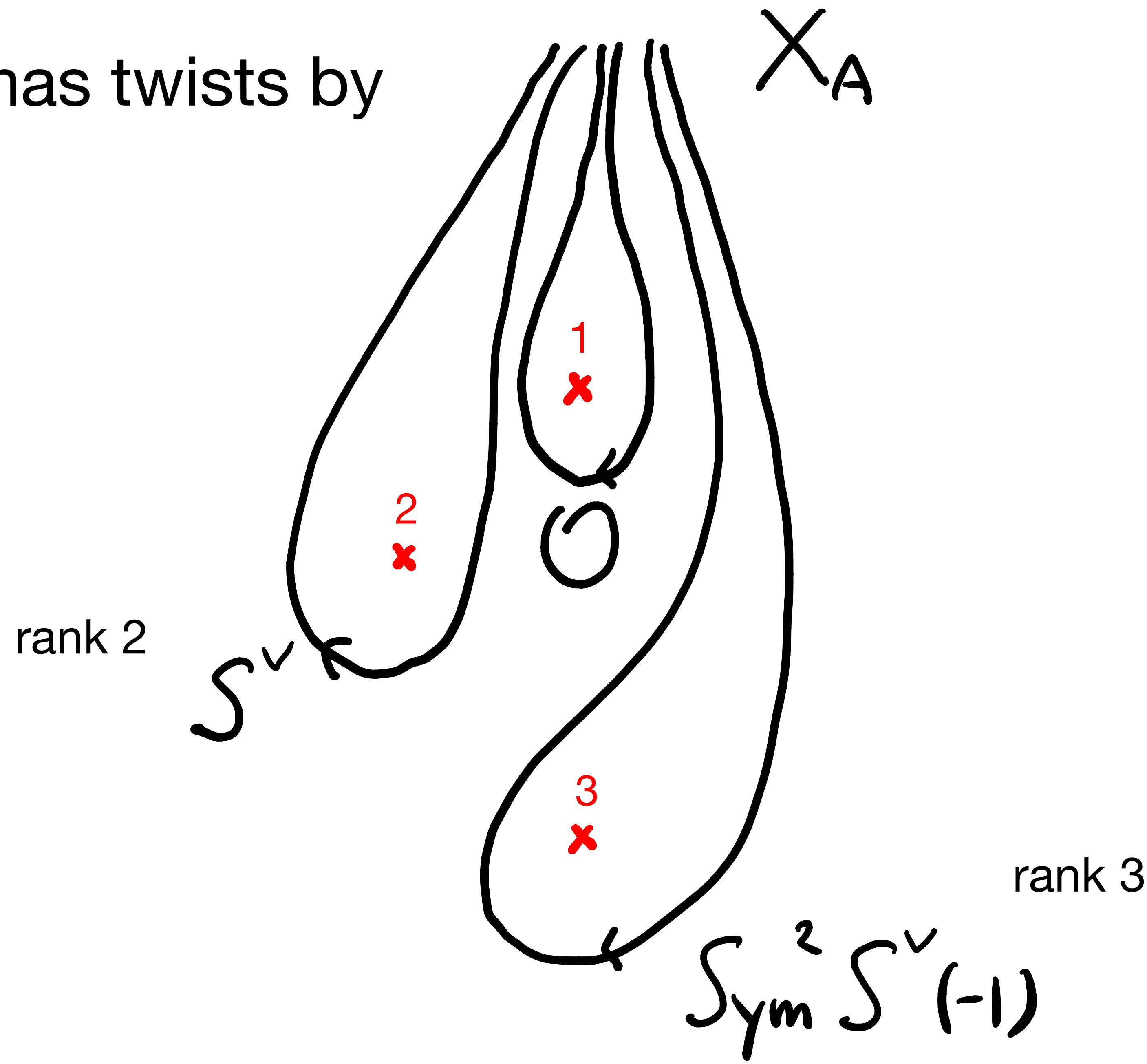
which is empty there.

In the phase $\zeta \gg 0$,

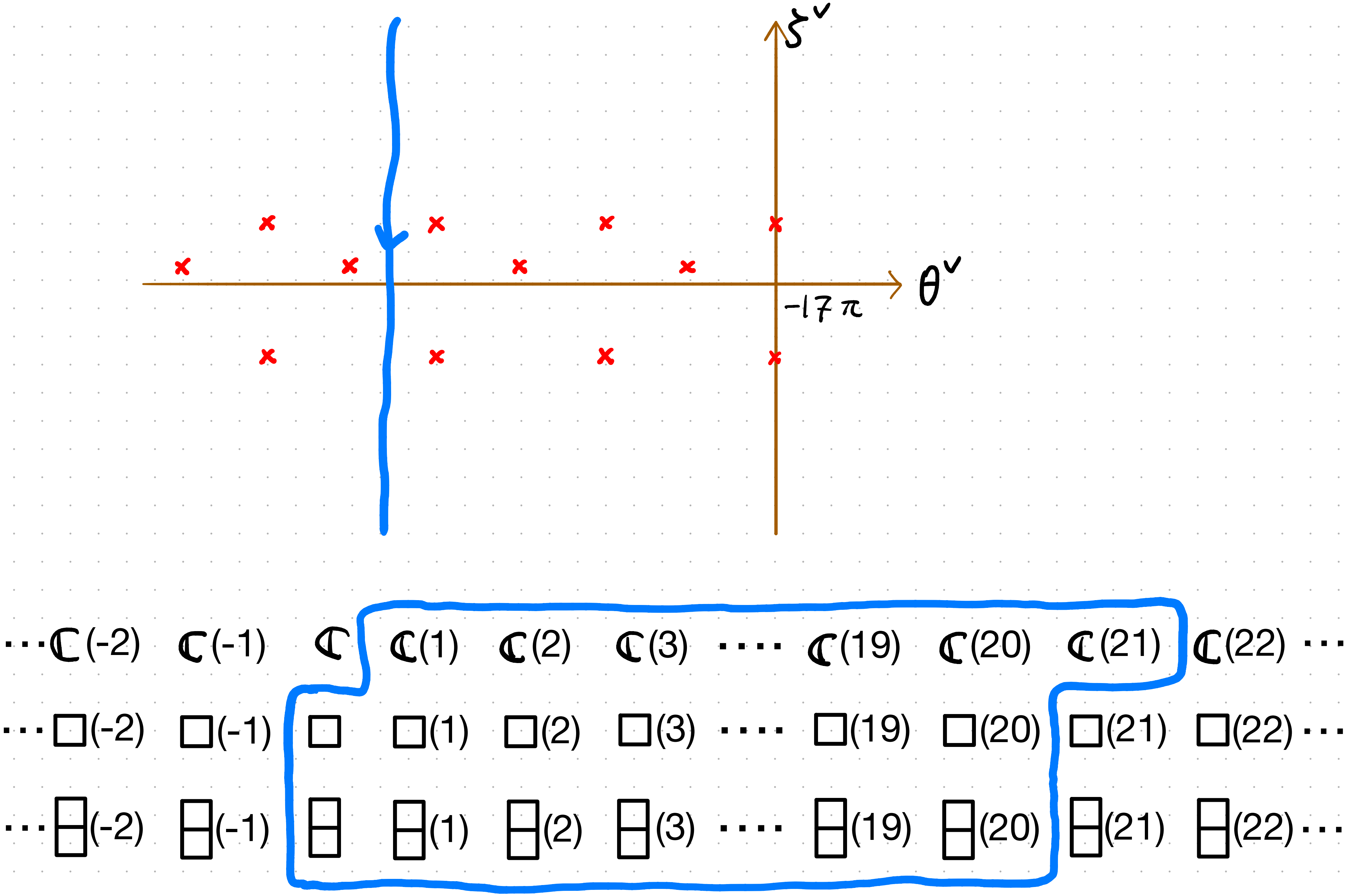
GLSM		non-linear σ-model	
$\mathcal{K}$	$\longmapsto$	$\mathcal{O}$	the structure sheaf of X_A
$\square \otimes \mathcal{K}$	$\longmapsto$	$\mathcal{S}^\vee$	a rank 2 vector bundle
$\square\square \otimes \mathcal{K}(-1)$	$\longmapsto$	$\mathrm{Sym}^2 \mathcal{S}^\vee(-1)$	rank 3

under the RG flow.

