BCOV cusp forms of lattice polarized K3 surfaces

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§1. BCOV formula of Calabi-Yau manifolds

'92 Cecotti, Fendly, Intligator and Vafa introduced a new index for $N = 2$ SFT in two dimensions,

$$\mathbf{F}_1 = \mathrm{Tr}_{\mathcal{H}}(-1)^F F, \qquad F : \text{Ferimion number operator}$$

(cf. Witten index $\mathrm{Tr}_{\mathcal{H}}(-1)^F$ is topological.)

This new index is not topological, but it was argued that

(1) $\mathbf{F}_1 = \mathbf{F}_1(t, \bar{t})$ splits "almost" to a product $F(t)\overline{F(t)}$, where $t_1, ..., t_r$ are holomorphic coordinates of the moduli of $N = 2$ theory.

(2) The spliting is not complete, but satisfies the **holomorphic anomaly equation**

$$\boxed{\frac{\partial}{\partial t_i}\frac{\partial}{\partial \bar{t}_j}\mathbf{F}_1 = \mathrm{Tr}(\mathcal{C}_i \mathcal{C}_{\bar{j}}) + \frac{\chi}{12} g_{i\bar{j}}}$$

$\mathcal{C}_i = (C_{i\ b}^{a})$ describes the operator algebra of the ground states

$g_{i\bar{j}}$: Zamolodchikov metric, $\chi := Tr(-1)^F$

- In case of $N=2$ σ-models on a Calabi-Yau 3 fold X, using the so-called special Kähler geometry on the (Kähler) moduli, the h.a.e. can be solved as

$$\mathbf{F}_1 = \frac{1}{2}\log\left\{e^{(3+h_X^{1,1}-\frac{\chi}{12})\mathcal{K}(t,\bar{t})}(\det g_{i\bar{j}})^{-1}|f|^2\right\}$$

- If we have a family of (mirror) CY 3 folds, which has a LCSL at o given by

$$x_1 = \cdots = x_r = 0,$$

we have the "topological limit" $\lim_{\bar{t}\to\infty}\mathbf{F}_1(t,\bar{t}) := \lim_{\lambda\to\infty}\mathbf{F}_1(t,\lambda\bar{t})$, where

$$\mathcal{K}(t,\bar{t}) \longrightarrow -\log(w_0(x)\overline{w_0(x)})$$

$$\det(g_{i\bar{j}})^{-1} \longrightarrow \left|\frac{\partial(x_1,\ldots,x_r)}{\partial(t_1,\ldots,t_r)}\right|^2$$

Definiton. [BCOV formula (of log form) for CY 3 folds]

$$F_1^{top}(t) = \frac{1}{2}\log\left\{\left(\frac{1}{w_0(x)}\right)^{3+h_X^{1,1}-\frac{\chi}{12}}\frac{\partial(x_1,\cdots,x_r)}{\partial(t_1,\cdots,t_r)}f(x)\right\}$$

$f(x)$: homolorphic functions which we determine by suitable boundary conditions

Bershadsky-Cecotti-Ooguri-Vafa ('93)

Discovery. **(BCOV '93)** **(1)** If we find a suitable $f(x)$, $F_1^{top}(t)$ gives the generating function of genus one Gromov-Witten invariants of X.

(2) This generalizes to the higher genus functions $\{(F_g^{top}(t), f_g(x))\}_{\geq 2}$ by

$$\frac{\partial}{\partial \bar{t}_i}\mathbf{F}_g(t,\bar{t}) = \frac{1}{2}\partial_{\bar{i}} S^{jk}\Big\{ D_j D_k \mathbf{F}_{g-1}(t,\bar{t}) + \sum_{r+s=g} D_j \mathbf{F}_r(t,\bar{t}) D_k \mathbf{F}_s(t,\bar{t}) \Big\}$$

with $\partial_{\bar{i}} S^{jk} = e^{2\mathcal{K}(t,\bar{t})} C_{\bar{i}\bar{j}\bar{k}} g^{\bar{j}j} g^{\bar{k}k}$.

- **This is still misterious (at least for me) after 30 years since the discovery!**
- **Also this motivates us studying the moduli spaces of CY manifolds.**

The subject of today: BCOV formula $F_1^{top}(t)$ for K3 surfaces

For K3 surfaces, there are no corrections in F_1^{top} from Gromov-Witten invariants. However, we observe nice modular forms from it.

