Manifolds with exceptional holonomy and mirrors of their submanifolds

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Holonomy group

The holonomy group $\mathrm{Hol}(g)$ is one of tools to study the structure of a Riemannian manifold (X, g).

Theorem (Berger, 1955)

Let (X, g) be a simply connected Riemannian manifold and it is

- *irreducible (i.e., (X, g) does not locally decompose into the product of Riemannian manifolds),*
- *and not locally symmetric (i.e., $\nabla R \neq 0$).*

Then the holonomy group $\mathrm{Hol}(g)$ is one of the following.

$$\mathrm{SO}(n), \ \mathrm{U}(n), \ \mathrm{SU}(n), \ \mathrm{Sp}(n)\mathrm{Sp}(1), \ \mathrm{Sp}(n), \ G_2, \ \mathrm{Spin}(7).$$

G_2 geometry

$$\begin{aligned} G_2 :=& \mathrm{Aut}(\mathbb{O}) \\ =& \{T \in \mathrm{GL}(\mathbb{O}) \mid T \text{ preserves the multiplication of } \mathbb{O}\}. \end{aligned}$$

Identify $\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}\mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7$.

Describe the multiplication of $\mathbb{O}$ by $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$:

$$\mathbb{R}^7 \times \mathbb{R}^7 \ni (u, v) \mapsto \varphi_0(u, v, \cdot)^\sharp \in \mathbb{R}^7 \qquad (u \perp v).$$

Then

$$G_2 = \{g \in GL(7, \mathbb{R}) \mid g^* \varphi_0 = \varphi_0\} \subset \mathrm{SO}(7).$$

- (X^7, g) : G_2-manifold $\overset{def}{\iff} \mathrm{Hol}(g) \subset G_2$ $(\Longrightarrow \mathrm{Ric}(g) = 0)$.
- $\mathrm{Hol}(g) \subset G_2 \Longrightarrow \exists \varphi \in \Omega^3(X^7)$ s.t. $\nabla \varphi = 0$.
- Fixing such a $\varphi \in \Omega^3(X^7)$, we call (X^7, φ, g) a G_2-manifold.
- G_2 geometry is characterized by a 3-form φ.

How to understand G_2 geometry?

$\{G_2\text{-manifolds}\}$

$\{\text{CY 3-manifolds}\} \longleftrightarrow \{G_2\text{-manifolds}\} \longleftrightarrow \{\text{3,4-dim. manifolds}\}$

- The analogy of Calabi-Yau 3-manifolds $(\mathrm{SU}(3) \subset G_2)$
  Y^6: a Calabi-Yau 3-mfd $\Longrightarrow S^1 \times Y^6$: a G_2-manifold
- We might consider higher dimensional analogues of the theory for 3,4-dim. manifolds.
  - Flat connections on 3-mfds ($\Longrightarrow$ Chern-Simons theory)
  - ASD connections on 4-mfds ($\Longrightarrow$ Donaldson theory)

  $\rightsquigarrow$ G_2-instanton

Calibrated geometry [Harvey-Lawson, 1982]

- calibration : a closed differential form $\varphi \in \Omega^k(X^n)$ on a Riemannian manifold (X^n, g) satisfying a certain condition.
- calibration $\Rightarrow$ calibrated submanifolds
  - Every compact calibrated submanifold is homologically volume minimizing, and the volume is topological.

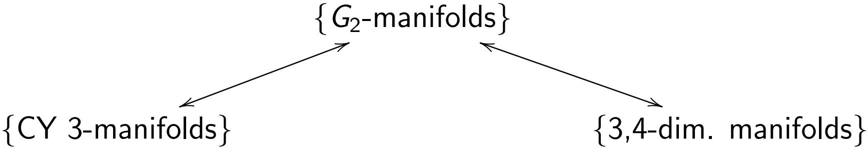
$\mathrm{Hol}(g)$ $(\subset)$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	G_2
(X, g)	X^{2n}:Kähler	X^{2n}:Calabi-Yau	X^7 : G_2-manifold
calibrated submfds	N^{2k}:complex submfds		A^3:associative submfds
		L^n:special Lag. submfds	C^4:coassociative submfds

- Red objects have obstructed deformations.
- Blue objects have unobstructed deformations.