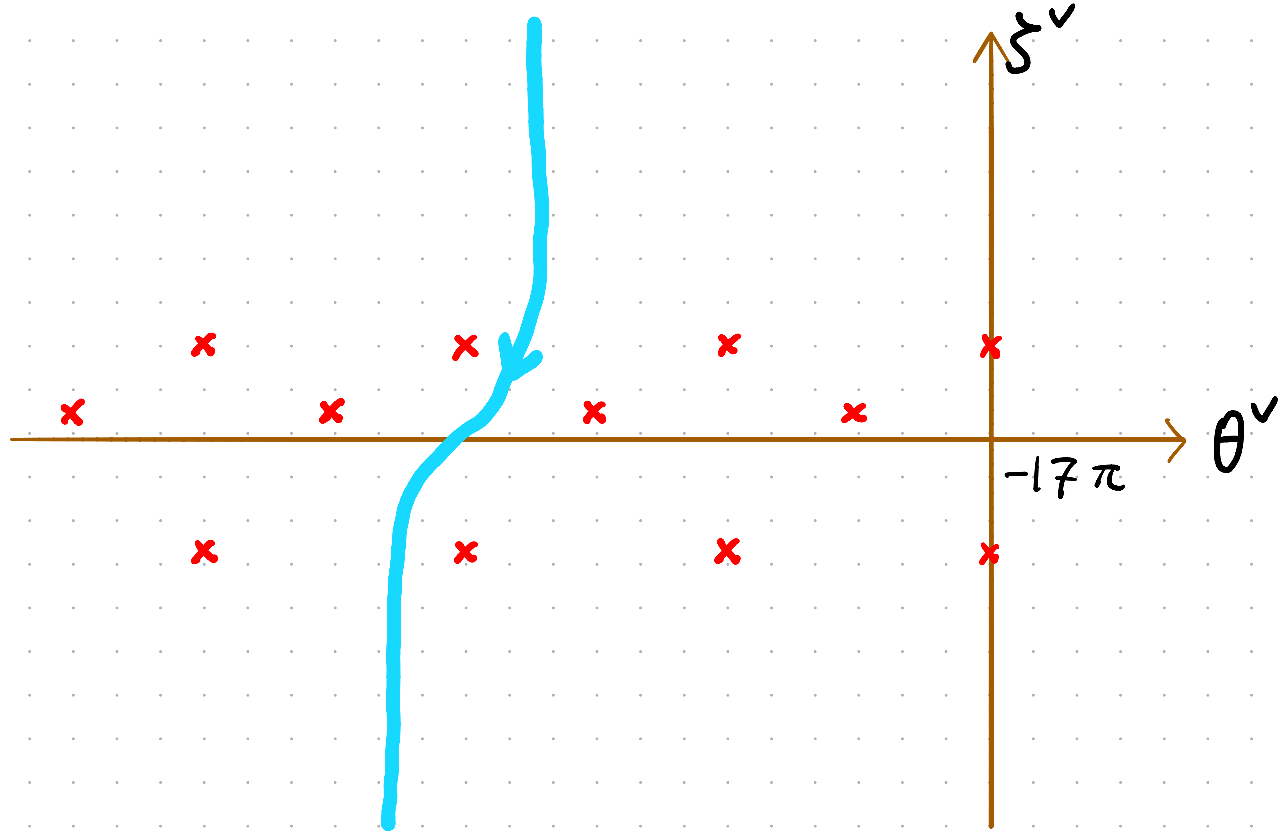
Seidel-Thomas twists by

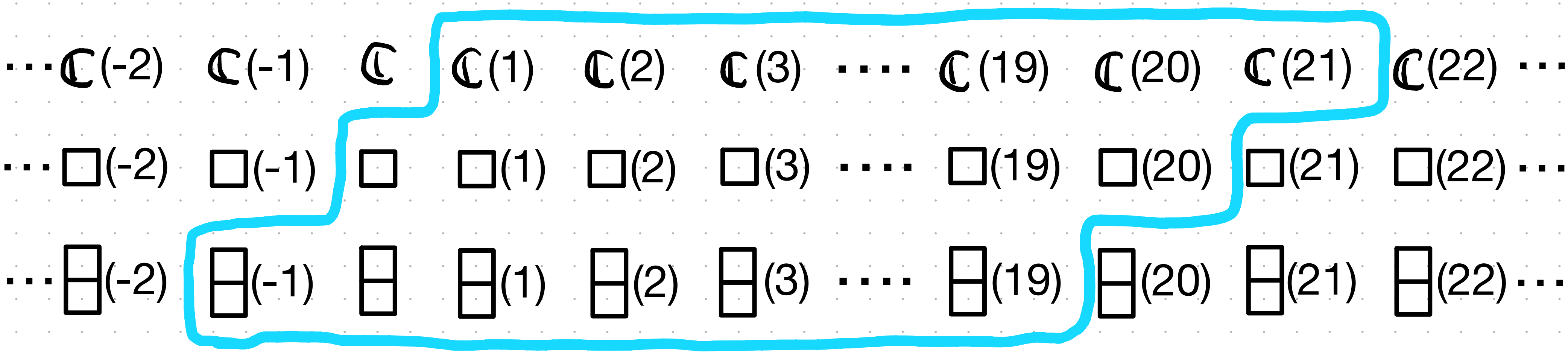


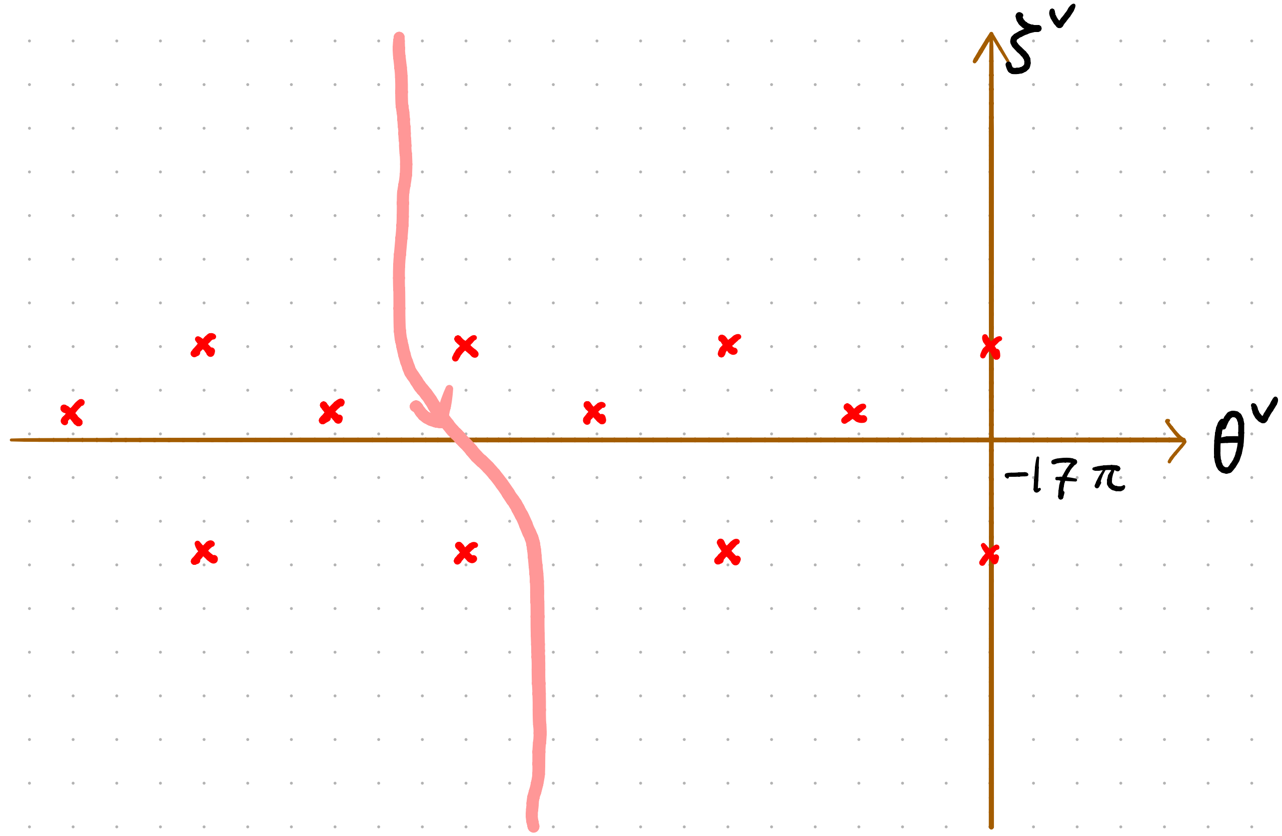
Dual Rødland model

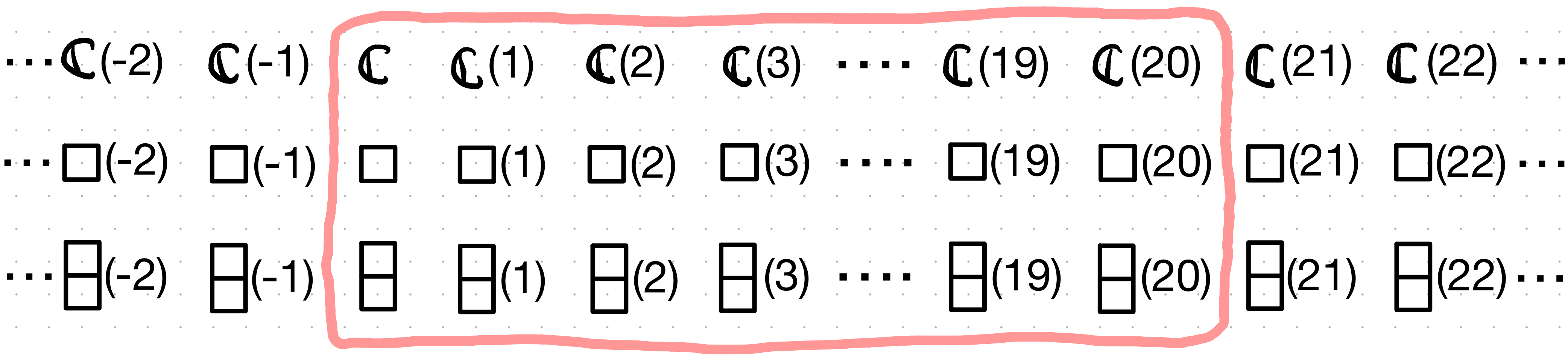


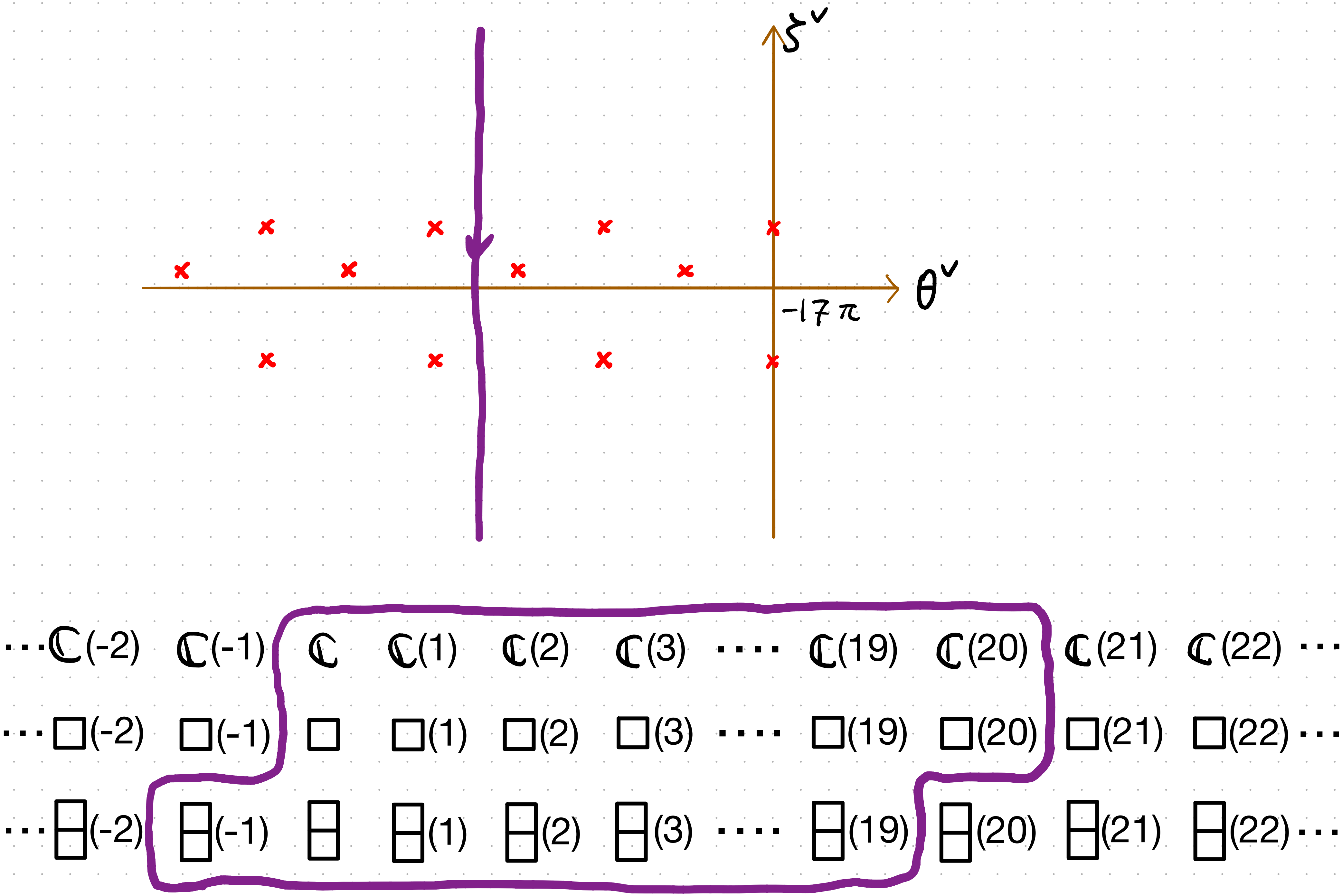
$\zeta^\vee$
$\theta^\vee$
-17π
$\cdots \mathbb{C}(-2)$ $\mathbb{C}(-1)$ $\mathbb{C}$ $\mathbb{C}(1)$ $\mathbb{C}(2)$ $\mathbb{C}(3)$ $\cdots$ $\mathbb{C}(19)$ $\mathbb{C}(20)$ $\mathbb{C}(21)$ $\mathbb{C}(22) \cdots$


$\xi^\vee$
$\theta^\vee$
-17π

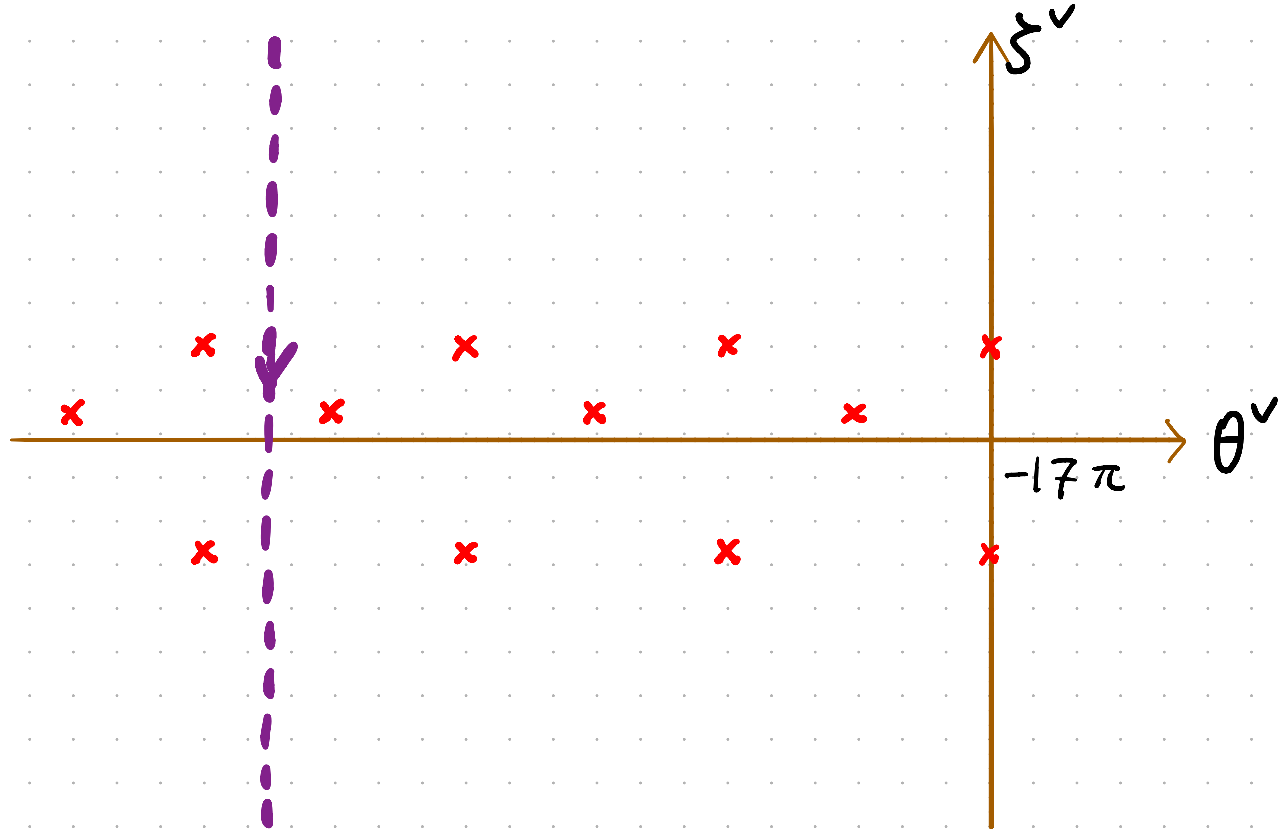

$\cdots \mathbb{C}(-2)$ $\mathbb{C}(-1)$ $\mathbb{C}$ $\mathbb{C}(1)$ $\mathbb{C}(2)$ $\mathbb{C}(3)$ $\cdots$ $\mathbb{C}(19)$ $\mathbb{C}(20)$ $\mathbb{C}(21)$ $\mathbb{C}(22) \cdots$
$\cdots$ □(-2) □(-1) □ □(1) □(2) □(3) $\cdots$ □(19) □(20) □(21) □(22) $\cdots$
$\cdots$ (-2) (-1) (1) (2) (3) $\cdots$ (19) (20) (21) (22) $\cdots$

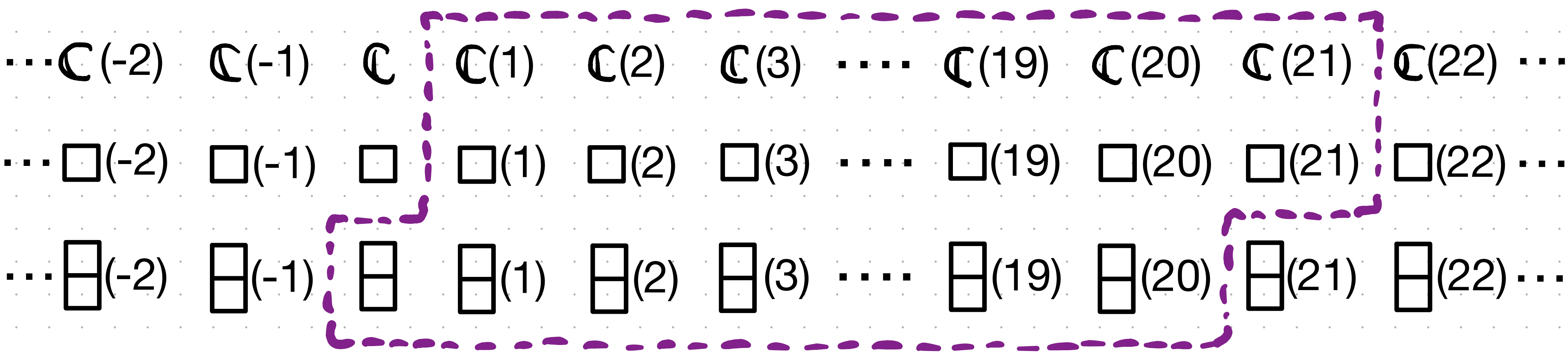

ζ∨
θ∨
−17π


⋯ ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ⋯ ℂ(19) ℂ(20) ℂ(21) ℂ(22) ⋯
(-2) (-1) (1) (2) (3) (19) (20) (21) (22)
(-2) (-1) (1) (2) (3) (19) (20) (21) (22)