. — work with Atsushi Kanazawa ArXiv:2303.04383, Adv.Math(2023).

§2. Lattice polarized K3 surfaces

X : a K3 surface (Kähler, $c_1(T_X)=0$, $H^1(X,\mathcal{O}_X)=0$)

$(H^2(X,\mathbb{Z}),(*,*)) \simeq L_{K3}$ where $L_{K3} := U^{\oplus 3} \oplus E_8(-1) \oplus E_8(-1)$

$\phi : H^2(X,\mathbb{Z}) \simeq L_{K3}$ is called a marking of K3

Definitions:

Fix a primitive embedding $\underset{(1,\rho-1)}{M} \hookrightarrow \underset{(3,19)}{L_{K3}}$

- (X,ϕ) : (marked) M-**polarized** K3 $\Leftrightarrow$ $\phi^{-1}(M) \subset Pic(X)$, $\phi^{-1}(C_M^{pol}) \subset Amp(X)$

- $(X_1,\phi_1) \sim (X_2,\phi_2)$ $\Leftrightarrow$ $\exists f : X_1 \to X_2$(isom.) s.t.

$$\begin{array}{ccc} H^2(X_1,\mathbb{Z}) & \xleftarrow[f^*]{\sim} & H^2(X_2,\mathbb{Z}) \\ \phi_1 \downarrow \wr & & \phi_1 \downarrow \wr \\ L_{K3} & \xleftarrow{\sim} & L_{K3} \\ \cup & & \cup \\ M & \underset{id}{=} & M \end{array}$$

$$\Omega_M = \Omega(M^\perp) := \{[w] \in \mathbb{P}(M^\perp \otimes \mathbb{C}) | (w.w) = 0, (w, \overline{w}) > 0\}^+$$

Period domain

$$\begin{array}{c}\text{Moduli space of} \\ M\text{-polarized K3 surfaces}\end{array} = \Omega_M / O(M, L_{K3})$$

$$O(M, L_{K3}) = \{g \in O(L_{K3}) |\ g|_M = id_M,\ g \text{ acts on } \Omega_M\}$$

Mirror symmetry (Dolgachev '96, Todorov '96)

When we have the decomposition: $M \oplus M^\perp = M \oplus U \oplus \check{M} \subset L_{K3}$,

$$M\textbf{-polarized K3 surfaces} \underset{\text{mirror sym.}}{\longleftrightarrow} \check{M}\textbf{-polarized K3 surfaces}$$

Remark (M-polarizable K3 surfaces, HLOY '01)

A slightly larger group acts on the period domain to obtain

$$\{\text{ isom. classes of } M\text{-polarizable K3 surfaces }\} = \Omega_M / O(M^\perp)_+$$

if $M \hookrightarrow L_{K3}$ is unique up to isom., where

$$O(M^\perp)_+ := \{g \in O(M^\perp) \mid g \text{ acts on } \Omega_M\}.$$

§3. Settings for the BCOV formula

0. Take an embedding $\check{M} \hookrightarrow L_{K3}$ s.t. $\check{M} \oplus \check{M}^{\perp} = \underset{(1,\check{r}-1)}{\check{M}} \oplus \underset{(1,1)}{U} \oplus \underset{(1,r-1)}{M} \subset L_{K3}$

1. Suppose we have a family of $\check{M}$-polarizable K3 surfaces s.t.

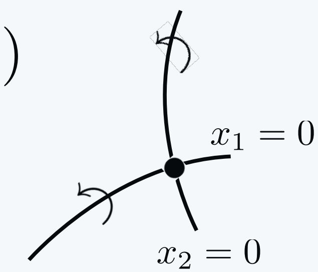
$$\begin{array}{ccc} \check{\mathfrak{X}} & \supset & \check{X}_x \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathcal{M} & \ni & x \end{array}$$

the associated local system $R^2\pi_*\mathbb{C}_{\check{\mathfrak{X}}}$ has a boundary point o, i.e., a LCSL, which is characterized by a certain local solutions

$$w_0(x), w^{(2)}(x), w_1^{(1)}(x), \ldots, w_r^{(1)}(x)$$

satisfying a quadratic relation

$$2w_0 w^{(2)} + (w^{(1)}, w^{(1)})_M = 0.$$



2. Then we can define the period map by

$$\begin{array}{cccl} \mathcal{P}: & \mathcal{M} & \longrightarrow & \Omega_{\check{M}} \\ & \cup\!\!\!\!\!\,\cup & & \cup\!\!\!\!\!\,\cup \\ & x & \mapsto & \mathcal{P}(x) := [w_0(x), w^{(2)}(x), w_1^{(1)}(x), \cdots, w_r^{(1)}(x)] \end{array}$$

$$= \left[\int_{\phi^{-1}(e)} \omega_x, \int_{\phi^{-1}(f)} \omega_x, \int_{\phi^{-1}(\gamma_1)} \omega_x, \cdots, \int_{\phi^{-1}(\gamma_r)} \omega_x\right]$$

