Topological Modular Forms and heterotic string theory

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Based on
[TY'21] arXiv: 2108.13542
[TY'23] arXiv: 2305.0619
[JF-Y, in preparation]

Goal of this talk : Explain our works on

Segal-Stolz-Teichner proposal. ('04, '11)

... a deep "conjecture" connecting

Homotopy theory (Math) and Physics.

My works, conceptually :

(A) Use Proposal to translate

Phys. problems → Math problems

and solve it mathematically.

(B) Use Proposal to get

Math ideas from physics.

(C) Try to attack Proposal.

Plan (Phys ⟷ Math)

§0. Introduction to Segal-Stolz-Teichner program

(A) §1. Vanishing of heterotic anomaly

[TY'21] & Topological Modular Forms (TMF)

(A) §2. Secondary anomaly

(B)

(C) [TY'23] & Anderson self-duality of TMF

(A) §3. Proof of 576-periodicity in SQFTs.

(B) WITHOUT using TMF.

[JF-Y, in preparation]

§0. Introduction to Segal-Stolz-Teichner program

Segal-Stolz-Teichner proposal ('04, '11)

(1) The "Space"

$$\{\text{2-dim, } \mathcal{N}=(0,1) \text{ SUSY unitary QFTs}\}$$

forms a $\overbrace{\text{spectrum}}^{(E_\infty,\ \Omega\text{-}}$ "SQFT".

↑ (in homotopy theory)
"refinement" of topological spaces

(2) We have

$$\text{SQFT} \overset{\text{of } E_\infty\text{-ring spectra}}{\simeq} \text{TMF}.$$

↑
Topological Modular Forms.

Why is SST-proposal interesting / important?

- Mathematically,

  TMF itself has been deep & important.
  ('90s ~ now)

  "topological ver." of Modular Forms.

  mixture of homotopy theory & algebraic geometry...

- Physically,

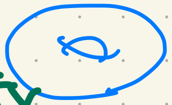
  SST-proposal has a lot of implications for SQFTs.

  such as torsion phenomena, duality, periodicity...
  (the topic of this talk.)

$$\underline{\text{Segal - Stolz - Teichner}} \ : \ \{2d\ \mathcal{N}=(0,1)\ \text{SQFT}\} \simeq \text{TMF}$$

a "topological refinement"

$$\begin{tikzcd}
\left\{ 2d\ \boxed{\mathcal{N}=(0,1)}\ \text{SQFT} \atop \text{central charge } c \right\} \arrow[r] & MF[\Delta^{-1}]_{-c} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\mathcal{T} \arrow[r, mapsto] & Z_{\mathcal{T}}(T^2, RR)
\end{tikzcd}$$



$\mathcal{N}=(0,1) \rightsquigarrow$ holomorphicity

$MF[\Delta^{-1}]_k$: weakly $\boxed{\text{holom}}$. $\mathbb{Z}$- modular forms of weight k

$\cap$

$\mathbb{Z}((q))$

$$\theta : \underset{\substack{\circlearrowleft \\ SL_2(\mathbb{Z})}}{\mathbb{H}} \to \mathbb{C} \ \text{holom}, \ \theta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \theta(\tau)$$

$$\theta(\tau) = \sum_n \underbrace{a_n}_{\in \mathbb{Z}} q^n \qquad q = e^{2\pi i \tau}$$

$$MF[\Delta^{-1}]_* = \mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}] / (c_4^3 - c_6^2 - 1728\Delta)$$

Math <u>TMF</u> is a spectrum which is a "topological" version of $MF[\Delta^{-1}]$ …

- Defined as a global section of an E_∞-sheaf on $M_{ell}/\mathbb{Z}$
- $\pi_* TMF \xrightarrow{\exists} (MF[\Delta^{-1}])_{*/2}$. $= \mathbb{Z}[C_4, C_6, \Delta, \Delta^{-1}] / (C_4^2 - C_6^2 - 1728\Delta)$

  inducing $\pi_* TMF \otimes \mathbb{Q} \cong MF[\Delta^{-1}]_{*/2} \otimes \mathbb{Q}$.

  but $\exists$ nontrivial cokernels.