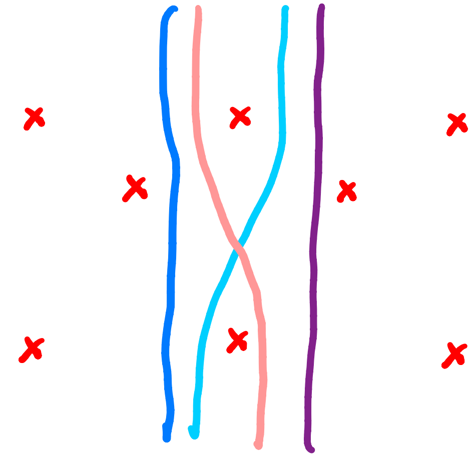


ζ∨
θ∨
−17π
⋯ ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ⋯ ℂ(19) ℂ(20) ℂ(21) ℂ(22) ⋯
⋯ □(-2) □(-1) □ □(1) □(2) □(3) ⋯ □(19) □(20) □(21) □(22) ⋯
⋯ (-2) (-1) (1) (2) (3) ⋯ (19) (20) (21) (22) ⋯


ζ^∨
θ^∨
−17π


⋯ ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ⋯ ℂ(19) ℂ(20) ℂ(21) ℂ(22) ⋯

For each window $\mathbf{w}$ such as

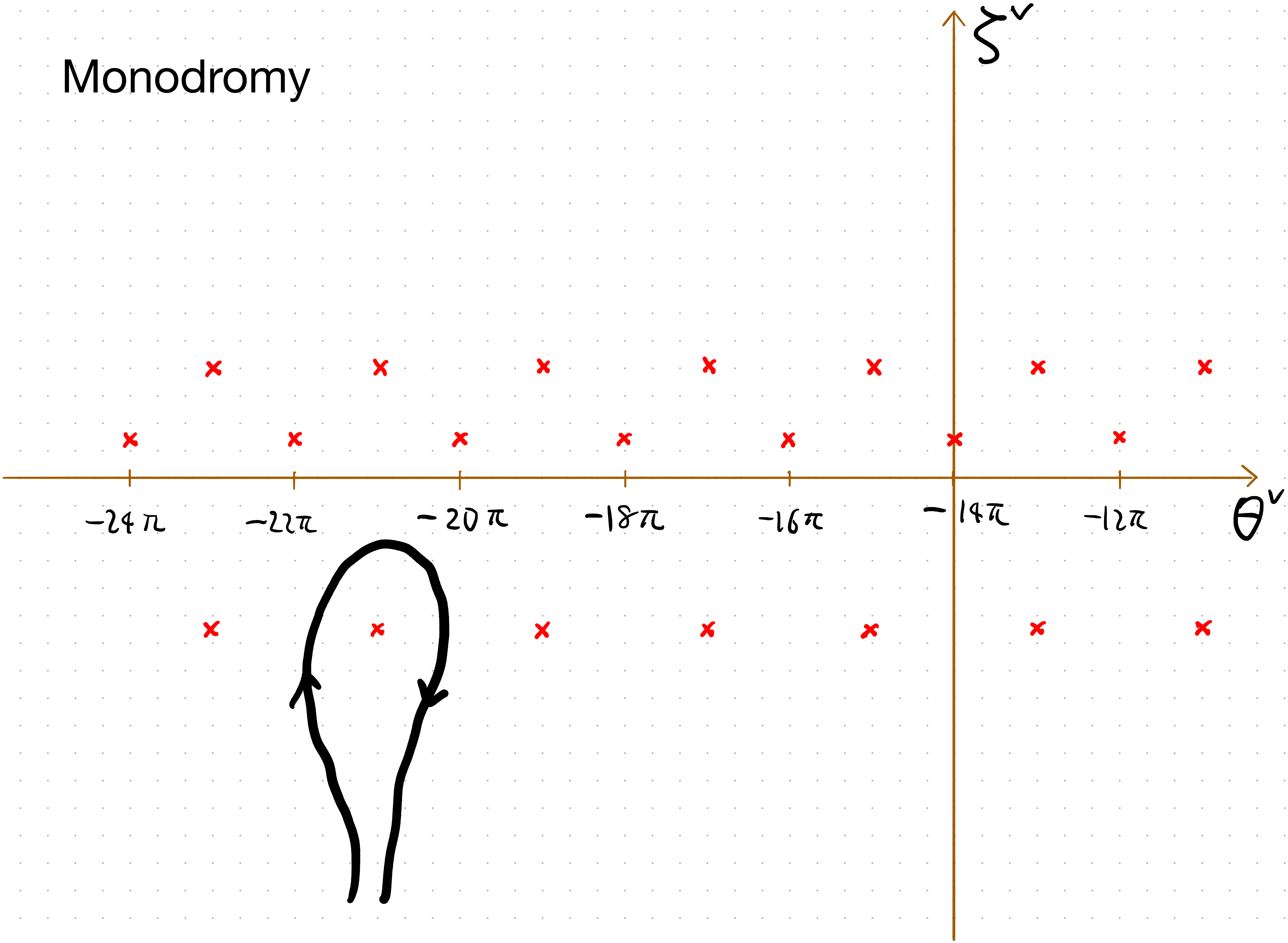


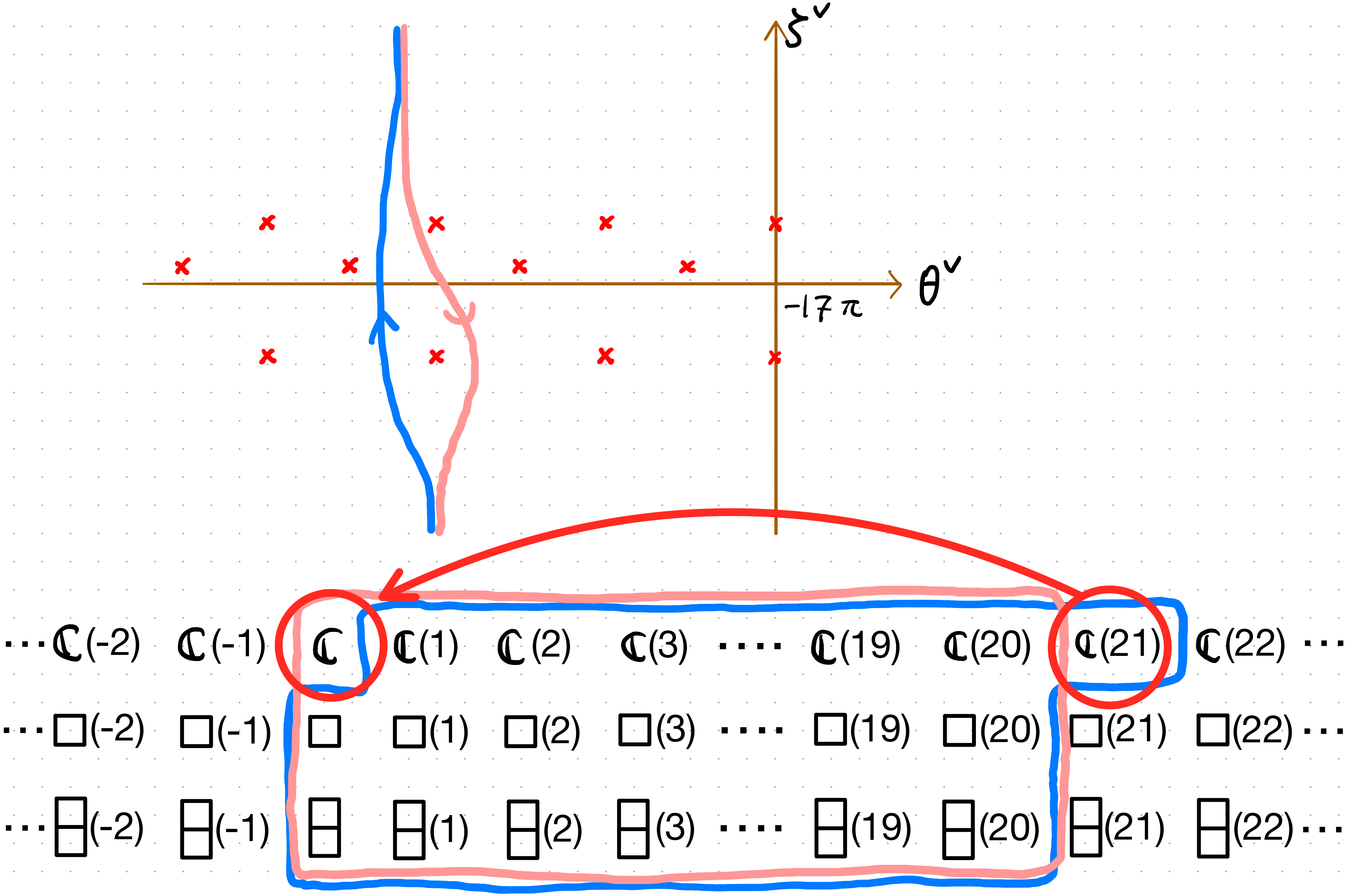
$$D_{G^\vee}(V^\vee, W^\vee) \supset S^\vee \supset J_{\mathbf{w}}$$

$$D^b(Y_A) \xleftarrow{\ \cong\ } J_{\mathbf{w}} \xrightarrow{\ \cong\ } D^b(X_A)$$

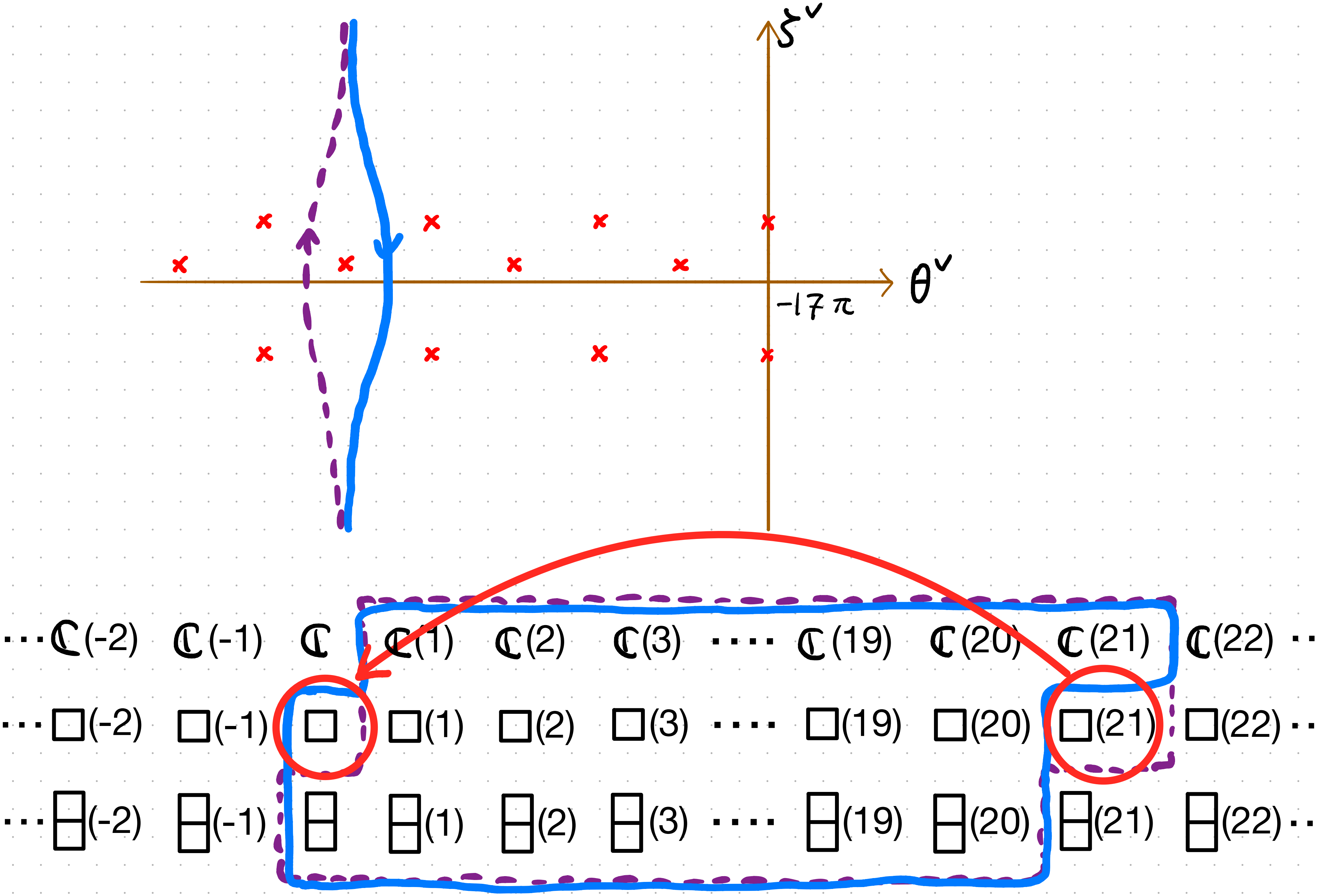
$$D^b(Y_A) \longleftarrow S^\vee \longrightarrow D^b(X_A)$$