$\{e, f, \gamma_1, \ldots, \gamma_r\}$ is a basis of $\check{M}^{\perp} = U \oplus M$

3. Define the mirror map by introducing the inhomogeneous coordinates

$$\mathcal{P}(x) = [w_0(x), w^{(2)}(x), w_1^{(1)}, \cdots, w_r^{(1)}] = [1, -\frac{1}{2}(t^2)_M, t_1, \cdots, t_r]$$

which describes the isomorphism

$$\begin{array}{ccc} \Omega_{\check{M}} & \xrightarrow{\sim} & M \otimes \mathbb{R} + \sqrt{-1}C_M \\ \Cup & & \Cup \\ \mathcal{P}(x) & \mapsto & (t_1, \cdots, t_r) \\ \circlearrowright & & \circlearrowright \\ O(\check{M}^\perp)_+ & & O(\check{M}^\perp)_+ \end{array}$$

: **Tube domain coordinates (= mirror map)**

Holomorphic functions on the tube domain $T_M := M \otimes \mathbb{R} + \sqrt{-1}C_M$ with natural transformation properties are called automorphic forms of $O(\check{M}^\perp)_+$.

4. **Automorphic form on T_M.**

(1)Write the linear action of $g \in O(\check{M}^\perp)_+$ by

$$g \cdot \big(1, -\frac{1}{2}(t^2)_M, t_1, \cdots, t_r\big) = \big(D(g,t), A(g,t), B_1(g,t), \cdots, B_r(g,t)\big).$$

This induces the action $g : (t_1, ..., t_r) \mapsto (g \cdot t_1, ..., g \cdot t_r)$ by

$$\boxed{g \cdot t := \frac{B_i(g,t)}{D(g,t)} \quad (i = 1, ..., r)}$$ ("Modular action")

(2) Homolorphic functions $F(t)$ on T_M satisfying

$$\boxed{F(g\cdot t) = D(g,t)^w F(t)} \quad (g \in O(\check{M}^\perp)_+)$$

are called **automorphic forms of weight** w.

Remark. The period integral $w_0(x) = w_0(x(t))$ with the mirror map $x = x(t)$ defines an automorphic form of weight one (with possibly a multiplier $v(g)$), i.e., it holds that

$$\boxed{w_0(x(g\cdot t)) = v(g)\, D(g,t)\, w_0(x(t))} \quad (|v(g)| = 1)$$

Definition (H.K. '23) We define **BCOV formula** by

$$\tau_{BCOV}(t) := \left\{ \left(\frac{1}{w_0(x)} \right)^{r+1} \frac{\partial(x_1,\cdots,x_r)}{\partial(t_1,\cdots,t_r)} \prod_i dis_i^{r_i} \prod_i x_i^{-1+a_i} \right\}$$

where r_i and a_i are parameters to be fixed by boundary conditions.

If $(\tau_{BCOV}(t))^{-1}$ defines a cusp form on $T_M = M \otimes \mathbb{R} + \sqrt{-1}C_M$, we call it **BCOV cusp form.**

Lemma.

The Jacobian factor $\dfrac{\partial(x_1, \cdots, x_r)}{\partial(t_1, \cdots, t_r)}$ has weight r (with possibly a multiplier system) w.r.t. $O(\check{M}^{\perp})_+ (= O(U \oplus M)_+)$.

Proof) Recall that $\Omega_{\check{M}} \simeq M \otimes \mathbb{R} + \sqrt{-1} C_M$ is describrd by a quadric

$$\{[u, v, z] \mid 2uv + (z, z)_M = 0\} \subset \mathbb{P}(\check{M}^{\perp} \otimes \mathbb{C}).$$

Using this, we can show that

$$\frac{u^r}{2} dt_1 \wedge dt_2 \wedge ... \wedge dt_r = Res\Big(\frac{d\mu_{\mathbb{P}^{r+1}}}{2uv + (z, z)_M}\Big)$$
$$= Res\Big(\frac{d\mu'_{\mathbb{P}^{r+1}}}{2u'v' + (z', z')_M}\Big) = \frac{u'^r}{2} dt'_1 \wedge dt'_2 \wedge ... \wedge dt'_r$$

Here we can identify $\dfrac{u'}{u}$ with the automorphic factor $D(g, t)$. $\square$

Proposition.

