Mirror Symmetry and Rigid Structures of Generalized K3 Surfaces

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Motivation from Physics

Aspinwall-Morrison identified the moduli space of non-linear σ-models on a K3 surface S with

$$\mathrm{Gr}_{2,2}^{po}(\mathbb{R}^{4,20}) = \mathrm{O}(4,20)/(\mathrm{SO}(2)\times\mathrm{SO}(2)\times\mathrm{O}(20))$$

(orthogonal pairs of positive oriented 2-planes in $\mathbb{R}^{4,20} \cong H^*(S,\mathbb{R})$)

Geometrically, to a K3 surface with a Kähler class ω, we associate two 2-planes

$$(\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma)\rangle_{\mathbb{R}}, \langle \mathrm{Re}(e^{i\omega}), \mathrm{Im}(e^{i\omega})\rangle_{\mathbb{R}}) \in \mathrm{Gr}_{2,2}^{po}(\mathbb{R}^{4,20}).$$

However, there are “non-geometric points”. (cf. Prof. Ooguri’s talk)

Mirror symmetry does not preserve the geometric points.

Motivation from Math

Two fields with similar interests:

- mirror symmetry
  duality between complex geometry and symplectic geometry
- generalized Calabi-Yau geometry
  unification of CY geometry and symplectic geometry

The relation has not been investigated.

Almost the only result in this direction was the study of generalized K3 surfaces (real 4-dim) by Huybrechts, showing the relationship with the moduli space of SCFT.

Based upon his work, mirror symmetry for K3 surface may be refined. A conventional formulation has some problems.

Generalized CY structures (4-dim)

M: C^∞-manifold underlying a K3 surface,
$A^{2*}_{\mathbb{C}}(M) = \oplus_{i=0}^{2} A^{2i}_{\mathbb{C}}(M)$ with Mukai pairing

$$\langle \varphi, \psi \rangle = \varphi_2 \wedge \psi_2 - \varphi_0 \wedge \psi_4 - \varphi_4 \wedge \psi_0 \in A^4_{\mathbb{C}}(M)$$

where φ_i denotes the degree i part of φ.

Definiton 2.1 (Hitchin)

A generalized CY structure on M is a closed form $\varphi \in A^{2*}_{\mathbb{C}}(M)$ such that

$$\langle \varphi, \varphi \rangle = 0, \quad \langle \varphi, \overline{\varphi} \rangle > 0$$

Generalized CY structures (4-dim)

- symplectic form ω, $\varphi = e^{i\omega}$.

$$\langle e^{i\omega}, e^{i\omega}\rangle = \langle 1 + i\omega - \frac{1}{2}\omega^2, 1 + i\omega - \frac{1}{2}\omega^2\rangle = 0,$$
$$\langle e^{i\omega}, e^{-i\omega}\rangle = 2\omega^2 > 0.$$

- hol 2-form w.r.t complex structure σ, $\varphi = \sigma$.

$$\langle \sigma, \sigma\rangle = 0,$$
$$\langle \sigma, \overline{\sigma}\rangle = \sigma \wedge \overline{\sigma} > 0.$$

B-field transform

A real closed 2-form $B \in A^2_{\mathbb{R}}(M)$ is called a B-field. The B-fields acts on $A^{2*}_{\mathbb{C}}(M)$ by the exterior product:

$$e^B\varphi = (1 + B + \frac{1}{2}B \wedge B) \wedge \varphi.$$

This action is orthogonal w.r.t. the Mukai pairing

$$\langle e^B\varphi, e^B\psi\rangle = \langle \varphi, \psi\rangle.$$

For a B-field B and a gCY structure φ, the B-field transform $e^B\varphi$ is also a gCY structure.

Classification of gCY structures

Theorem 2.2 (Hitchin)

Let φ be a gCY structure.