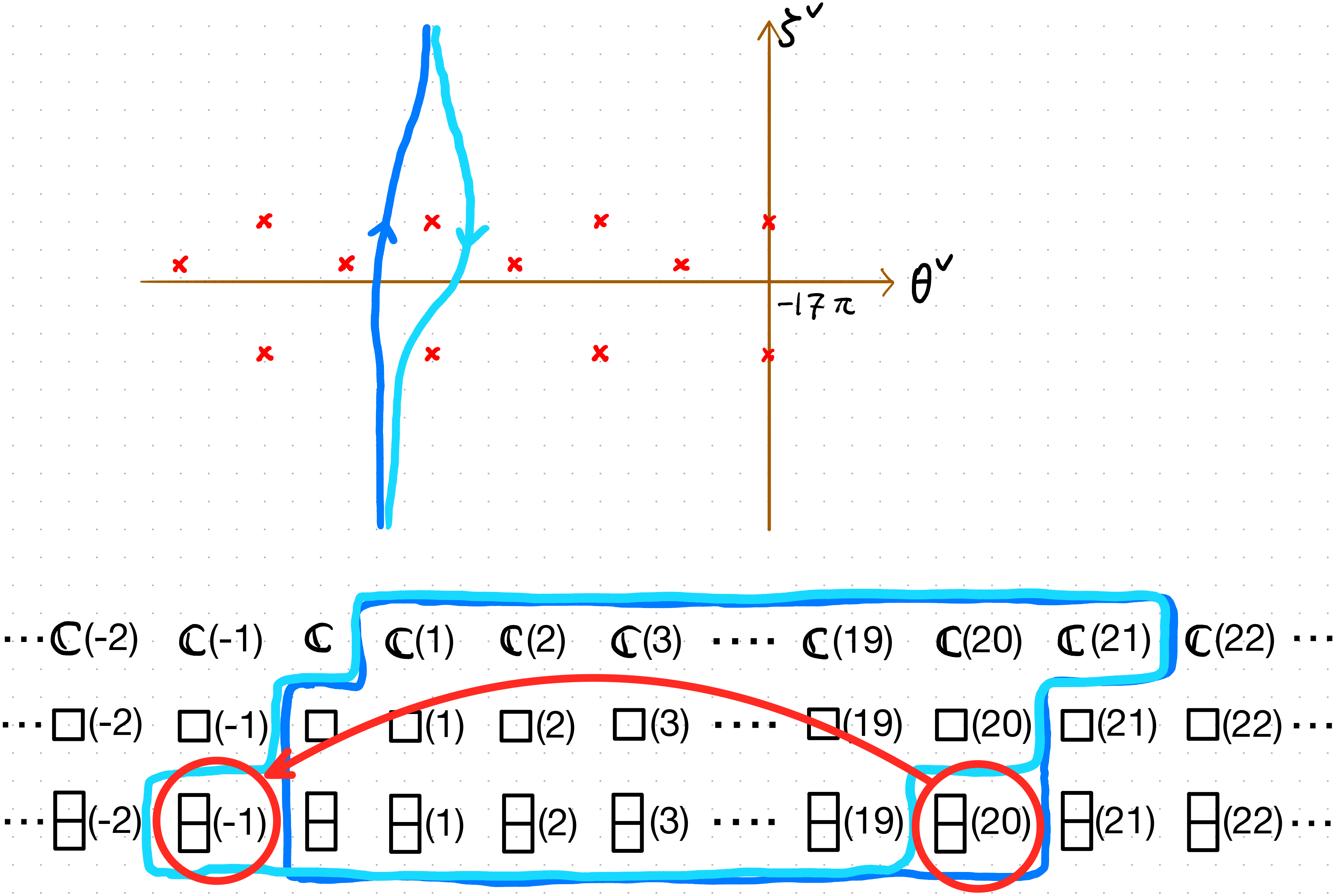
Monodromy





$\xi^\vee$
$\theta^\vee$
-17π
$\cdots \mathbb{C}(-2)$ $\mathbb{C}(-1)$ $\mathbb{C}$ $\mathbb{C}(1)$ $\mathbb{C}(2)$ $\mathbb{C}(3)$ $\cdots$ $\mathbb{C}(19)$ $\mathbb{C}(20)$ $\mathbb{C}(21)$ $\mathbb{C}(22) \cdots$
$\cdots \square(-2)$ $\square(-1)$ $\square$ $\square(1)$ $\square(2)$ $\square(3)$ $\cdots$ $\square(19)$ $\square(20)$ $\square(21)$ $\square(22) \cdots$
(-2) (-1) (1) (2) (3) (19) (20) (21) (22)


ζ^∨
θ^∨
−17π
⋯ ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ⋯ ℂ(19) ℂ(20) ℂ(21) ℂ(22) ⋯
⋯ □(-2) □(-1) □ □(1) □(2) □(3) ⋯ □(19) □(20) □(21) □(22) ⋯
⋯ ⊟(-2) ⊟(-1) ⊟ ⊟(1) ⊟(2) ⊟(3) ⋯ ⊟(19) ⊟(20) ⊟(21) ⊟(22) ⋯



ζ∨
θ∨
-17π
ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ℂ(19) ℂ(20) ℂ(21) ℂ(22)
(-2) (-1) (1) (2) (3) (19) (20) (21) (22)
(-2) (-1) (1) (2) (3) (19) (20) (21) (22)

The replacement in the phase $\zeta^\vee \gg 0$ can be done

with the branes $\mathcal{K}^\vee$, $\Box \otimes \mathcal{K}^\vee$, $\boxminus \otimes \mathcal{K}^\vee(-1)$

where $\mathcal{K}^\vee$ is

$$\mathbb{C} \underset{A(p)+[\tilde{x}\tilde{x}]}{\overset{a}{\rightleftarrows}} \mathbb{C}(1)^{\oplus 21} \underset{A(p)+[\tilde{x}\tilde{x}]}{\overset{a}{\rightleftarrows}} \mathbb{C}(2)^{\oplus \binom{21}{2}} \underset{A(p)+[\tilde{x}\tilde{x}]}{\overset{a}{\rightleftarrows}} \cdots \underset{A(p)+[\tilde{x}\tilde{x}]}{\overset{a}{\rightleftarrows}} \mathbb{C}(20)^{\oplus \binom{21}{3}} \underset{A(p)+[\tilde{x}\tilde{x}]}{\overset{a}{\rightleftarrows}} \mathbb{C}(21)$$

$$Q = \sum_{i,j} \left(a_{ij} \eta^{ij} + \left(A^{ij}(p) + [\tilde{x}^i \tilde{x}^j] \right) \bar{\eta}_{ij} \right) ; \quad \{ \eta_{ij}, \bar{\eta}^{kl} \} = \epsilon^{kl}_{ij}$$

which are empty, $\mathcal{K}^\vee \longmapsto 0$, as

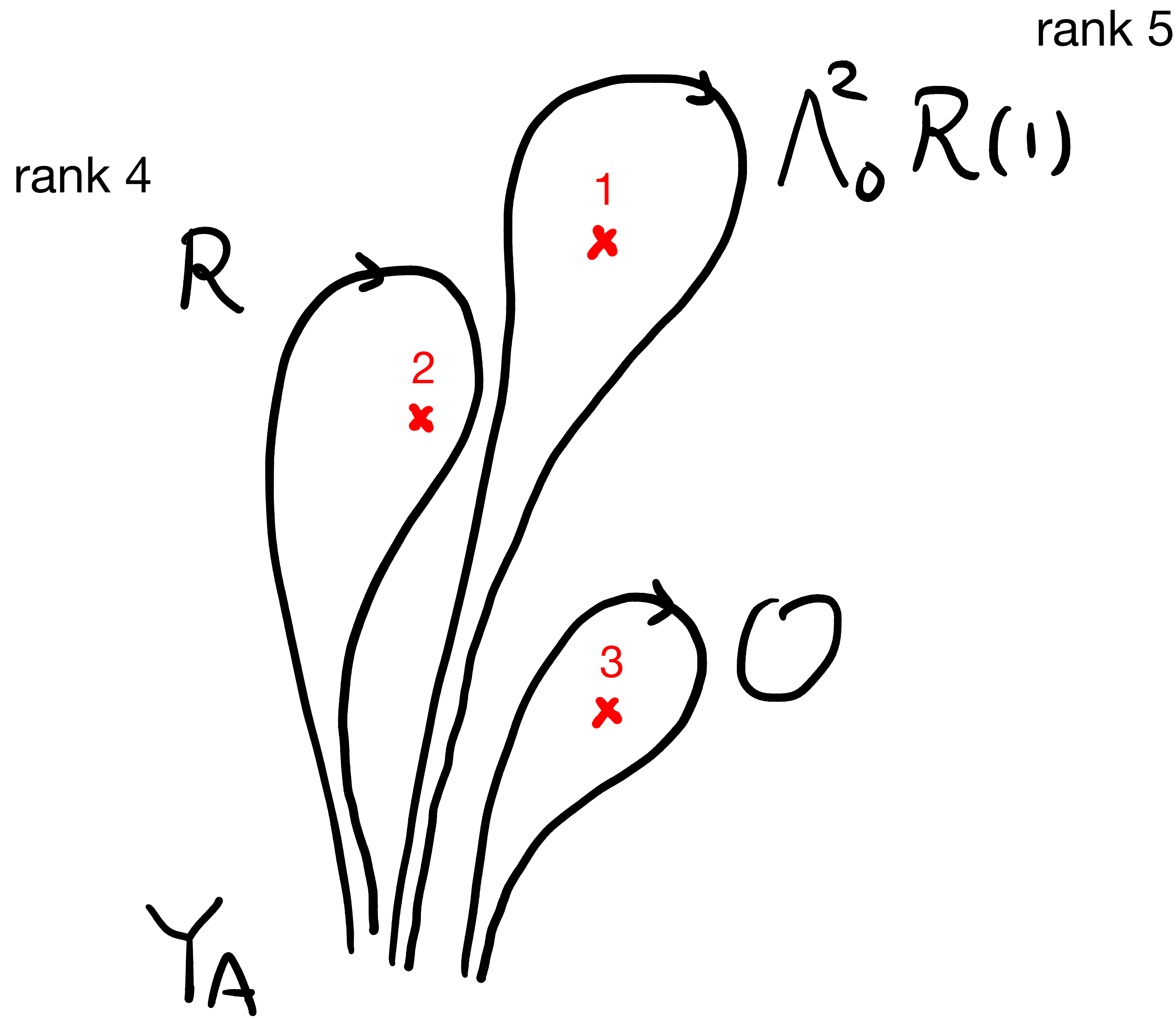
$$\{ Q, Q^\dagger \} = \| a \|^2 + \| A(p) + [\tilde{x}\tilde{x}] \|^2 > 0 \quad \text{if} \quad \zeta^\vee \gg 0.$$

In the phase $\zeta^\vee \ll 0$

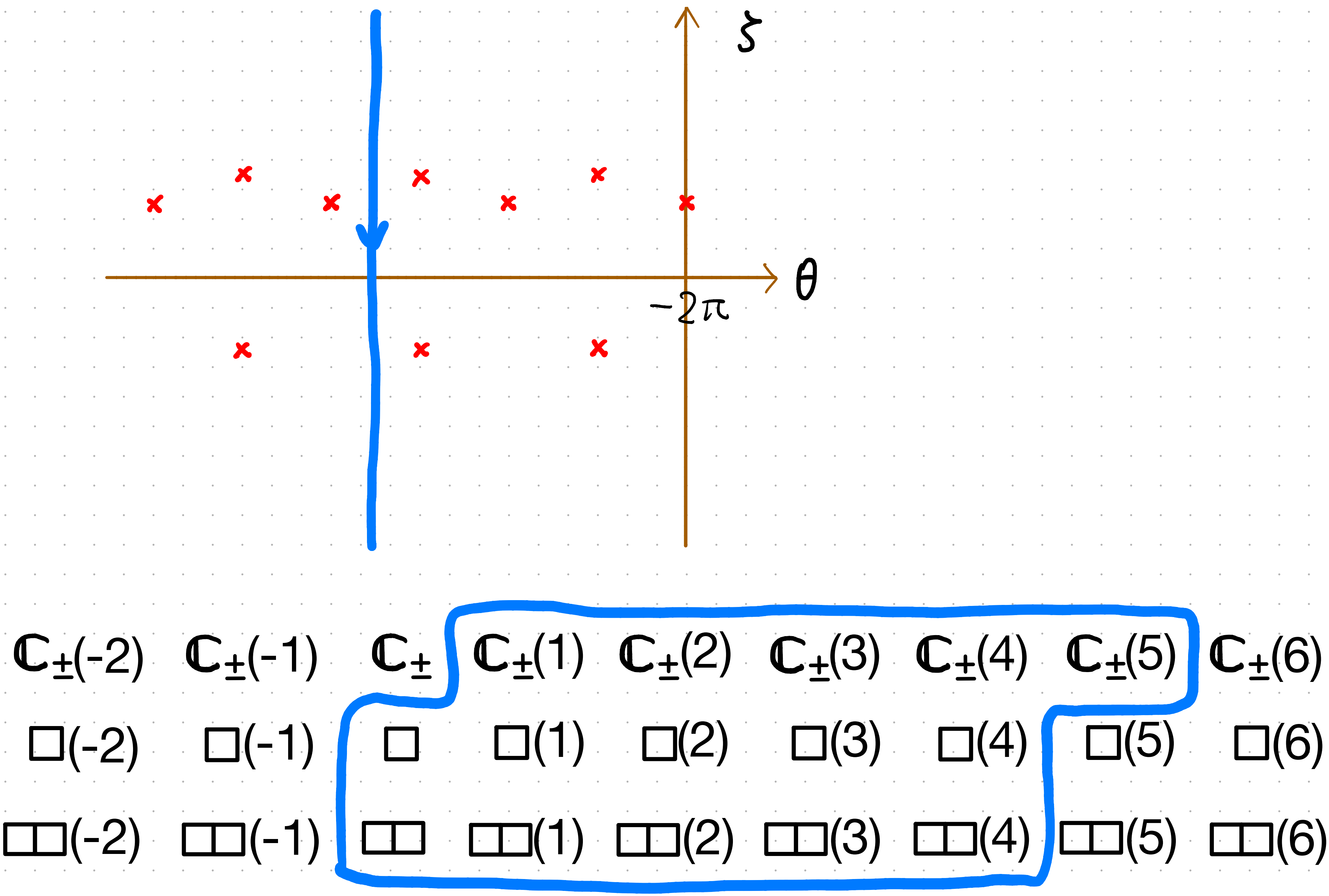
GLSM		non-linear σ-model	
$K^\vee$	$\longmapsto$	$\mathcal{O}$	the structure sheaf of Y_A
$\square \otimes K^\vee$	$\longmapsto$	R	a rank 4 vector bundle
$\begin{array}{\|c\|}\hline \\ \hline \\ \hline\end{array} \otimes K^\vee(-1)$	$\longmapsto$	$\Lambda^2_0 R \ (1)$	rank 5

under the RG flow.