Calibrated submanifolds might be useful to understand a manifold.

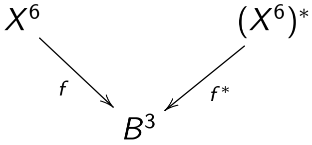
- Gromov-Witten invariant "counts" pseudoholomorphic curves.
  $\rightsquigarrow$ Can we "count" associative submanifolds?
- - Casson invariant "counts" flat connections.
  - Donaldson invariant "counts" ASD connections.

  $\rightsquigarrow$ Can we "count" G_2-instantons?
- Mirror symmetry for Calabi-Yau 3-manifolds
  $\rightsquigarrow$ Mirror symmetry for G_2-manifolds?



Mirror symmetry

- Strominger–Yau–Zaslow (SYZ conjecture):
  mirror symmetry of Calabi-Yau 3-folds would be explained in terms of special Lagrangian (SL) dual T^3-fibrations (including singular fibers).



  Smooth fibers $f^{-1}(b), (f^*)^{-1}(b)$ are "dual" SL T^3.

- Leung–Yau–Zaslow:
  Given a SL dual T^3-fibration, SL submanifolds correspond to deformed Hermitian Yang–Mills (dHYM) connections (or LYZ connections?) via the real Fourier–Mukai transform.

A similar argument works for G_2-manifolds.

- Lee–Leung:
  Given a coassociative dual T^4-fibration, (co)associative submanifolds correspond to deformed Donaldson–Thomas (dDT) connections (or LL connections?) via the real Fourier–Mukai transform.

calibrated submanifold	"mirror"
special Lagrangian	dHYM connection
(co)associative	dDT connection

Definition

- (X^7, φ, g): a G_2-manifold,
- $(L, h) \to X$: a smooth complex Hermitian line bundle.

A Hermitian connection ∇ of (L, h) is called a deformed Donaldson–Thomas (dDT) connection (deformed G_2-instanton) if

$$\frac{1}{6}F_\nabla^3 + F_\nabla \wedge *\varphi = 0,$$

where $F_\nabla = d_\nabla \circ d_\nabla \in \sqrt{-1}\Omega^2$ is a curvature of ∇.

- dDT connection can also be considered as an analogue of the G_2-instanton (DT connection): $F_\nabla \wedge *\varphi = 0$.
- We expect that dDT connections will have similar properties to associative submanifolds and G_2-instantons.
- Examples of dDT connections:
  - flat connections ($F_\nabla = 0$)
  - dHYM conn. (mirror of "SL $\Longrightarrow$ associative")

$\{$dHYM conn.$\}$ $\subset$ $\{$dDT conn.$\}$

mirror $\updownarrow$ mirror $\updownarrow$ analogue

$\{$SL submfds$\}$ $\subset$ $\{$associative submfds$\}$ $\{G_2$-instantons$\}$

analogue of GW inv.? higher dimensional gauge theory?

- Can we "count" dDT connections to define invariants?
  - How about the moduli (deformation) theory?
  - Do dDT connections behave similarly to associative submfds and G_2-instantons?
    $\Longrightarrow$ Check if they have similar properties.

Properties of associative submanifolds

(1) The moduli space is 0-dimensional and canonically orientable if we perturb the G_2-structure.
(2) associator equality $\Rightarrow$ a G_2-structure is a calibration, and we can characterize associative submanifolds by the vanishing of a tensor, which is useful in deformation theory.
(3) Homologically volume minimizing. The volume is topological.
(4) ciritical points of the Chern-Simons type functional.

(G_2-instantons have similar properties to the above.)

Theorem (K.-Yamamoto)

Similar properties to the above also hold for dDT connections. (The analogy for (4) is proved by Karigiannis-Leung.)

We will state below that the "mirrors" of (2) and (3) hold true.

"Volume" for connections

- (X^n, g): a compact connected oriented Riemannian manifold,
- $(L, h) \to X$: a smooth complex Hermitian line bundle,
- $\mathcal{A}_0 = \{$Hermitian connections of $(L, h)\}$.