The inverse power $(\tau_{BCOV}(t))^{-1}$ of the BCOV formula

$$\tau_{BCOV} = \left(\frac{1}{w_0(x)}\right)^{r+1} \frac{\partial(x_1, \cdots, x_r)}{\partial(t_1, \cdots, t_r)} \prod_k dis_k^{r_k} \prod_i x_i^{-1+a_i}$$

has weight one with respect to $O(\check{M}^{\perp})_+$.

Proof) The period integral $w_0(x(t))$ has weight **one** as we remarked.

Since, the Jacobian has weight **r**, the weight of $(\tau_{BCOV})^{-1}$ is one. $\square$

Boundary conditions:

We determine the parameters r_k and a_i by the following reularity conditions:

(1) **Conifold regularity** $\cdots$ a regularity at the discriminant loci $\{dis_k(x) = 0\}$.

$\rightarrow$ **it turns out** $\boxed{\mathbf{r}_k = -\frac{1}{2}}$ **in general**

(2) **Orbifold regularity** $\cdots$ a regularity from the so-called orbifold points.

$\rightarrow$ **case by case**

Example 1. $(6) \subset \mathbb{P}^3(3,1,1,1)$ ($M_2 = \langle 2 \rangle$-polarized K3 surface) $\to M_2 \oplus U \oplus \check{M}_2 \subset L_{K3}$

$\check{M}_2 = \langle -2 \rangle \oplus U \oplus E_8(-1)^{\oplus 2}$-polarizable K3 surfaces
(a Picard rank 19 family of K3 surfaces)

1. Picard-Fuchs equation $\{\theta_x^3 - 8x(6\theta_x + 5)(6\theta_x + 3)(\theta_x + 1)\}w(x) = 0$
2. mirror map $x(t) = \dfrac{1}{j(t)}$, $w_0(x) = E_4(t)^{\frac{1}{2}}$
3. $\left(\dfrac{1}{w_0(t)}\right)^2 C_{xx}\left(\dfrac{dx}{dt}\right)^2 = 2$, where $C_{xx} = \dfrac{2}{x^2(1-1728x)}$ is the Griffiths-Yukawa coup. (mirror)
4. $\tau_{BCOV}(t) = \left(\dfrac{1}{w_0(t)}\right)^2 \left(\dfrac{dx}{dt}\right) dis_0^{r_0} x^{-1+a}$, where $dis_0 = 1 - 1727x$

Form the 3rd relation, we have $\dfrac{dx}{dt} = w_0(x)x(1-1728x)^{\frac{1}{2}}$.

Using this (and after a little calculations), we find

$$\boxed{(\tau_{BCOV}(t))^{-1} = \left(\eta(t)^{24}\right)^{\frac{1}{6}}} \leftarrow \text{a cusp form!}$$

for $r_0 = -\dfrac{1}{2}$ and $a = -\dfrac{1}{6}$ (justified by the orbifold regularity).

In this (trivial) case, we obtain a **BCOV cusp form** from $\tau_{\mathbf{BCOV(t)}}$.

Example 2. ($M_{20} \oplus U \oplus \check{M}_{20}$ from the list in Lian and Yau '93)

$\check{M}_{20} = \langle -20 \rangle \oplus U \oplus E_8(-1)^{\oplus 2}$-polarizable K3 surfaces

(a Picard rank 19 family of K3 surfaces)

1. Picard-Fuchs equation
$$\left\{\theta_x^3 - 2x(2\theta_x+1)(3\theta_x^2+3\theta_x+1) - x^2(4\theta_x+3)(4\theta_x+4)(4\theta_x+5)\right\} w = 0$$
2. mirror map $x(t) = q - 4q^2 - 6q^3 + 56q^4 - 45q^5 + \cdots$ (**Thompson series** of $\Gamma_0(10)_+$)
3. $$\Big(\frac{1}{w_0(t)}\Big)^2 C_{xx}\Big(\frac{dx}{dt}\Big)^2 = 20, \text{ where } C_{xx} = \frac{20}{x^2(1+4x)(1-16x)}$$

$$\Gamma_0(n) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid c \equiv 0 \bmod n\right\}$$

Proposition.