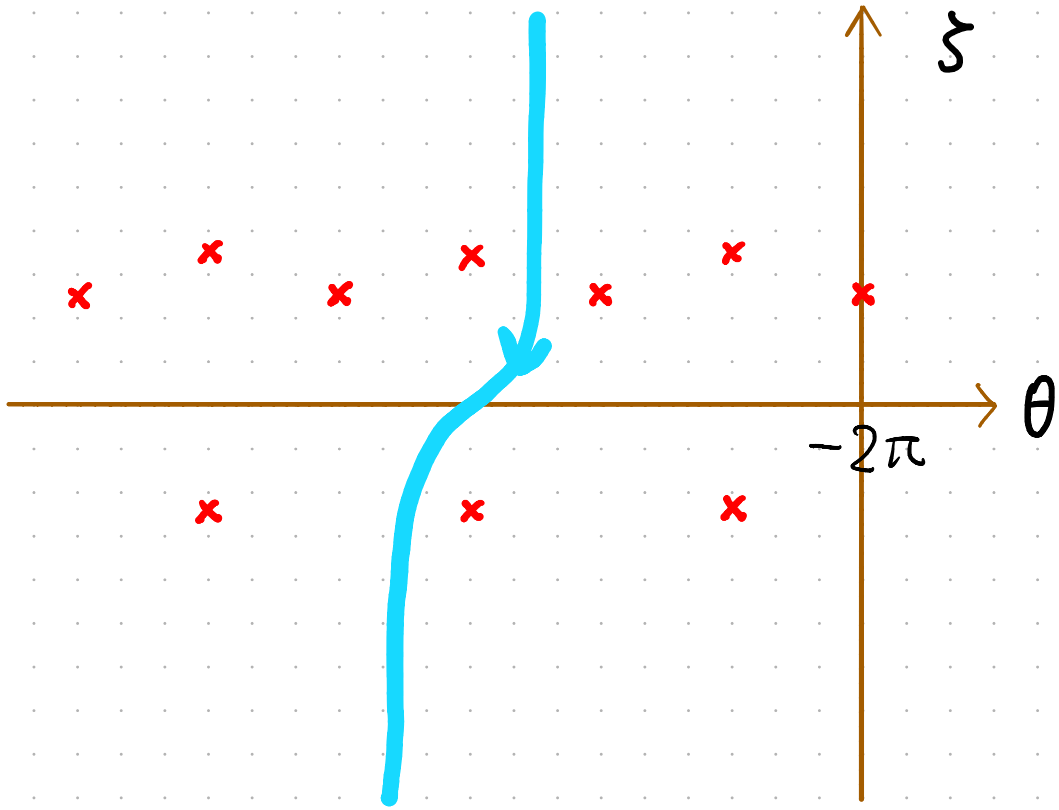
Seidel-Thomas twists by



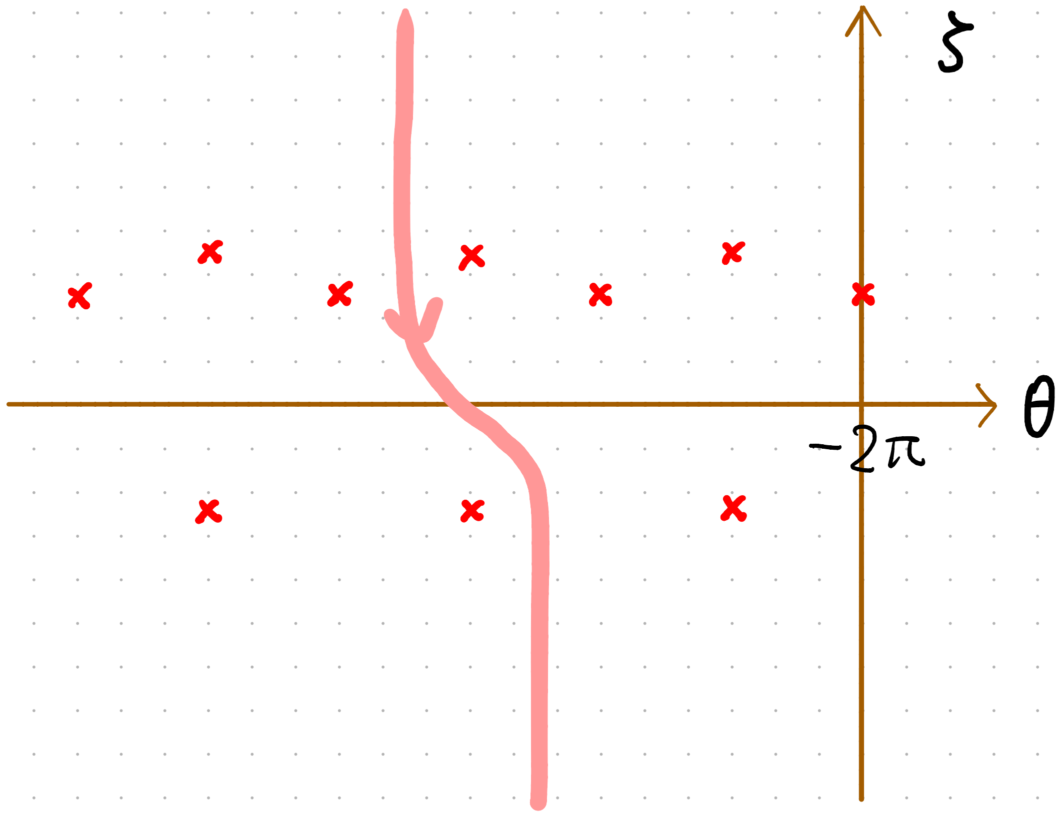
Hosono-Takagi model



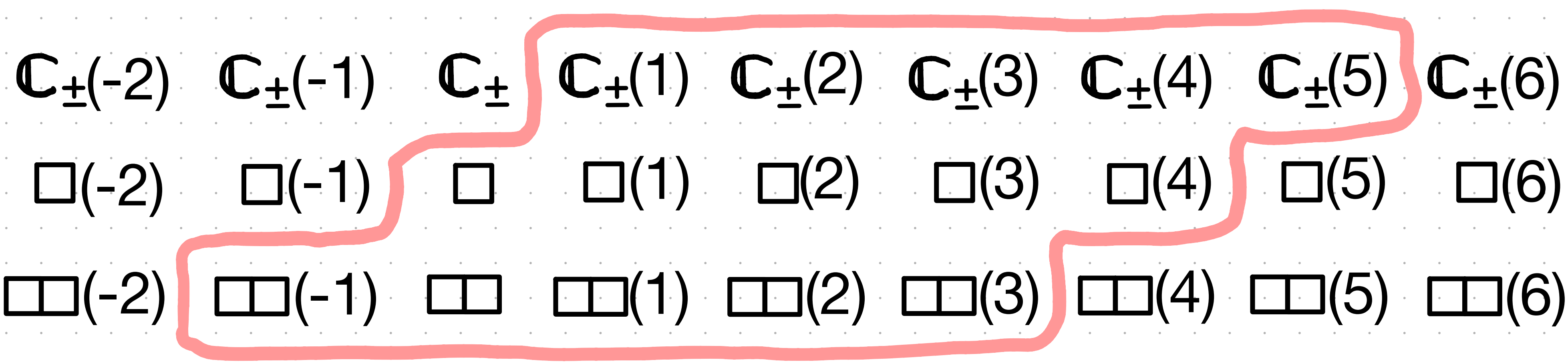
ζ
θ
−2π
ℂ±(-2) ℂ±(-1) ℂ± ℂ±(1) ℂ±(2) ℂ±(3) ℂ±(4) ℂ±(5) ℂ±(6)
□(-2) □(-1) □ □(1) □(2) □(3) □(4) □(5) □(6)
□□(-2) □□(-1) □□ □□(1) □□(2) □□(3) □□(4) □□(5) □□(6)



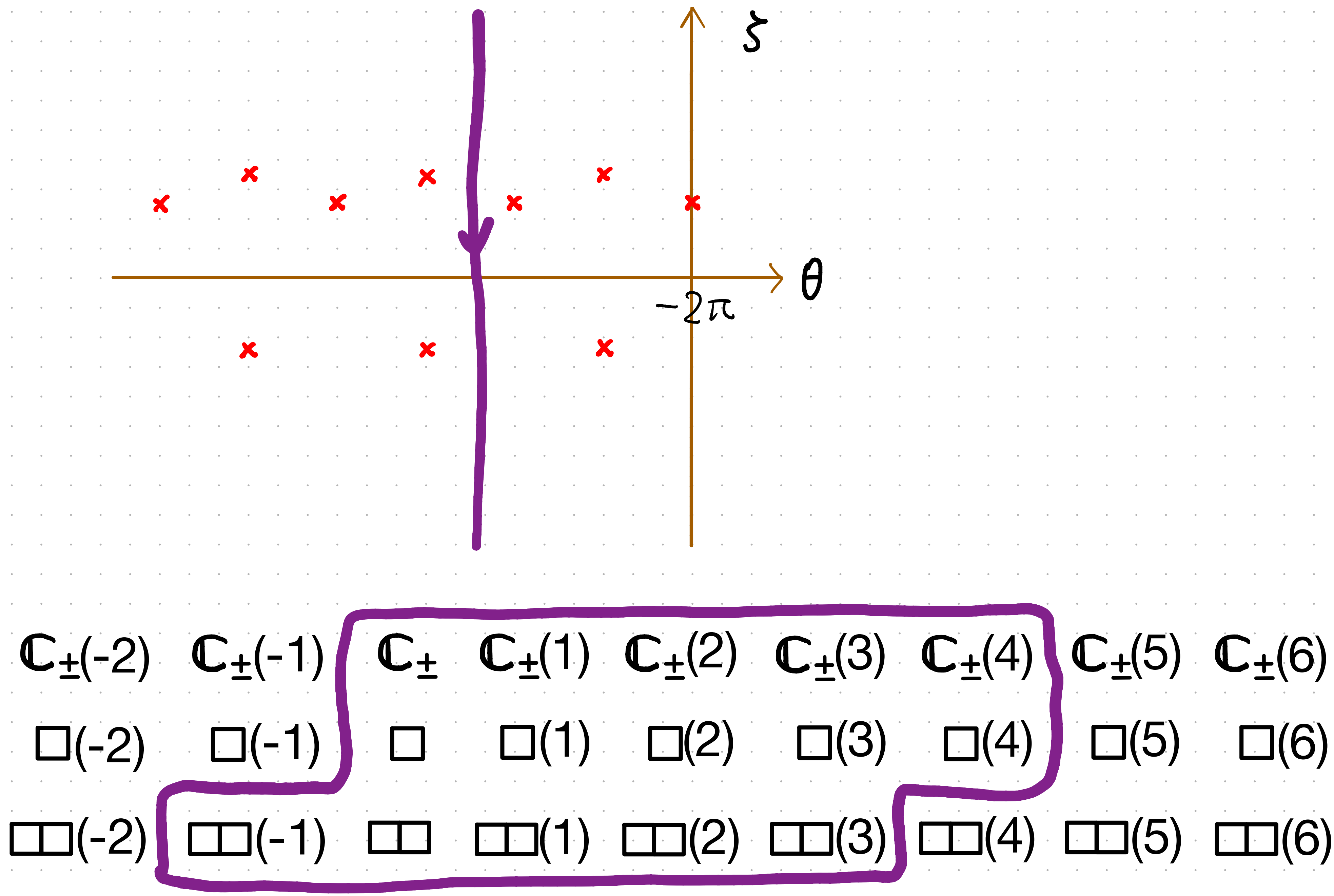
$\mathbb{C}_{\pm}(-2)$	$\mathbb{C}_{\pm}(-1)$	$\mathbb{C}_{\pm}$	$\mathbb{C}_{\pm}(1)$	$\mathbb{C}_{\pm}(2)$	$\mathbb{C}_{\pm}(3)$	$\mathbb{C}_{\pm}(4)$	$\mathbb{C}_{\pm}(5)$	$\mathbb{C}_{\pm}(6)$
□(-2)	□(-1)	□	□(1)	□(2)	□(3)	□(4)	□(5)	□(6)
□□(-2)	□□(-1)	□□	□□(1)	□□(2)	□□(3)	□□(4)	□□(5)	□□(6)



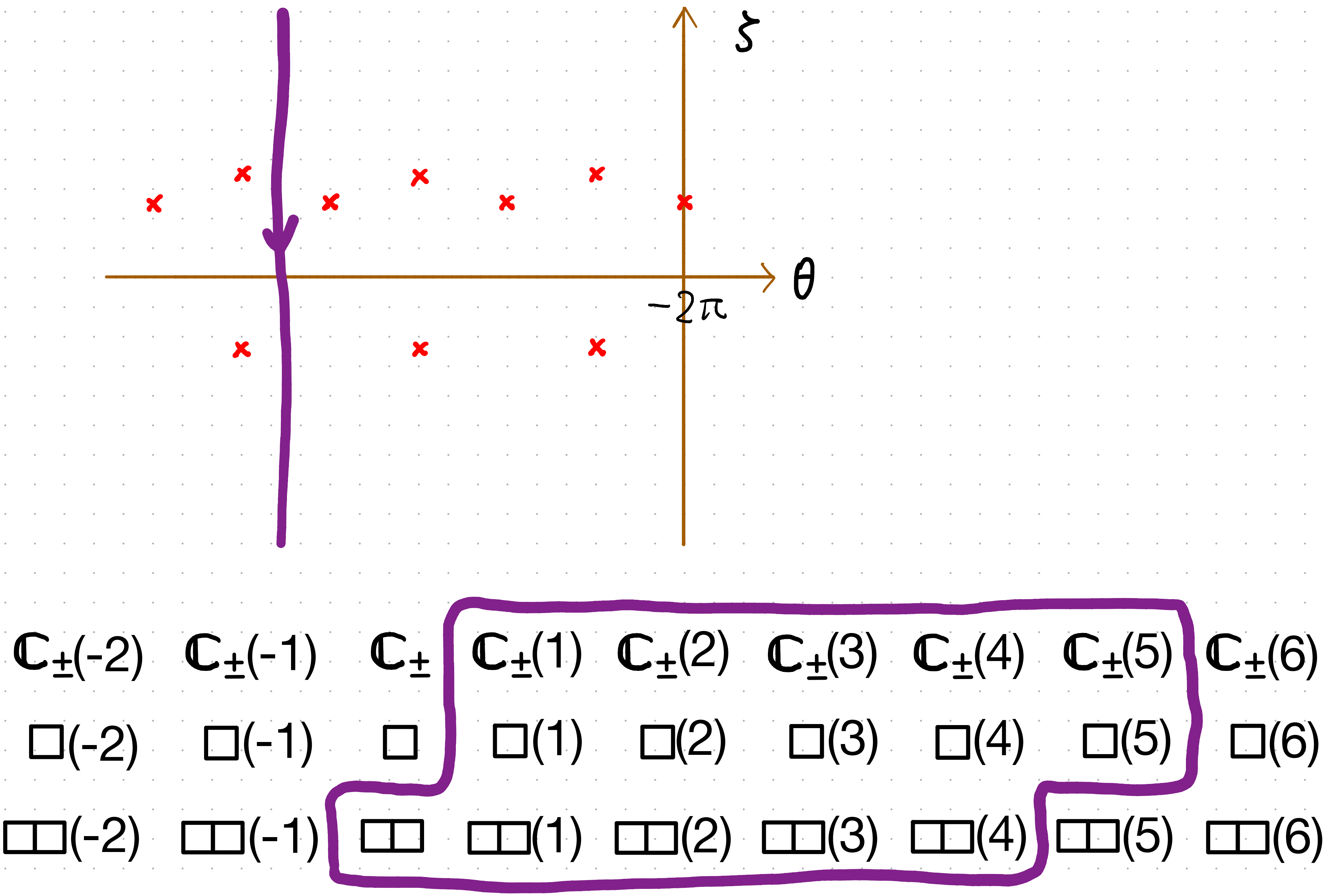
ζ
θ
−2π



$\mathbb{C}_\pm(-2)$ $\mathbb{C}_\pm(-1)$ $\mathbb{C}_\pm$ $\mathbb{C}_\pm(1)$ $\mathbb{C}_\pm(2)$ $\mathbb{C}_\pm(3)$ $\mathbb{C}_\pm(4)$ $\mathbb{C}_\pm(5)$ $\mathbb{C}_\pm(6)$
□(-2) □(-1) □ □(1) □(2) □(3) □(4) □(5) □(6)
□□(-2) □□(-1) □□ □□(1) □□(2) □□(3) □□(4) □□(5) □□(6)