- (type A) $\varphi_0 \neq 0$: $\exists$ a symplectic form ω, a B-field B,

$$\varphi = e^B(\varphi_0 e^{i\omega}) = \varphi_0 e^{B+i\omega}$$

- (type B) $\varphi_0 = 0$: $\exists$ a hol 2-form σ (w.r.t. a complex str), a B-field B,

$$\varphi = e^B\sigma = \sigma + \sigma \wedge B \ (= \sigma + \sigma \wedge B^{0,2})$$

Definiton 2.3

gCY structures φ, φ' are isomorphic if $\exists$ an exact B-field B and $f \in \mathrm{Diff}_*(M)$ such that $\varphi = e^B f^*\varphi'$.

$$\mathrm{Diff}_*(M) = \mathrm{Ker}(\mathrm{Diff}(M) \to O(H^2(M,\mathbb{Z}))).$$

Unification of A- and B-structures

A fascinating aspect of gCY structures is the occurrence of the complex structure σ and symplectic structure $e^{i\omega}$ in the same moduli.

Example 2.4 (Hitchin)

For a hol 2-form σ, $\mathrm{Re}(\sigma)$ and $\mathrm{Im}(\sigma)$ are symplectic forms. A family of gCY structures of type A

$$\varphi_t = te^{\frac{1}{t}(\mathrm{Re}(\sigma)+i\mathrm{Im}(\sigma))} = t(1 + \frac{1}{t}\sigma + \frac{1}{2t^2}\sigma^2) = t + \sigma$$

converges, as $t \to 0$, to the gCY structure σ of type B.

The B-fields interpolate between gCY structures of type A and B.

Kähler structure

For a gCY structure φ, define a distribution P_φ of real 2-planes by

$$P_\varphi = \langle \mathrm{Re}(\varphi), \mathrm{Im}(\varphi) \rangle_{\mathbb{R}}.$$

gCY structures φ and φ' are called orthogonal if P_φ and $P_{\varphi'}$ are pointwise orthogonal. (stronger than $\langle \varphi, \varphi' \rangle = 0$.)

Definiton 2.5

A gCY structure φ is called Kähler if $\exists$ another gCY structure φ' orthogonal to φ.

A Kähler structure for $\varphi = \sigma$ is of the form $\varphi' = \varphi_0' e^{B+i\omega}$. The orthogonality reads

$$\sigma \wedge B = \sigma \wedge \omega = 0.$$

i.e. B is a closed real $(1,1)$-form and $\pm\omega$ is a Kähler form.

HyperKähler structure

Recall a Kähler form ω on a K3 surface is called hyperKähler if $\exists\, C \in \mathbb{R}$

$$2\omega^2 = C\sigma \wedge \overline{\sigma}.$$

Definiton 2.6

A gCY structure φ is hyperKähler if $\exists$ a Kähler structure φ' such that

$$\langle \varphi, \overline{\varphi} \rangle = \langle \varphi', \overline{\varphi'} \rangle.$$

- $\langle e^{i\omega}, e^{-i\omega} \rangle = 2\omega^2$, $\langle \sigma, \overline{\sigma} \rangle = \sigma \wedge \overline{\sigma}$.
- If φ' is a (hyper)Kähler structure for φ, then $e^B\varphi'$ is a (hyper)Kähler structure for $e^B\varphi$.
- A gCY structure is not always (hyper)Kähler.

Classification of hyperKähler structures

(Details are not important.)

- $\varphi = \sigma$: a hyperKähler structure is $\varphi' = \lambda e^{B+i\omega}$, where B is a closed $(1,1)$-form and $\pm\omega$ is a hyperKähler form such that

$$2|\lambda|^2\omega^2 = \sigma \wedge \overline{\sigma}.$$

- $\varphi = \lambda e^{i\omega}$: a hyperKähler structure is either
  - $\varphi' = \sigma$, where $\pm\omega$ is a hyperKähler form,
  - $\varphi' = \lambda' e^{B'+i\omega'}$ such that
    - $\omega \wedge \omega' = \omega \wedge B' = \omega' \wedge B = 0$, $B'^2 = \omega^2 + \omega'^2$,
    - $|\lambda|^2\omega^2 = |\lambda'|^2\omega'^2$.