Define the "volume functional" $V : \mathcal{A}_0 \to \mathbb{R}$ by

$$V(\nabla) := \int_X v(\nabla) \mathrm{vol}_g,$$

$$v(\nabla) := \sqrt{\det \left(\mathrm{id}_{TX} + (-\sqrt{-1} F_\nabla)^\sharp \right)}$$

$$= \sqrt{1 + |F_\nabla|^2 + \left| \frac{F_\nabla^2}{2!} \right|^2 + \left| \frac{F_\nabla^3}{3!} \right|^2 + \cdots}$$

- V corresponds to the (standard) volume functional for submanifolds via the real FM.
- V is called the Dirac-Born-Infeld (DBI) action in physics.

Theorem ("Mirror" of associator equality, K.-Yamamoto)

Let (X^7, φ, g) be a G_2-manifold. For any $\nabla \in \mathcal{A}_0$, we have

$$\left(1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right)^2 + \left|*\varphi \wedge F_\nabla + \frac{1}{6}F_\nabla^3\right|^2 + \frac{1}{4}|\varphi \wedge *(F_\nabla)^2|^2 = v(\nabla)^2,$$

In particular,

$$\left|1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right| \leq v(\nabla)$$

for any $\nabla \in \mathcal{A}_0$. The equality holds if and only if ∇ is dDT.

- For any dDT connection ∇, ∇ is a global minimizer of V and $V(\nabla)$ is topological.
  $\left(\int_X \left(1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right) \mathrm{vol}_g = \mathrm{Vol}(X) + (-2\pi^2 c_1(L)^2 \cup [\varphi]) \cdot [X]\right)$.
- This is the "mirror" of the fact that every compact associative (calibrated) submanifold is homologically volume minimizing, and the volume is topological.

Corollary

Suppose that L is a flat line bundle. Then, any dDT connection is a flat connection. In particular, the moduli space of dDT connections is $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$.

Let ∇_0 be a flat connection and ∇ be any dDT connection. Then,

$$\int_X \sqrt{1+|F_\nabla|^2+\left|\frac{F_\nabla^2}{2!}\right|^2+\left|\frac{F_\nabla^3}{3!}\right|^2}\mathrm{vol}_g = V(\nabla) = V(\nabla_0) = \int_X \mathrm{vol}_g,$$

which implies that $F_\nabla = 0$.

Minimal connections

- (X^n, g): a (not necessarily compact) connected oriented Riemannian manifold,
- $(L, h) \to X$: a smooth complex Hermitian line bundle,
- $\mathcal{A}_0 := \{\text{Hermitian connections of } (L, h)\}$.

Since $v(\nabla) \geq 1$ for $\forall \nabla \in \mathcal{A}_0$, we define the normalized volume functional $V^0 : \mathcal{A}_0 \to [0, \infty]$ by

$$V^0(\nabla) = \int_X (v(\nabla) - 1)\mathrm{vol}_g.$$

We see that

$$V^0(\nabla) = 0 \qquad \Longleftrightarrow \qquad F_\nabla = 0.$$

Definition

Critical points of V^0 (or V) are called minimal connections.

- Every dDT conn is a global minimizer of $V \implies$ minimal.

Proposition

$$\nabla \in \mathcal{A}_0 \textit{ is minimal} \Longleftrightarrow \delta_\nabla F_\nabla = 0 \Longrightarrow \underbrace{(d\delta_\nabla + \delta_\nabla d)}_{=:\Delta_\nabla} F_\nabla = 0.$$

- $\delta_\nabla : \Omega^k \to \Omega^{k-1}$: ∇-dependent differential operator like the codifferential d^*.
- This is a similar characterization to Yang–Mills connections.

- To "count" dDT connections, we need to consider the compactification of the moduli space.
- To do this, we should know how bubbles occur. For Yang–Mills connections, it is known from
  (1) Price's monotonicity formula,
  (2) ε-regularity theorem of Uhlenbeck-Nakajima.

  Can we show these analogies for minimal connections?

  $\Longrightarrow$ (1) is (probably) OK.