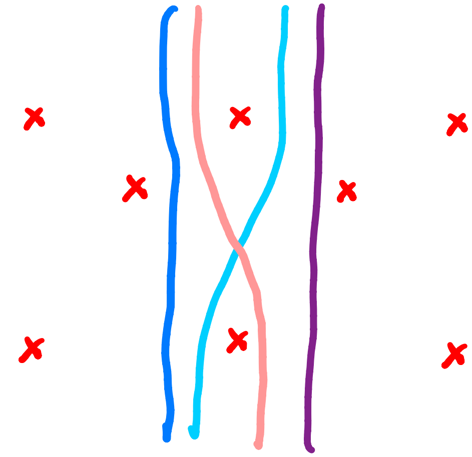


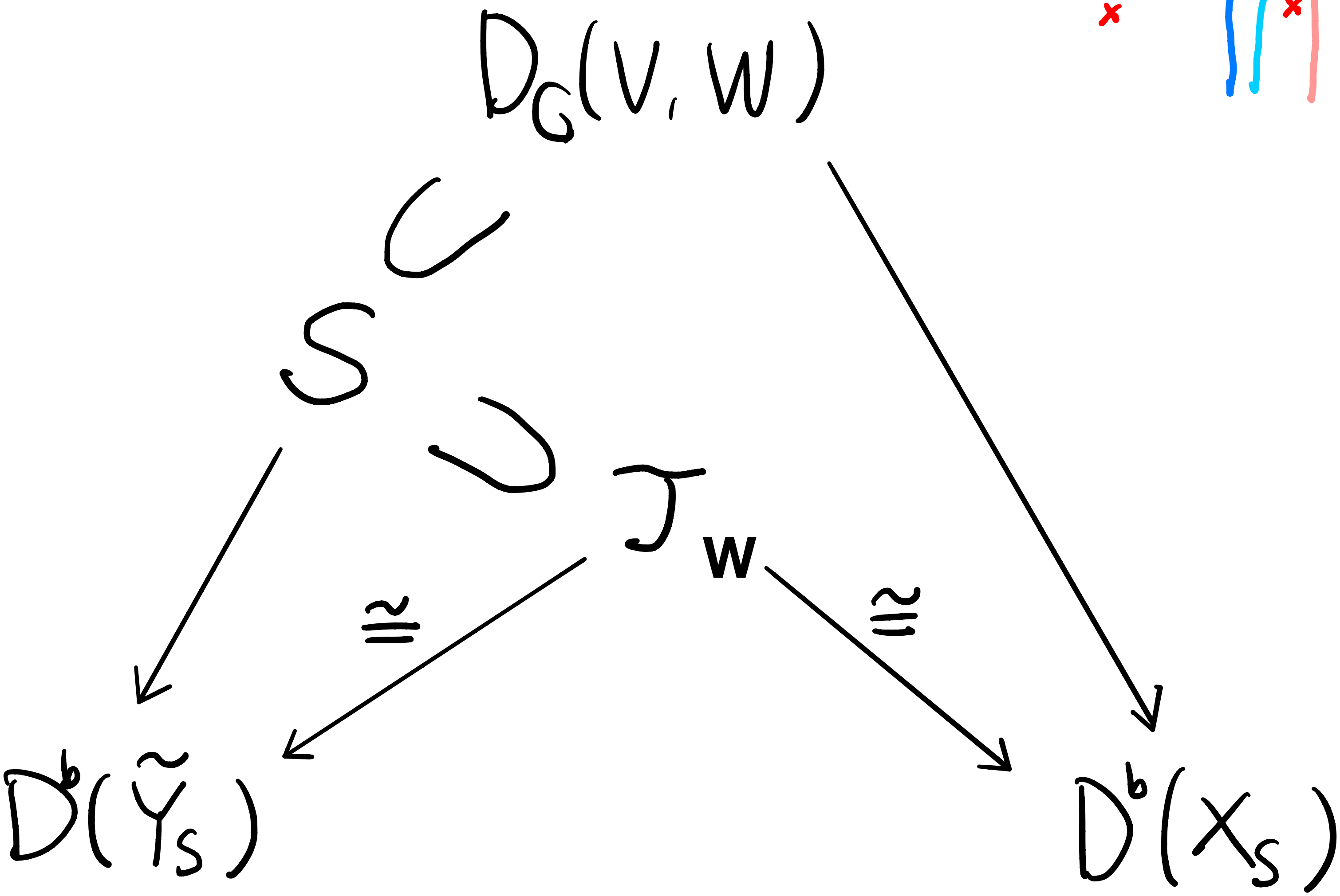
ζ
θ
−2π
ℂ±(-2) ℂ±(-1) ℂ± ℂ±(1) ℂ±(2) ℂ±(3) ℂ±(4) ℂ±(5) ℂ±(6)
□(-2) □(-1) □ □(1) □(2) □(3) □(4) □(5) □(6)
□□(-2) □□(-1) □□ □□(1) □□(2) □□(3) □□(4) □□(5) □□(6)



ζ
θ
−2π
ℂ±(-2) ℂ±(-1) ℂ± ℂ±(1) ℂ±(2) ℂ±(3) ℂ±(4) ℂ±(5) ℂ±(6)
□(-2) □(-1) □ □(1) □(2) □(3) □(4) □(5) □(6)
□□(-2) □□(-1) □□ □□(1) □□(2) □□(3) □□(4) □□(5) □□(6)