  <u>e.g.</u>
$$\begin{tikzcd}
\pi_{\pm 24} TMF \arrow[r] & MF[\Delta^{-1}]_{\pm 12} \\
\nexists \arrow[r, dashed, mapsto] & \Delta , \Delta^{-1} \arrow[u, phantom, sloped, "\in"] \\
\exists \text{ lift} \arrow[r, mapsto] & 24\Delta , 24\Delta^{-1}
\end{tikzcd}$$

- $\exists$ many 2, 3 - power torsions in $\pi_* TMF$
- 576 - periodic $\pi_* TMF \cong \pi_{*+576} TMF$

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$						$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$			
				$\mathbb{Z}_8$			$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$					$\mathbb{Z}_2$	$\mathbb{Z}_2$
d	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$				$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$			
		$\mathbb{Z}_2$			$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$			
d	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$		$\mathbb{Z}_{(2)}^2$				$\mathbb{Z}_{(2)}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$		$\mathbb{Z}_{(2)}^2$			
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$				$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$			$\mathbb{Z}_2$	$\mathbb{Z}_2$	
d	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$		$\mathbb{Z}_{(2)}^2$				$\mathbb{Z}_{(2)}^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$		$\mathbb{Z}_{(2)}^3$			
	$\mathbb{Z}_{(2)}$		$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$			$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_4$			
d	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
$\pi_d(\mathrm{tmf})^3_{(2)}$	$\mathbb{Z}_{(2)}^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$		$\mathbb{Z}_{(2)}^3$				$\mathbb{Z}_{(2)}^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$		$\mathbb{Z}_{(2)}^3$			
		$\mathbb{Z}_2^2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$		$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$			$\mathbb{Z}_2$				
d	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$		$\mathbb{Z}_{(2)}^4$				$\mathbb{Z}_{(2)}^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$		$\mathbb{Z}_{(2)}^4$			
	$\mathbb{Z}_2$					$\mathbb{Z}_2$					$\mathbb{Z}_2$					
d	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$		$\mathbb{Z}_{(2)}^5$				$\mathbb{Z}_{(2)}^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$		$\mathbb{Z}_{(2)}^5$			
	$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$		$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2^2$					$\mathbb{Z}_4$	$\mathbb{Z}_2$
d	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$		$\mathbb{Z}_{(2)}^5$				$\mathbb{Z}_{(2)}^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$		$\mathbb{Z}_{(2)}^5$			
		$\mathbb{Z}_2$			$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$		$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
d	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^6$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$		$\mathbb{Z}_{(2)}^6$				$\mathbb{Z}_{(2)}^6$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$		$\mathbb{Z}_{(2)}^6$			
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$				$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$				$\mathbb{Z}_2$	
d	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^6$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$		$\mathbb{Z}_{(2)}^6$				$\mathbb{Z}_{(2)}^7$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^7$		$\mathbb{Z}_{(2)}^7$			
	$\mathbb{Z}_{(2)}$			$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$			$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$			
d	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^7$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^7$		$\mathbb{Z}_{(2)}^7$				$\mathbb{Z}_{(2)}^7$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2^7$		$\mathbb{Z}_{(2)}^7$			
		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$				$\mathbb{Z}_{(2)}$							
d	176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}^8$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8$		$\mathbb{Z}_{(2)}^8$				$\mathbb{Z}_{(2)}^8$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8$		$\mathbb{Z}_{(2)}^8$			

← Easy

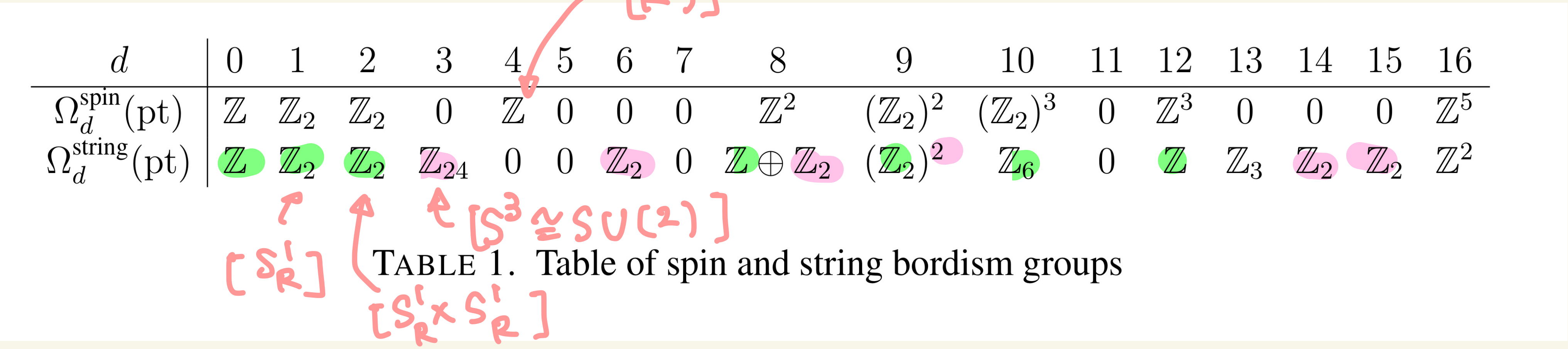
← Hard!