Theorem (K., Monotonicity formula)

- (X^n, g): *an oriented Riemannian manifold, with* $\dim X = n = 2m+1$ *and* $\mathrm{Ric}(g) \geq 0$. *Fix* $p \in X$.
- $(L, h) \to X$: *a smooth complex Hermitian line bundle.*

Then $\exists a = a(n, p, g) \geq 0, 0 < \exists r'_p < \mathrm{inj}_g(p)$, $\exists$ *a function* $\Theta : [0, \infty) \to \mathbb{R}$ *s.t. for any minimal connection* ∇

$$(0, r'_p] \to \mathbb{R}, \qquad \rho \mapsto \frac{e^{a\rho^2}}{\rho} \int_{B_\rho(p)} (v(\nabla) - 1)\mathrm{vol}_g + 2a\Theta(\rho)$$

is non-decreasing.

(Outline of the proof)

- We first show the "integration by parts formula" for min. conn ∇.

$$\int_X (\Delta_\nabla f_1) \cdot f_2 \cdot v(\nabla)\mathrm{vol}_g = \int_X f_1 \cdot (\Delta_\nabla f_2) \cdot v(\nabla)\mathrm{vol}_g,$$

where $f_1, f_2 \in \Omega^0$, one of which is compactly supported.

- Set $f_1 = 1$, $f_2 =$ "cut off function" and compute $\Delta_\nabla f_2$.
- After some calculations, we see that the monotonically is obtained if the following is satisfied:

(1) $0 < \exists r'_p < \mathrm{inj}_g(p)$, $\forall \tau \in [0, r'_p]$,

$$n \int_{B_\tau(p)} \mathrm{vol}_g \geq \tau \frac{\partial}{\partial \tau} \int_{B_\tau(p)} \mathrm{vol}_g, \qquad \omega_n \tau^n \geq \int_{B_\tau(p)} \mathrm{vol}_g.$$

(2) $(\mathrm{tr}\, G_\nabla^{-1} - 1)v(\nabla) - n + 1 \geq 0.$

(1) is satisfied if $\mathrm{Ric}(g) \geq 0$ (relative volume comparison theorem).
(2) is an algebraic condition. It is satisfied if $\dim X = n = 2m + 1$.

Corollary

Let $(L, h) \longrightarrow \mathbb{R}^{2m+1}$ *be a (necessarily trivial) smooth complex Hermitian line bundle over* $(\mathbb{R}^{2m+1}, g_0)$*, where* g_0 *is the standard flat metric.*
If ∇ *is minimal with* $V^0(\nabla) < \infty$*, then* ∇ *is flat. (i.e.* $F_\nabla = 0$*.)*

(proof) We can take $a = 0$ and $r'_p = \infty$ for $(\mathbb{R}^{2m+1}, g_0)$.
If $F_\nabla \neq 0$, $\exists p \in \mathbb{R}^{2m+1}, \exists R_0 > 0$ s.t.

$$\frac{1}{R_0} \int_{B_{R_0}(p)} (v(\nabla) - 1)\mathrm{vol}_{g_0} > 0.$$

By monotonicity formula, for $\forall R \geq R_0$,

$$0 < \frac{1}{R_0} \int_{B_{R_0}(p)} (v(\nabla) - 1)\mathrm{vol}_g \leq \frac{1}{R} \int_{B_R(p)} (v(\nabla) - 1)\mathrm{vol}_g \to 0 \ (R \to \infty),$$

which is a contradiction.

Future work

- Roughly, we could show

$$\frac{e^{a\rho^2}}{\rho^\kappa} \int_{B_\rho(p)} (v(\nabla) - 1)\mathrm{vol}_g$$

is non-decreasing for $\kappa = 1$.

  - I am not sure $\kappa = 1$ the best for the monotonicity. That is, we might be able to prove the monotonicity for $\kappa > 1$.
  - In fact, for dDT connections on a G_2-manifold, (recall that G_2-manifolds are 7-dim.) we can take $\kappa = 13/7 > 1$.
  - For Yang–Mills connections, there is an analogous monotonicity formula. In that case, κ is taken to be "scaling invariant" (in a certain sense). There are no such a property for our case.
- Can we show "ε-regularity theorem" to study "blowup set"?
- Can we construct nontrivial examples of minimal/dDT connections?

I would like to thank the organizers and everyone involved for holding such a great conference.

Thank you so much!