Any hyperKähler structure is a B-field transform of one of the above cases. There are 3 cases:

(type A, type B), (type B, type A), (type A, type A)

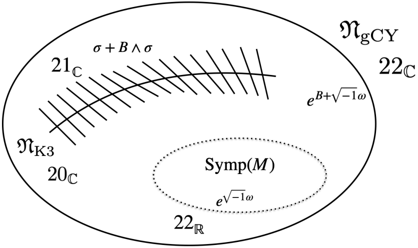
Period domains and period maps

$\mathfrak{N}_{\mathrm{gCY}} = \{\mathbb{C}\varphi\}/\cong$: moduli space of gCY structures of hyperKähler type

Theorem 2.7 (Huybrechts)

$$\begin{array}{ccc} \mathfrak{N}_{\mathrm{gCY}} & \xrightarrow[\mathbb{C}\varphi \to [\phi]]{\mathfrak{per}_{\mathrm{gCY}}} & \widetilde{\mathfrak{D}} = \{[\varphi] \in \mathbb{P}(H^*(M,\mathbb{C})) \mid \langle \varphi, \varphi \rangle = 0, \langle \varphi, \overline{\varphi} \rangle > 0\} \\ \cup & & \cup \\ \mathfrak{N}_{\mathrm{K3}} & \xrightarrow[\mathbb{C}\sigma \to [\sigma]]{\mathfrak{per}_{\mathrm{K3}}} & \mathfrak{D} = \{[\sigma] \in \mathbb{P}(H^2(M,\mathbb{C})) \mid \langle \sigma, \sigma \rangle = 0, \langle \sigma, \overline{\sigma} \rangle > 0\} \end{array}$$

$\mathfrak{per}_{\mathrm{gCY}}$: étale surjective



Generalized K3 surfaces

Definiton 2.8

A generalized K3 surface is a pair (φ, φ') of gCY structures such that φ is a hyperKähler structure for φ'.

- A K3 surface M_σ with a hyperKähler form ω is considered as a gK3 surface $(e^{i\omega}, \sigma)$.
- gK3 surfaces (φ, φ') and (ψ, ψ') are called isomorphic if $\exists$ $f \in \mathrm{Diff}_*(M)$ and exact $B \in A^2(M)$ such that

$$(\varphi, \varphi') = e^B f^*(\psi, \psi') = (e^B f^* \psi, e^B f^* \psi').$$

Isom classes are classified by cohomology classes.

gK3 surfaces and SCFT moduli space

Theorem 2.9 (Huybrechts)

$\mathfrak{M}_{\mathrm{HK}} = \left(\mathrm{Met}^{\mathrm{HK}}(M)/\mathrm{Diff}_*(M)\right) \times H^2(M,\mathbb{R})$:
moduli space of the B-field shifts of the hyperKähler metrics

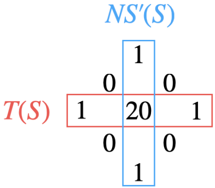
$$\begin{array}{ccccc}
\mathfrak{M}_{\mathrm{K3}} \times H^2(M,\mathbb{R}) & \overset{\iota}{\hookrightarrow} & \mathfrak{M}_{\mathrm{gK3}} & \xrightarrow[(\varphi,\varphi')\mapsto(P_{[\varphi]},P_{[\varphi']})]{\mathrm{per}_{\mathrm{gK3}}} & \mathrm{Gr}^{po}_{2,2}(H^*(M,\mathbb{R})) = \mathfrak{M}_{(2,2)} \\
& \searrow^{S^2} & \downarrow & & \downarrow{\scriptstyle S^2\times S^2} \\
& & \mathfrak{M}_{\mathrm{HK}} & \xrightarrow{\mathrm{per}_{\mathrm{HK}}} & \mathrm{Gr}^{po}_{4}(H^*(M,\mathbb{R})) = \mathfrak{M}_{(4,4)}
\end{array}$$

Mirror symmetry for K3 surfaces is an involution of the SCFT moduli space (Aspinwall-Morrison). Ready to discuss mirror symmetry.

K3 surfaces and lattices

Mirror symmetry for a (classical) K3 surface S is very subtle because the complex and Kähler structures are somewhat mixed in $H^2(S,\mathbb{C})$.