TABLE 2. Table of $\pi_d(\mathrm{tmf})_{(2)}$. For each d it is a direct sum of the entries on the first row and the second row. The second row is periodic with period 192.

$(\mathrm{TMF} = \mathrm{tmf}[\Delta^{-24}])$

Witten genus

- we have $\mathrm{MString} \xrightarrow{\mathrm{Wit}} \mathrm{TMF}$:



d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\Omega_d^{\mathrm{spin}}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	0	$\mathbb{Z}^3$	0	0	0	$\mathbb{Z}^5$
$\Omega_d^{\mathrm{string}}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_6$	0	$\mathbb{Z}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$

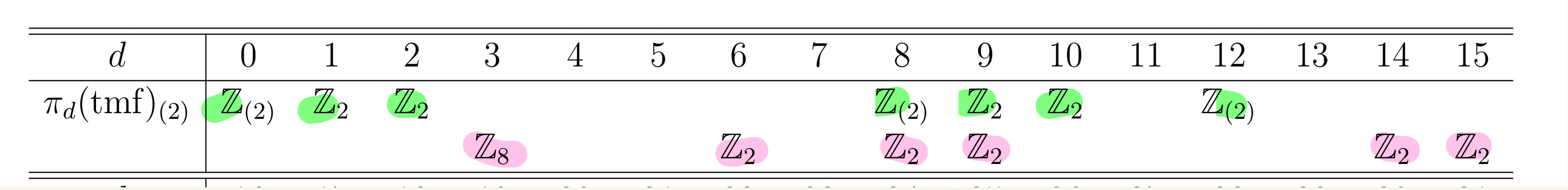
[K3]

$[S^1_R]$

$[S^1_R \times S^1_R]$

$[S^3 \cong SU(2)]$

TABLE 1. Table of spin and string bordism groups



d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\pi_d(\mathrm{tmf})_{(2)}$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$						$\mathbb{Z}_{(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_{(2)}$			
				$\mathbb{Z}_8$			$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$					$\mathbb{Z}_2$	$\mathbb{Z}_2$

Physically corresponds to

$$\{\text{String manifolds}\} \xrightarrow{\substack{\text{SUSY} \\ \sigma\text{-model}}} \left\{\begin{matrix} 2d\ \mathcal{N}=(0,1) \\ \text{SQFTs} \end{matrix}\right\}$$

Part I. Vanishing of heterotic anomaly and Topological Modular Forms [TY '21]

Main Result of [TY '21] (: Physical statement.)

> Heterotic String theories are anomaly-free.

Strategy We use proposals

Physics ⟷ Math (Homotopy theory)

- Segal-Stolz-Teichner
- Freed-Hopkins ('16)

to translate Phys. Q to Math. Q.
and solve it mathematically.

We use the following diagram :

worldsheet $\qquad\qquad$ spacetime

$c = \frac{d+22}{2}$ $\qquad\qquad$ d-dim

Phys

$$\begin{tikzcd}
\left\{\begin{array}{c} \text{2d SCFTs} \\ \mathcal{N}=(0,1) \end{array}\right\} \arrow[r, "\text{heterotic}", "\text{compactification}"'] \arrow[d, "\begin{array}{c}\text{Segal-Stolz-Teichner}\\ \text{Proposal}\\ \text{['04, '11]}\end{array}"] & \left\{\begin{array}{c} \text{QFTs on} \\ \text{String manifolds} \end{array}\right\} \arrow[d, "\begin{array}{c}\text{extract}\\ \text{anomalies}\end{array}", "\begin{array}{c}\text{Freed-Hopkins}\\ \text{proposal}\\ \text{['16]}\end{array}"'] \\
\pi_{-(d+22)}\mathrm{TMF} \arrow[r, "\alpha"] & \pi_{-(d+2)} I_{\mathbb{Z}}\mathrm{MString}
\end{tikzcd}$$

Math

Topological Modular Forms $\qquad\qquad$ Anderson dual to String bordism

⇝ **Phys. Problem** Show that heterotic anomaly vanishes.