The conifold and orbifold regularities uniquely determine the parameters in $\tau_{\rm BCOV}$ as $r_0 = r_1 = -\frac{1}{2}$ and $a = -\frac{3}{4}$. Then, we have

$$\tau_{\rm BCOV}(t) = \left(\frac{1}{w_0(x)}\right)^2 \frac{dx}{dt} dis_0^{r_0} dis_1^{r_1} x^{-1+a} = \frac{1}{\eta_1(t)\eta_2(t)\eta_5(t)\eta_{10}(t)},$$

and $(\tau_{\rm BCOV}(t))^{-1}$ defines a BCOV cusp form on $\mathbb{H}_+$ w.r.t. $\Gamma_0(10)_+$.

Here we define $\eta_k(t) := \eta(kt)$.

Similar calculations apply to other cases of the $\check{M}_{2n}$-polarizable K3 surfaces in the list of Lian and Yau ('93). We can **verify the following results** for all cases in the list, which we state as a conjecture in general:

Conjecture. (H.K. '23)

For families of $\check{M}_{2n} = \langle -2n \rangle \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$-polarizable K3 surfaces over $\mathbb{P}^1$, we have the BCOV cusp forms

$$(\tau_{_{BCOV}}(t))^{-1} = \eta_{_{BCOV}}(t)$$

with the eta products,

$$\eta_{\rm BCOV}(t) = \left(\prod_{r|n} \eta_r(t)^{\pm 1} \right)^w ,$$

where $+1$ is taken when $(r, n/r) \neq 1$ and -1 when $(r, n/r) = 1$.

Supporting evidence. The eta product $\eta_{_{BCOV}}(t)$ defines a cusp form of the genus zero group $\Gamma_0(n)_+$ if #cusps($\Gamma_0(n)_+$)=1.

List of genus zero groups of type $\Gamma_0(n)_+$ (Conway-Norton '79).

n	type	c	**n**	type	c	**n**	type	c	**n**	type	c	**n**	type	c
1	1A	1	14	14A	1	27	27A	3*	42	42A	1	62	62AB	1
2	2A	1	15	15A	1	28	28B	2	44	44AB	2	66	66A	1
3	3A	1	16	16C	3	29	29A	1	45	45A	2	69	69AB	1
4	4A	2	17	17A	1	30	30B	1	46	46CD	1	70	70A	1
5	5A	1	18	18B	2	31	31AB	1	47	47AB	1	71	71AB	1
6	6A	1	19	19A	1	32	32A	4	49	49Z	4*	78	78A	1
7	7A	1	20	20A	2	33	33B	1	50	50A	3*	87	87AB	1
8	8A	2	21	21A	1	34	34A	1	51	51A	1	92	92AB	2
9	9A	2	22	22A	1	35	35A	1	54	54A	3*	94	94AB	1
10	10A	1	23	23AB	1	36	36A	4	55	55A	1	95	95AB	1
11	11A	1	24	24B	2	38	38A	1	56	56A	2	105	105A	1
12	12A	2	25	25A	3*	39	39A	1	59	59AB	1	110	110A	1
13	13A	1	26	26A	1	41	41A	1	60	60B	2	119	119AB	1

c := the number of cusps in $\mathbb{H}/\Gamma_0(n)_+$
type: the name for the conjugacy classes of the Monster group

Some selected examples of the eta-products $\eta_{BCOV}(t)$:

$\underline{\Gamma_0(10)_+}$ $$\eta_{BCOV}(t) = \eta_1(t)\eta_2(t)\eta_5(t)\eta_{10}(t)$$

$\underline{\Gamma_0(16)_+}$ $$\eta_{BCOV}(t) = \frac{\eta_2(t)^4\eta_4(t)^4\eta_8(t)^4}{\eta_1(t)^4\eta_{16}(t)^4}$$

$\underline{\Gamma_0(29)_+}$ $$\eta_{BCOV}(t) = \eta_1(t)^2\eta_{29}(t)^2$$

$\underline{\Gamma_0(36)_+}$ $$\eta_{BCOV}(t) = \frac{\eta_2(t)^4\eta_3(t)^4\eta_6(t)^4\eta_{12}(t)^4\eta_{18}(t)^4}{\eta_1(t)^4\eta_4(t)^4\eta_9(t)^4\eta_{36}(t)^4}$$