A conventional formulation of mirror symmetry is given in terms of sublattices of $H^*(S,\mathbb{Z}) \cong U^{\oplus 4} \oplus E_8^{\oplus 2}$.



alg 2-cycles trans 2-cycles

$$NS(S) \oplus T(S) \subset H^2(S,\mathbb{Z})$$

alg cycles

$$NS'(S) = H^0(S,\mathbb{Z}) \oplus NS(S) \oplus H^4(S,\mathbb{Z})$$
$$\cong NS(S) \oplus U$$

Mirror symmetry for K3 surfaces

Definiton 3.1 (Dolgachev)

Given $M \subset \Lambda_{K3} = U^{\oplus 3} \oplus E_8^{\oplus 2}$ of sign $(1, \mu)$, assume that $\exists\, N$ such that

$$M^{\perp} = N \oplus U.$$

Then the family $\mathcal{S}$ of M-pol K3 surfaces and the family $\mathcal{S}^{\vee}$ of N-pol K3 surfaces are mirror symmetric.

For generic M-pol K3 surface S and N-pol K3 surface $S^{\vee}$,

$$NS'(S) \cong M \oplus U \cong T(S^{\vee}), \quad T(S) \cong N \oplus U \cong NS'(S^{\vee}),$$

duality of algebraic and transcendental cycles, Yukawa couplings.

Mirror symmetry for quartic surfaces

$M = \langle 4 \rangle$ and $N = \langle -4 \rangle \oplus U \oplus E_8^{\oplus 2}$, $\langle k \rangle = (\mathbb{Z}v, v^2 = k)$

- M-pol K3 surfaces = quartic surfaces $S_4 \subset \mathbb{CP}^3$.
- N-pol K3 surfaces = minimal resolution of

$$\left\{ x_1^4 + x_2^4 + x_3^4 + x_4^4 + \mu x_1 x_2 x_3 x_4 = 0 \right\} / G,$$

$$G = \{\mathrm{diag}[\alpha_1, \alpha_2, \alpha_3, \alpha_4] \mid \alpha_i^4 = \textstyle\prod_{j=1}^4 \alpha_j = 1\} \cong (\mathbb{Z}/4\mathbb{Z})^{\oplus 2}.$$

Mirror symmetry for K3 surfaces

The conventional formulation have several problems:

1. $NS'(S)$ and $T(S)$ are not symmetric.
2. The assumption $M^{\perp} = N \oplus U$ does not hold in general:
    - singular K3 surface, where $T(S)$ is of sign $(2, 0)$.

	singular K3 surface	??
Kähler	20-dim	0-dim
complex	0-dim	20-dim

    - $M^{\perp} = N \oplus U(k)$

The problems are caused by $H^0(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}) \cong U$.

Algebraic and transcendental lattices of gK3 surface

We define sublattices of $H^*(M,\mathbb{Z})$ reflecting a gCY structure.

Definiton 3.2

The algebraic and transcendental lattices of a gK3 surface $X = (\varphi, \varphi')$ are defined respectively by

$$\widetilde{NS}(X) = \{\delta \in H^*(M,\mathbb{Z}) \mid \langle \delta, [\varphi'] \rangle = 0\},$$

$$\widetilde{T}(X) = \{\delta \in H^*(M,\mathbb{Z}) \mid \langle \delta, [\varphi] \rangle = 0\}.$$

- $\widetilde{NS}(X)$ and $\widetilde{T}(X)$ are defined on an equal footing.

$$2 \leq \mathrm{rank}(\widetilde{NS}(X)),\ \mathrm{rank}(\widetilde{T}(X)) \leq 22.$$

- $\widetilde{NS}(X) \cap \widetilde{T}(X)$ may be non-trivial.
- In general, pt and $[M]$ are no longer "algebraic".

Complex and Kähler rigidity

Definiton 4.1

A gK3 surface $X = (\varphi, \varphi')$ is called

- complex rigid if φ' is of type B and $\mathrm{rank}(\widetilde{NS}(X)) = 22$.
- Kähler rigid if φ is of type A and $\mathrm{rank}(\widetilde{T}(X)) = 22$.

Theorem 4.2

A complex rigid gK3 surface is of the form $e^{B'}(\lambda e^{B+i\omega}, \sigma)$:

- M_σ: singular K3 surface
- $B \in H^{1,1}(M_\sigma, \mathbb{R})$,
- $B' \in H^2(M, \mathbb{Q})$,
- $\pm\omega$ is a Kähler form w.r.t. σ.