⟺ **Math Problem** (roughly): Show $\alpha = 0$.
under some conditions on α. implied by Physics.

Anderson duals.

Freed-Hopkins' proposal '16 + "Anomaly inflow"

$$\left\{ \begin{array}{l} \text{anomalies in } d\text{-dim QFT} \\ \quad \text{on } B\text{-mfds} \end{array} \right\} \Big/ \underset{\text{deformation}}{\sim} \quad \xleftrightarrow{\text{bij}} \quad \pi_{-(d+2)} I_{\mathbb{Z}} MTB$$

$\underbrace{I_{\mathbb{Z}}}_{\text{Anderson dual}} MTB$ is a spectrum which fits into

$$0 \to \mathrm{Ext}(\Omega^B_{*-1}, \mathbb{Z}) \to \pi_{-*} I_{\mathbb{Z}} MTB \to \mathrm{Hom}(\Omega^B_{*}, \mathbb{Z}) \to 0$$

(exact).

(c.f. UCT for $H\mathbb{Z}$)

Assumptions for α :

①
$$\begin{tikzcd}
TMF \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \Sigma^{-20} I_{\mathbb{Z}} MString \\
KO((q)) \arrow[r, "\exists \alpha_{spin}"'] & \Sigma^{-20} I_{\mathbb{Z}} MSpin \arrow[u]
\end{tikzcd}$$

… fermions detect anomalies

② α_{spin} is a MSpin-module map

… compatibility with compactification

③
$$\begin{tikzcd}
\alpha_{spin}(pt) : & KO((q))^{20}(pt) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & (I_{\mathbb{Z}} MSpin)^{0}(pt) \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] \\
& \mathbb{Z}((q)) \arrow[d, phantom, "\ni"] & \mathbb{Z}. \\
\text{is given by} & \phi(q) \arrow[r, mapsto] & \Delta(q)\phi(q)\big|_{q^0\text{-coeff}}.
\end{tikzcd}$$

… formula for anomaly poly (80's)

Thm [TY21]

Under assumptions ①②③, $\alpha = 0$.

Actually, once we have translated the problem,

Math proof is straightforward & clear.!

Keys : • $\forall \phi \in MF[\Delta^{-1}]_2$, $\phi(q)|_{q^0} = 0$.

(algebraic fact.)

• $\pi_{-21} TMF = 0$.

⤳ absence of **torsion** anomaly ...

§2 Secondary anomaly & Anderson self-duality of TMF

[TY'23]

by the result of [TY21] + ε, ∃! lift $\tilde{\alpha}$ of α_{spin} :

$$\begin{tikzcd}
 & TMF \arrow[ld, dashed, "\exists ! \tilde{\alpha}"'] \arrow[d, "\alpha_{spin}"] \arrow[rd, "0 \text{ by} [TY21]"] & \\
\Sigma^{-20} I_{\mathbb{Z}} MSpin/MString \arrow[r] \arrow[ru, phantom, "\circlearrowright"] & \Sigma^{-20} I_{\mathbb{Z}} MSpin \arrow[r] \arrow[ru, phantom, "\circlearrowright"] & \Sigma^{-20} I_{\mathbb{Z}} MString
\end{tikzcd}$$

(: diag of MString - modules.)

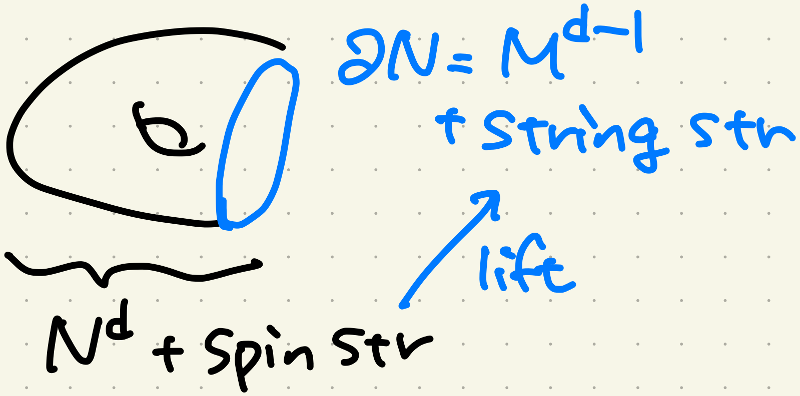
which we call the secondary anomaly transformation.

$\tilde{\alpha}$ turns out to be directly related to

the Anderson duality in TMF.