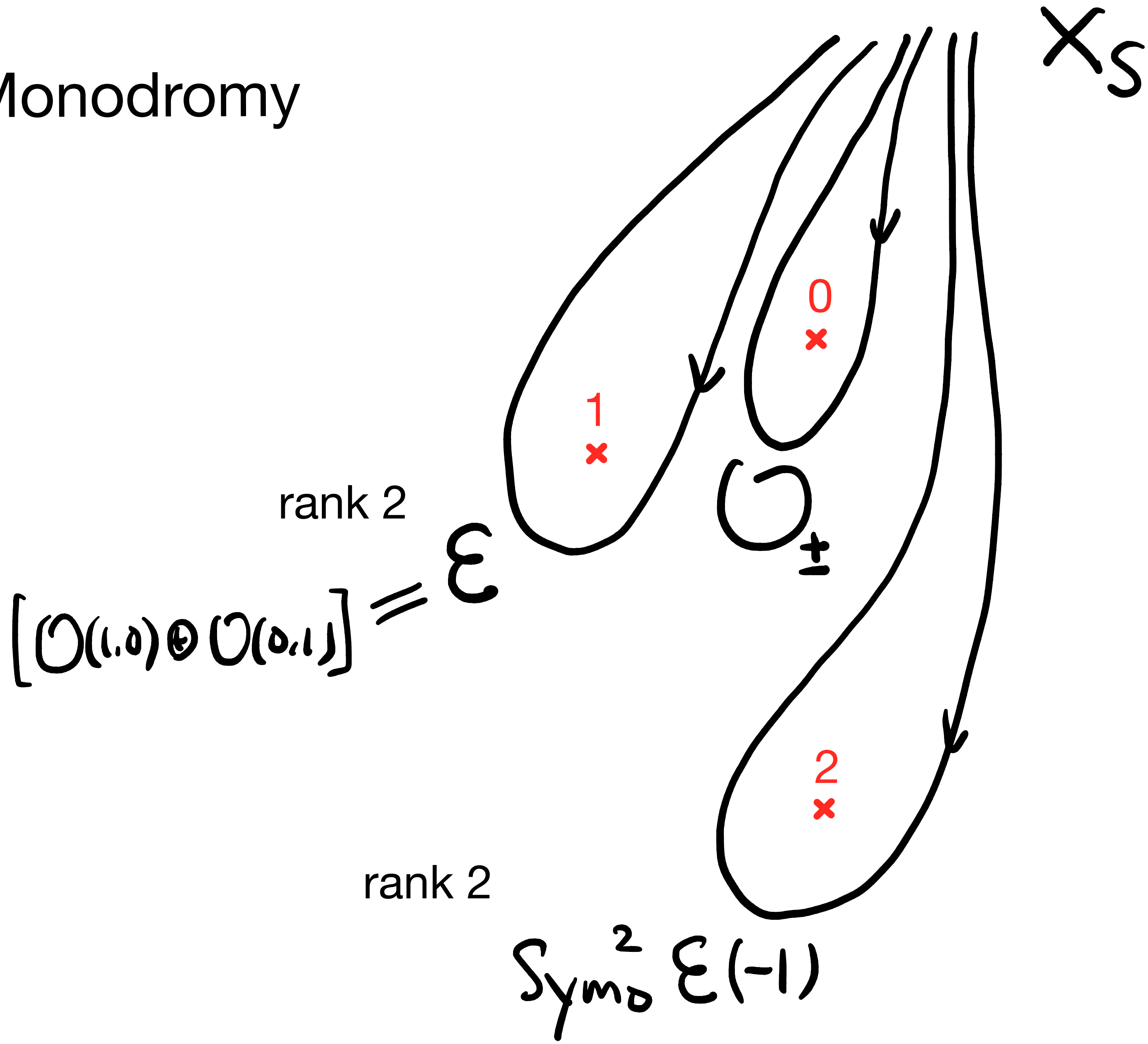
Then, we can make equivalences





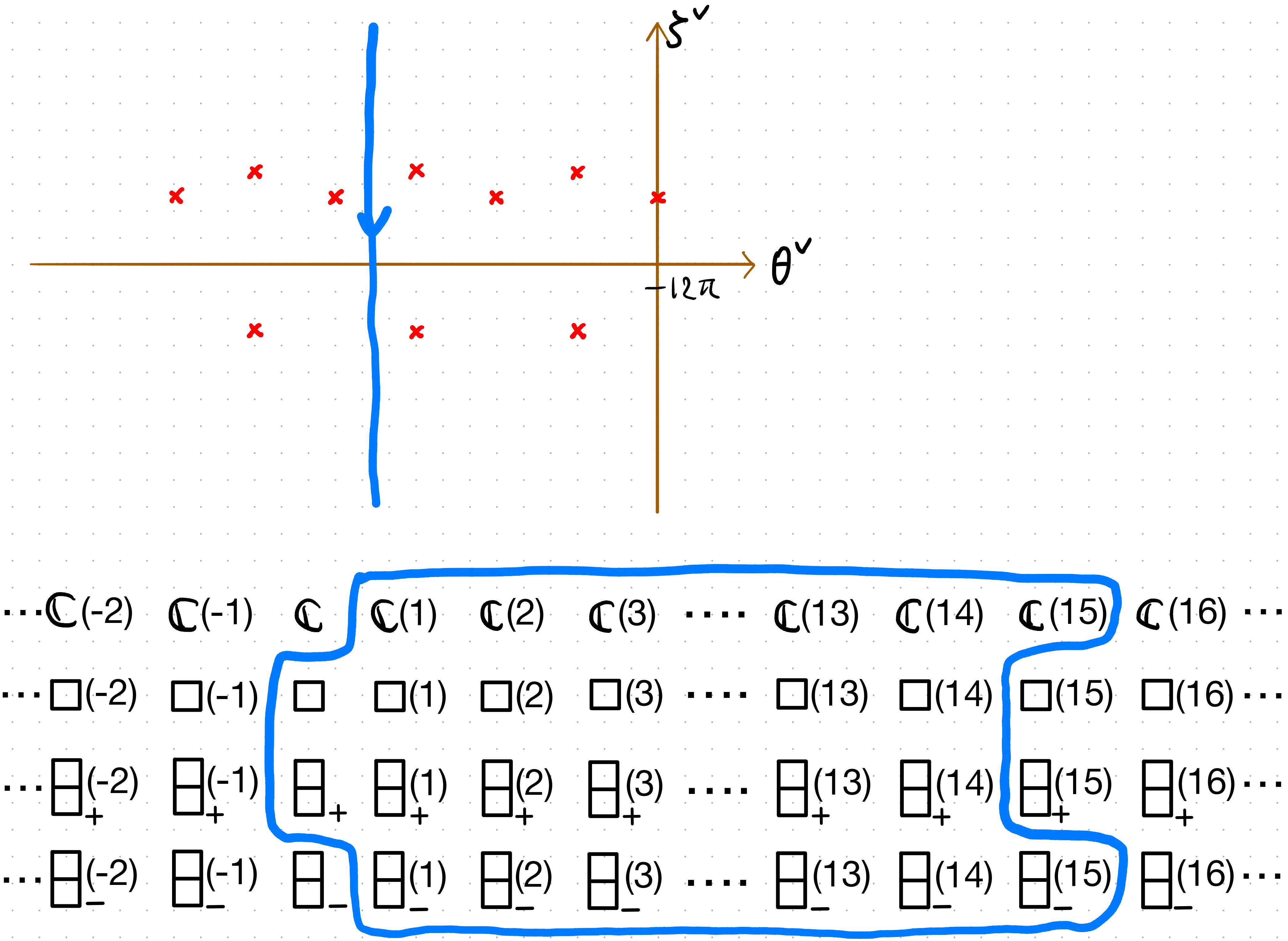
c.f. Rennemo 2015

Monodromy

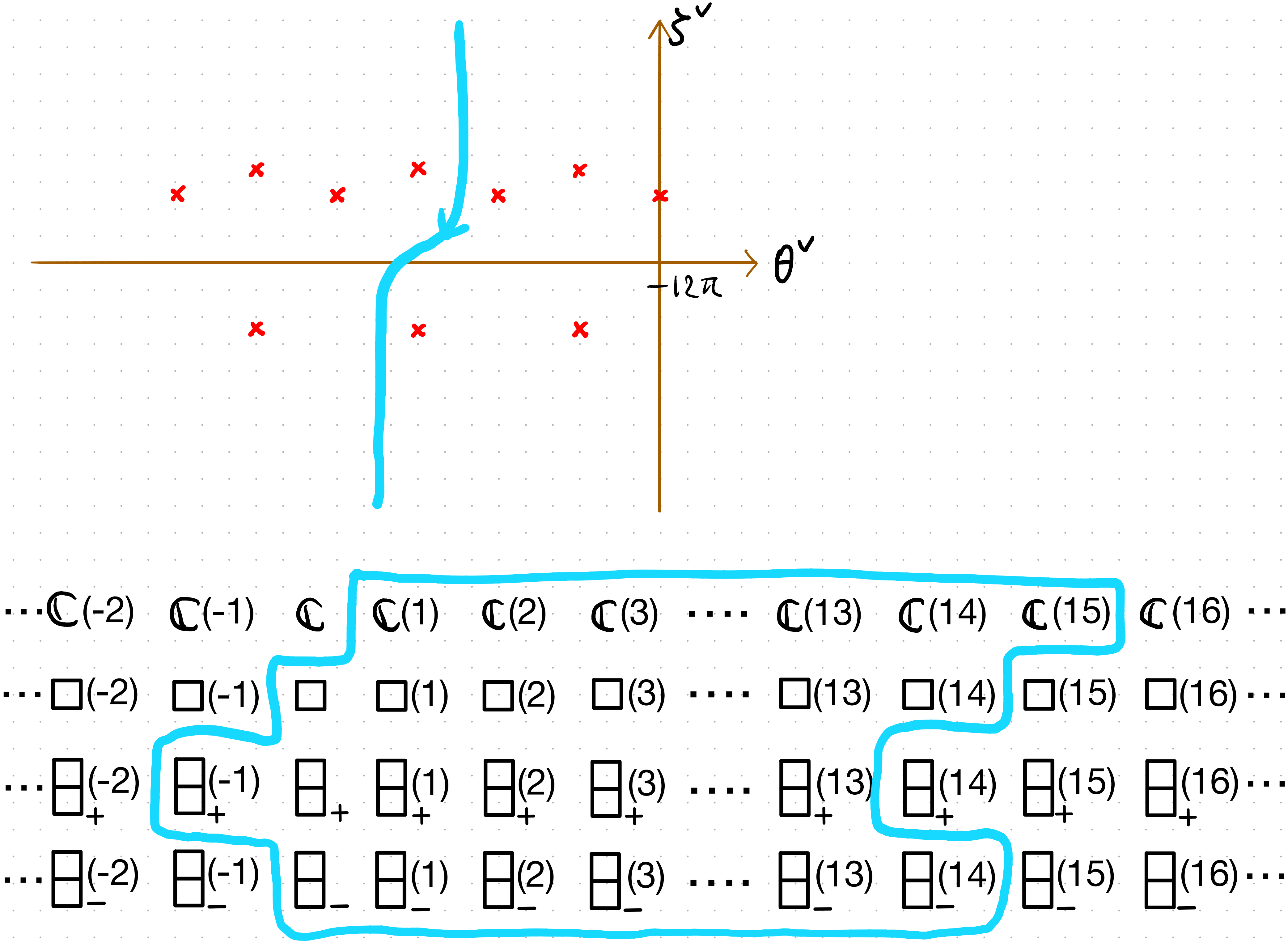


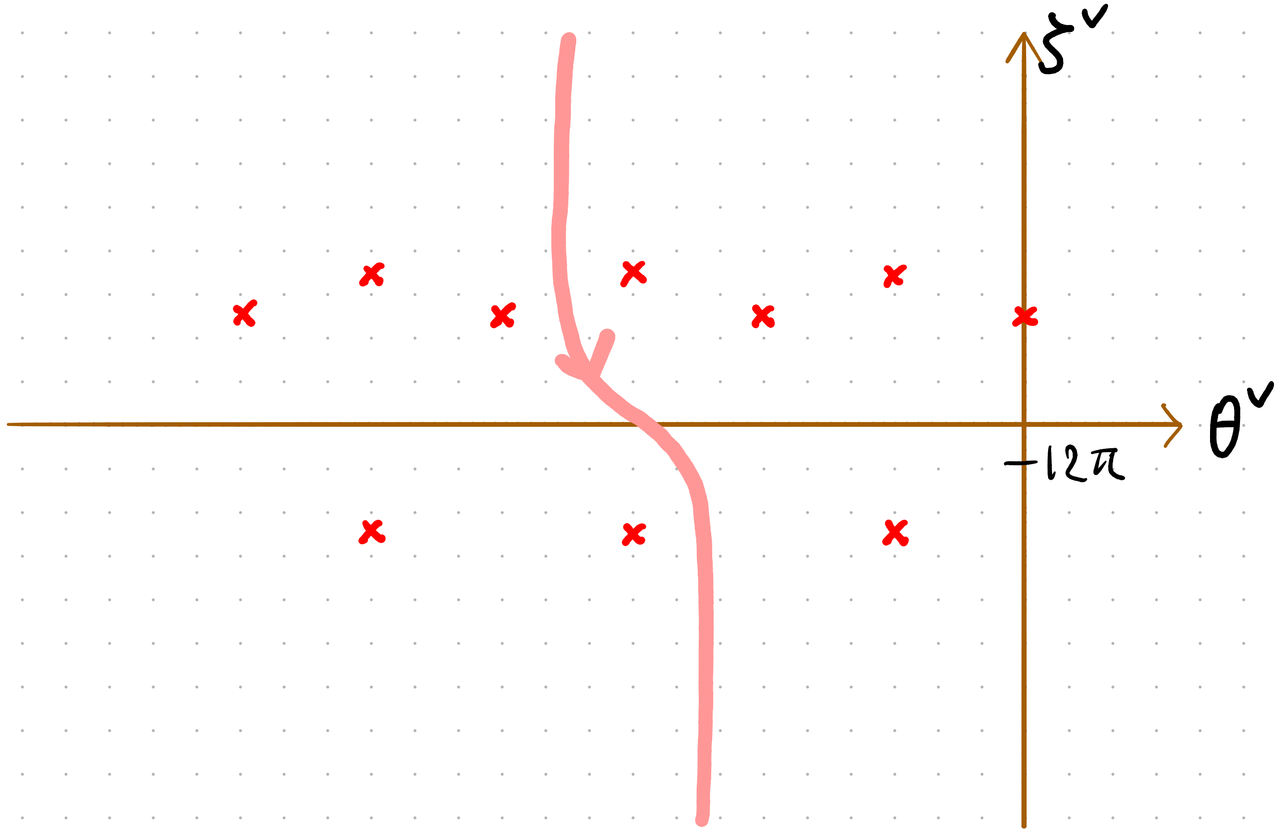
X_s
0
1
2
rank 2
$[\mathcal{O}(1,0)\oplus\mathcal{O}(0,1)] = \mathcal{E}$
$\mathcal{O}_\pm$
rank 2
$Sym_0^2\,\mathcal{E}(-1)$