Example of Kähler rigidity

S : K3 surface, $NS(S) = \mathbb{Z}H, H^2 = 2n > 0$.

$$v_1 = (1, 0, -n), \quad v_2 = (0, H, 0) \in NS'(S)$$

Then

$$\begin{aligned} e^{iH} &= (1, iH, -n) \\ &= v_1 + iv_2 \in (\mathbb{Z}v_1 + \mathbb{Z}v_2)_{\mathbb{C}} \subsetneq NS'(S)_{\mathbb{C}}. \end{aligned}$$

$$\begin{aligned} e^{i\epsilon H} &= (1, i\epsilon H, -\epsilon^2 n) \\ &= (1, 0, -\epsilon^2 n) + i\epsilon(0, H, 0) \\ &= (1, 0, 0) - \epsilon^2(0, 0, n) + i\epsilon(0, H, 0) \in NS'(S)_{\mathbb{C}} \end{aligned}$$

Cannot continuously deform e^{iH} in such a way that $\operatorname{rank}(\widetilde{T}(e^{i\epsilon H}, \sigma)) = 22$.
Lesson: consider the integral structure of $e^{i\omega}$, not ω itself.

Mukai lattice polarization

Definiton 4.3 (Mukai lattice polarization)

For $\kappa, \lambda \geq 2$ such that $\kappa + \lambda = 24$, and even lattices K and L of signature $(2, \kappa - 2)$ and $(2, \lambda - 2)$, a pair (X, j) of

- a gK3 surface $X = (\varphi, \varphi')$,
- a primitive embedding $j : K \oplus L \hookrightarrow H^*(M, \mathbb{Z})$ such that $K \subset \widetilde{NS}(X)$ and $L \subset \widetilde{T}(X)$

is called a (K, L)-polarized gK3 surface.

"polarization $\subset$ lattice polarization $\subset$ Mukai lattice polarization"

Mirror symmetry for gK3 surfaces

Definiton 4.4

The family $\mathcal{X}$ of (K, L)-pol gK3 surfaces and the family $\mathcal{Y}$ of (L, K)-pol gK3 surfaces are mirror symmetric.

For generic (K, L)-pol gK3 surface X and (L, K)-pol gK3 surface Y,

$$\widetilde{NS}(X) \cong K \cong \widetilde{T}(Y), \quad \widetilde{T}(X) \cong L \cong \widetilde{NS}(Y),$$

duality between algebraic and transcendental cycles w.r.t. gCY structures.

MS for complex and Kähler rigid gK3 surfaces

For $n > 0$, consider $K = \langle -2n \rangle^{\oplus 2} \oplus U \oplus E_8^{\oplus 2}$, $\quad L = \langle 2n \rangle^{\oplus 2}$.

- The family $\mathcal{X}$ of (K, L)-pol gK3 surfaces is given by

$$\mathcal{X} = \{X = (e^{B+i\omega}, \sigma)\}$$

  where $T(M_\sigma) = L$, and $B, \omega \in NS(M_\sigma)_{\mathbb{R}}$. They are singular K3 surfaces with complexified Kähler parameters $B + i\omega \in NS(M_\sigma)_{\mathbb{C}}$.

- The family $\mathcal{Y}$ of (L, K)-pol gK3 surfaces has a 19-dim subfamily of K3 surfaces of the form

$$\{Y = (e^{iH}, \sigma^\vee)\}$$

  where $NS(M_{\sigma^\vee}) = \mathbb{Z}H$ such that $H^2 = 2n$.

MS for complex and Kähler rgid gK3 surfaces

In summary, for $K = \langle -2n \rangle^{\oplus 2} \oplus U \oplus E_8^{\oplus 2}$, $L = \langle 2n \rangle^{\oplus 2}$,

- (K, L)-pol gK3 surfaces = singular K3 surfaces
- (L, K)-pol gK3 surfaces ⊃ pol K3 surfaces (S, H) with $H^2 = 2n$

	(K, L)-pol gK3	(L, K)-pol gK3
A-deform	20-dim	0-dim
B-deform	0-dim	20-dim

The new formulation is compatible with Aspinwall-Morrison's description of the moduli space $\mathfrak{M}_{(2,2)} = \mathrm{Gr}_{2,2}^{po}(H^*(M, \mathbb{R}))$ and mirror symmetry.

謝謝! Thank you!

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