$\underline{\Gamma_0(94)_+}$ $$\eta_{BCOV}(t) = \eta_1(t)\eta_2(t)\eta_{47}(t)\eta_{94}(t)$$

$\vdots$

• If we assume the conjecture, K3 differential operators follow:

If we postulate the cojecture, then the following relations follow:

a) $w_0(x) = x^\gamma \eta_{BCOV}(t)$

b) $\dfrac{1}{x(t)} = T_n(t) + c_n$ (the Thompson series of $\Gamma_0(n)_+$)

for all the genus zero group $\Gamma_0(n)_+$.

1. We determine γ by requiring the q-series expansion

$$w_0(x) = 1 + a_1 q + a_2 q^2 + \cdots$$

2. Substituting the inverse series $q = x + s_1 x + s_2 x^2 + \cdots$ of $1/x(t) = T_n(t) + c_n$ into the above q series of $w_0(x)$, we obtain

$$w_0(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \qquad (*)$$

Searching differential operators which annihilate the series $(*)$, we **find** 3rd order differential operators for all genus one groups $\Gamma_0(n)_+$.

$\rightarrow$ **K3 differential operators**

Proposition. (H.K.2023)
Assume the conjecture, then we have K3 differential operators of 3rd order for **all** genus zero groups of type $\Gamma_0(n)_+$.

An example of K3 differential operator: (for the case $\Gamma_0(36)_+$)

$$\begin{aligned}\mathcal{D}_{36A} =& \theta_x^3 - x(3\theta_x+1)\left(3\theta_x^2+2\theta_x+1\right) - 6x^2\theta_x\left(12\theta_x^2-3\theta_x-1\right)\\ &+2x^3\theta_x\left(284\theta_x^2+405\theta_x+199\right)+6x^4\theta_x\left(1156\theta_x^2+75\theta_x+89\right)\\ &-6x^5\theta_x\left(11927\theta_x^2+104401\theta_x+4939\right)\\ &+18x^6\left(8968\theta_x^3+11586\theta_x^2+5960\theta_x+2553\right)\\ &+18x^7\left(117888\theta_x^3+141184\theta_x^2-5086\theta_x-19947\right)\\ &-27x^8\left(301098\theta_x^3+446288\theta_x^2+70408\theta_x-6990\right)\\ &-27x^9\left(19871\theta_x^3+39147\theta_x^2+9715\theta_x+29949\right)\\ &+486x^{10}\left(2664\theta_x^3+4503\theta_x^2+2623\theta_x+561\right)\\ &+486x^{11}\left(2892\theta_x^3+6453\theta_x^2+5465\theta_x+1657\right)+360126x^{12}(\theta_x+1)^3.\end{aligned}$$

$$\left\{\begin{array}{cccccc} -1 & 0 & \frac{1}{3} & -1-\frac{2\sqrt{3}}{3} & -1+\frac{2\sqrt{3}}{3} & \infty \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right\} \quad \text{Riemann's } \mathcal{P} \text{ scheme}$$

the number of the cusps is 4, which coincides with the general formula.

§4. Clingher-Doran's family of K3 surfaces

- Clingher and Doran ('12) studied a special quartic $\{f=0\} \subset \mathbb{P}^3$ with

$$f = y^2zw - 4x^3z + 3\alpha\, xzw^2 + \beta\, zw^3 + \gamma\, xz^2w - \frac{1}{2}(\delta\, z^2w^2 + w^4).$$

- They found that

(1) When $\gamma \neq 0$, $\{f=0\}$ is a $\check{\mathbf{M}} = \mathbf{U} \oplus \mathbf{E_8}(-\mathbf{1}) \oplus \mathbf{E_7}(-\mathbf{1})$-polarized K3 surface.

(2) The parameter space

$$\mathcal{M}_{\mathrm{CD}} := \{[\alpha,\beta,\gamma,\delta] \in \mathbb{WP}^3(2,3,5,6) \mid \gamma \neq 0 \,\mathrm{or}\, \delta \neq 0\}$$

describes a **coase** moduli space of the $\check{M}$-polarized K3 surfaces.