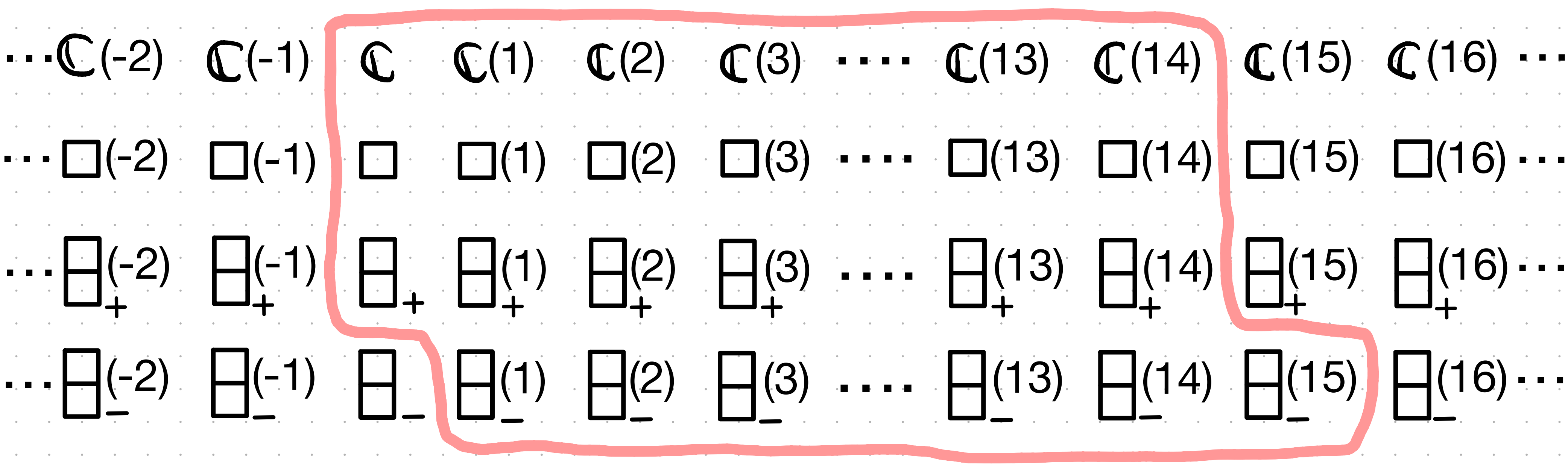
Dual Hosono-Takagi

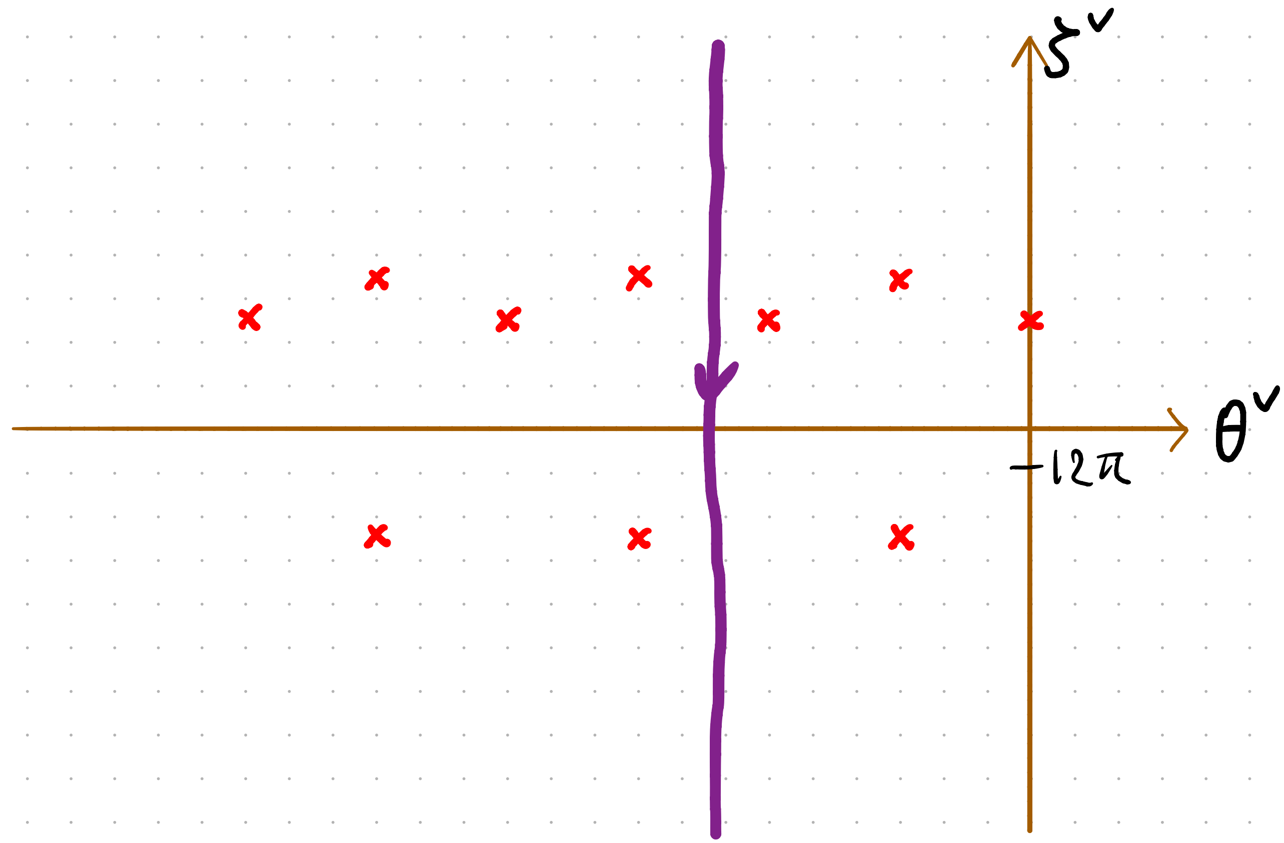


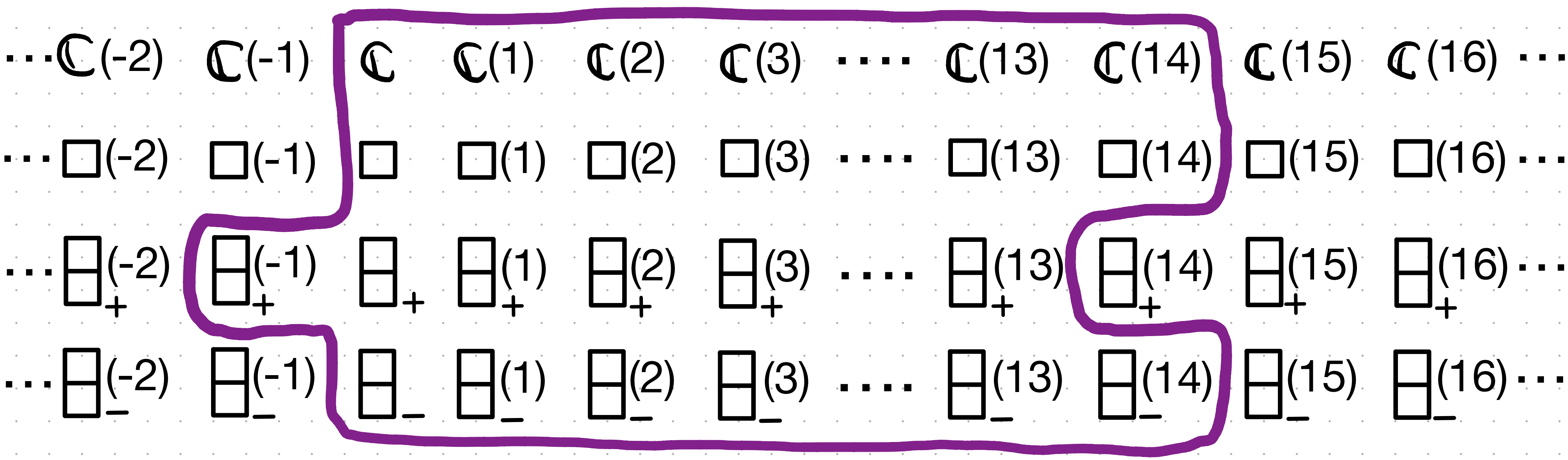
ζ∨
θ∨
-12π
⋯ ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ⋯ ℂ(13) ℂ(14) ℂ(15) ℂ(16) ⋯
⋯ □(-2) □(-1) □ □(1) □(2) □(3) ⋯ □(13) □(14) □(15) □(16) ⋯
(-2) (-1) (1) (2) (3) (13) (14) (15) (16) +
(-2) (-1) (1) (2) (3) (13) (14) (15) (16) −

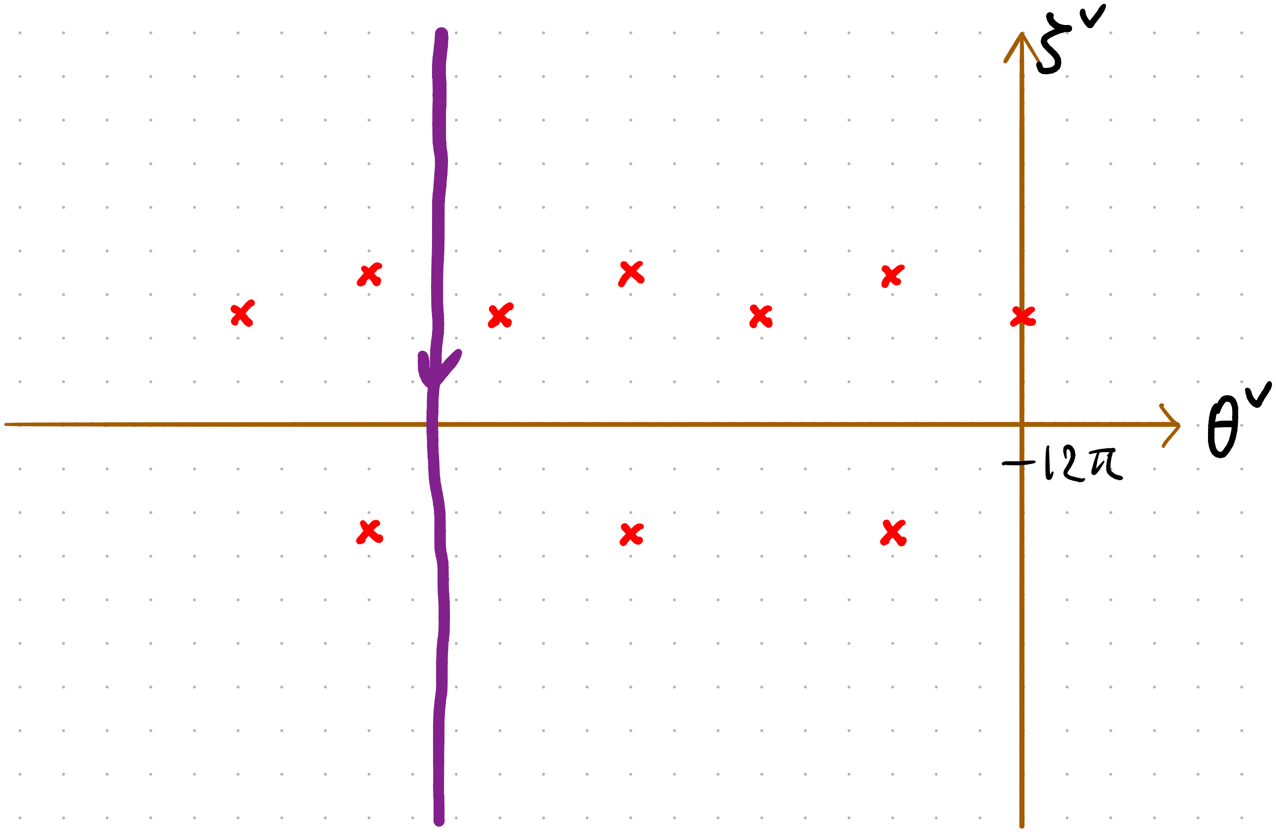

ζ∨
θ∨
−12π
··· ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ···· ℂ(13) ℂ(14) ℂ(15) ℂ(16) ···


ζ∨
θ∨
−12π

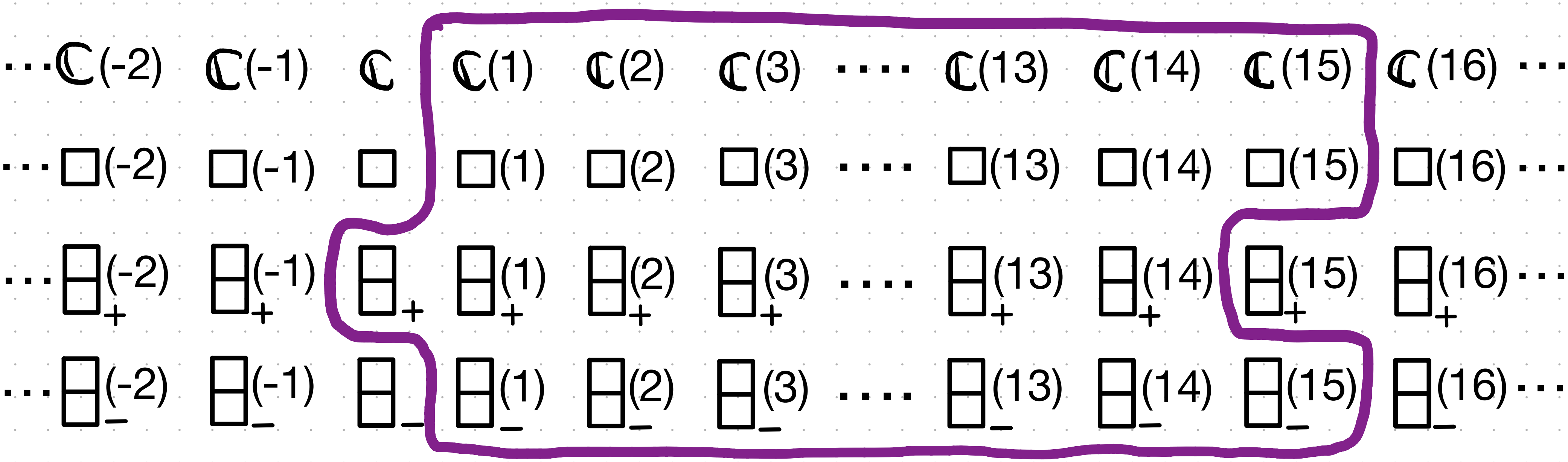

···ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ···· ℂ(13) ℂ(14) ℂ(15) ℂ(16) ···
···□(-2) □(-1) □ □(1) □(2) □(3) ···· □(13) □(14) □(15) □(16) ···


ζ^∨
θ^∨
-12π


ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ℂ(13) ℂ(14) ℂ(15) ℂ(16)
(-2) (-1) (1) (2) (3) (13) (14) (15) (16)
+
−

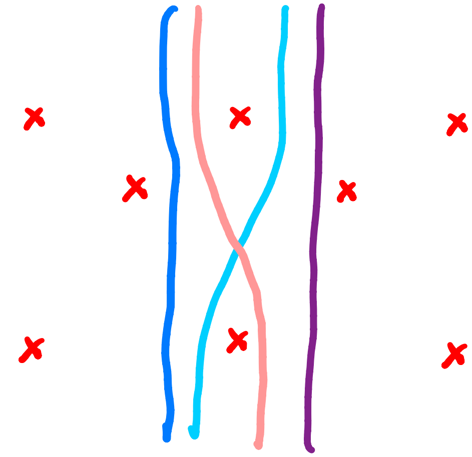


ζ∨
θ∨
-12π



⋯ ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ⋯ ℂ(13) ℂ(14) ℂ(15) ℂ(16) ⋯
⋯ □(-2) □(-1) □ □(1) □(2) □(3) ⋯ □(13) □(14) □(15) □(16) ⋯
(-2) (-1) (1) (2) (3) (13) (14) (15) (16)
+
(-2) (-1) (1) (2) (3) (13) (14) (15) (16)
−

For each window **W** such as

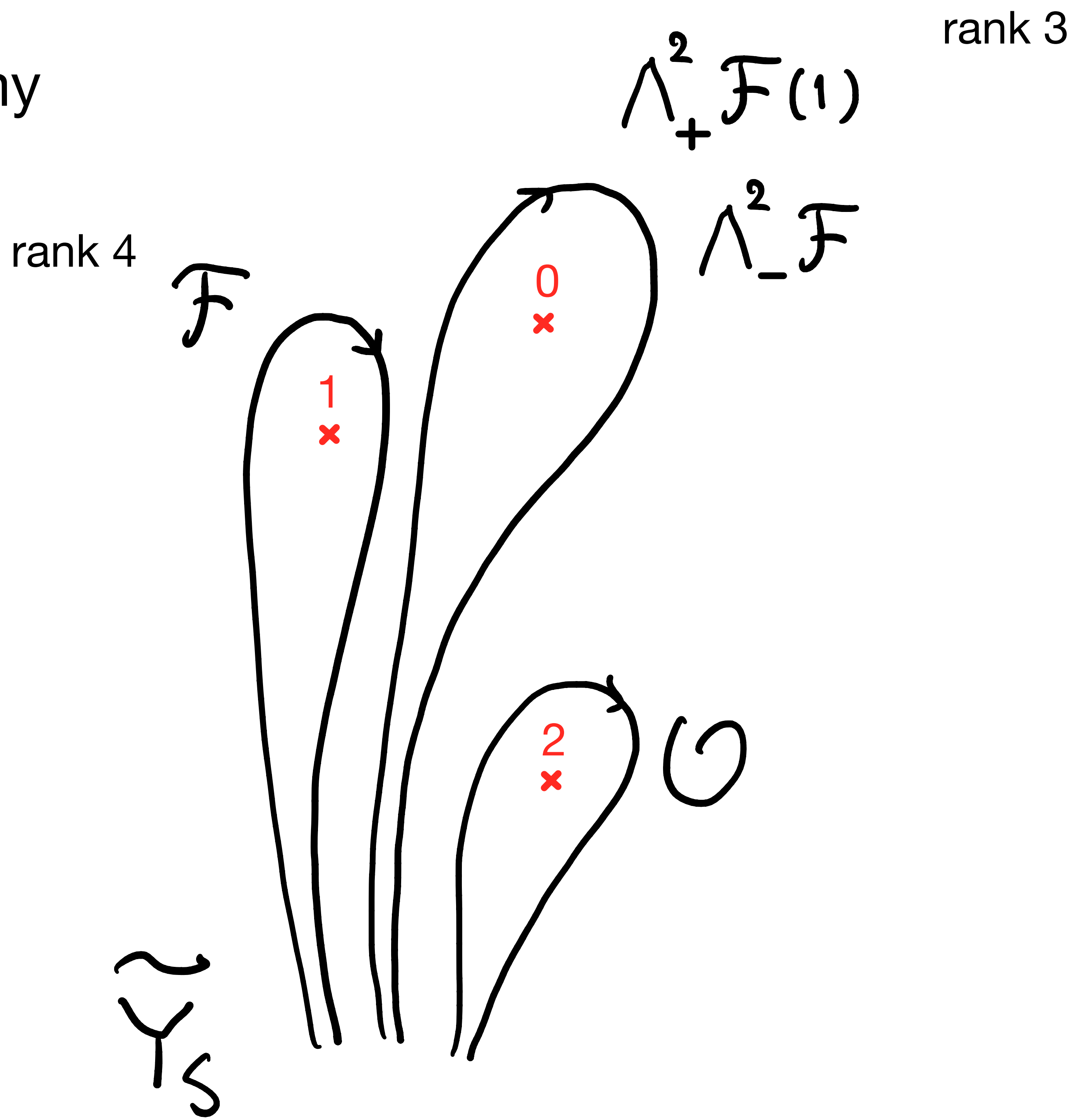


$$D_{G^\vee}(V^\vee, W^\vee) \supset S^\vee \supset \tilde{J}^\vee_{\mathbf{W}}$$

$$D^b(\tilde{Y}_S) \xleftarrow{\;\cong\;} \tilde{J}^\vee_{\mathbf{W}} \xrightarrow{\;\cong\;} D^b(X_S)$$

$$S^\vee \to D^b(\tilde{Y}_S), \qquad S^\vee \to D^b(X_S)$$

Monodromy



rank 3
$\Lambda^2_+ \mathcal{F}(1)$
$\Lambda^2_- \mathcal{F}$
rank 4
$\mathcal{F}$
0
1
2
$\mathcal{O}$
$\tilde{Y}_s$