Note. (i) $\Omega_{\check{M}} = \{[w] \in \mathbb{P}(\check{M}^{\perp} \otimes \mathbb{C}) \mid (w,w)=0, (w,\overline{w})>0\}^{+}$

$\simeq \mathbb{H}_2$ **the Siegel upper half space of genus two**

(ii) $O(\check{M}^{\perp})_{+}/\{\pm I_5\} \simeq Sp(4,\mathbb{Z})/\{\pm I_4\}$

(iii) $\mathcal{P} : \mathcal{M}_{CD} \to \mathbb{H}_2$ (period map)

cf. Weierstrass normal form: $y^2 = 4x^3 - g_2x - g_3$ with $[g_2, g_3] \in \mathbb{WP}^1(2,3)$

Theorem. (Clingher-Doran, '13)

$$\mathcal{P}^{-1}(\tau) = \left[\, \mathcal{E}_4(\tau),\, \mathcal{E}_6(\tau),\, 2^{12}3^5\chi_{10}(\tau),\, 2^{12}3^6\chi_{12}(\tau) \,\right]$$

where $\mathcal{E}_4$ and $\mathcal{E}_6$ are genus two Eisenstein series of weight four and six, and χ_{10} and χ_{12} are Igusa's cusp forms of weight ten and twelve, respectively.

Problem: Determine the BCOV cusp form in this case

To calculate the BCOV cusp forms, we need a family of K3 surfaces with a special boundary point (LCSL).

Results:

1. We can represent $\{f = 0\} \subset \mathbb{P}^3$ by $\{f_\Delta = 0\} \subset \mathbb{P}_\Delta$, Δ: reflexive polytope.
2. Using $\mathrm{Aut}(\mathbb{P}_\Delta) \supsetneqq (\mathbb{C}^*)^3$, we can transform $\{f_\Delta = 0\}$ to $\{F_\Delta = 0\}$ for which we find Picard-Fuchs equations and a LCSL.

–In fact, this is exactly in the frame work of the **extended GKZ system** introduced in HKTY ('93) and HLY ('95).

Proposition. (H.K.'23)

(1) The **conifold regularity** condition determines the parameters $r_k = -\dfrac{1}{2}$.

(2) There are **two** orbifold points A and B. Imposing the **orbifold regularity** for each, we obtain,

$$(\tau_{BCOV}(t))^{-1} = \begin{cases} \left(\chi_{10}(\tau)\right)^{\frac{1}{10}} & \text{for } A \\ \left(3\chi_{12}(\tau) + \chi_{10}(\tau)\mathcal{E}_4(\tau)^{\frac{1}{2}}\right)^{\frac{1}{12}} & \text{for } B \end{cases}.$$

Remark.

(i) When $\tau_{12} \to 0$ in $\tau = \left(\begin{smallmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{smallmatrix}\right)$,

$$\chi_{10}(\tau) \longrightarrow 0, \quad \chi_{12}(\tau) \longrightarrow \eta(\tau_{11})^{24}\eta(\tau_{22})^{24}$$

(ii)When $\tau_{12} \to 0$, the Picard lattice of $\check{M}$-polarized K3 surfaces extends to $U \oplus E_8(-1)^{\oplus 2}$, or the orthogonal latiice reduces

$$\check{M}^{\perp} = U^{\oplus 2} \oplus \langle -2 \rangle \ \longrightarrow \ U^{\oplus 2}$$

§5. Summary and some other aspects

Summary: BCOV formula of K3 surfaces $\to$ BCOV cusp forms

$$\tau_{BCOV} = \left(\frac{1}{w_0(x)}\right)^{r+1} \frac{\partial(x_1, \cdots, x_r)}{\partial(t_1, \cdots, t_r)} \prod_k dis_k^{r_k} \prod_i x_i^{-1+a_i}$$

1. (Vector-valued) quasi-automorphic forms follow from $\tau_{BCOV}(t)$:
   — for elliptic curves, we have $(\tau_{BCOV}(\tau))^{-1} = \eta(\tau)^2$ and

$$\boxed{\frac{\partial}{\partial\tau} \log(\tau_{BCOV}(\tau))^{-1} = \frac{1}{12} E_2(\tau)}$$

   — for K3 surfaces, we have the propagators

$$\boxed{S^a(t) = \sum_b K^{ab} \frac{\partial}{\partial t_b} \log(\tau_{BCOV}(t))^{-1}}$$

2. Conjectured relation to the Ray-Singer anlytic torsion.
3. $\tau_{BCOV}(t)$ for Calabi-Yau 3 folds and $\{(F_g(t), f_g(t))\}_{g\geq 2}$