- MSpin/MString : relative Spin/String bordism.

$$\Omega_d^{\mathrm{Spin/String}} = \left\{ \qquad \right\} \Big/ \sim \text{ bordism}$$



$$\Omega_d^{\mathrm{Spin/String}} \cong \pi_d(\mathrm{MSpin/MString})$$

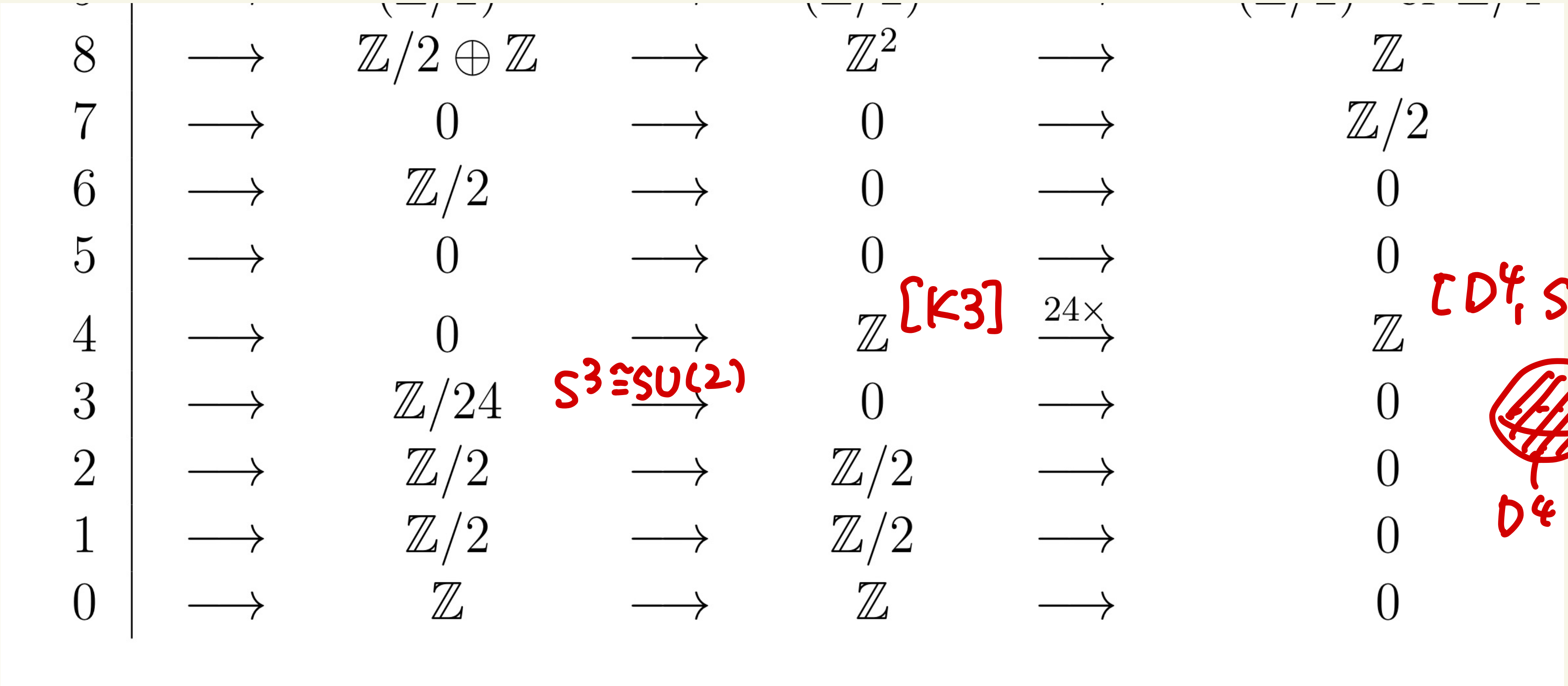
homotopy fiber exact sequence :

$$\cdots \to \Omega_d^{\mathrm{String}} \to \Omega_d^{\mathrm{Spin}} \to \Omega_d^{\mathrm{Spin/String}} \to \Omega_{d-1}^{\mathrm{String}} \to \cdots$$

(exact)

∴ $\Omega_*^{\mathrm{Spin/String}}$ = difference between Ω_*^{Spin} & $\Omega_*^{\mathrm{String}}$.

$\Omega_*^{Spin/String}$ = difference between Ω_*^{Spin} & Ω_*^{String} :



d		Ω_d^{String}		Ω_d^{Spin}		$\Omega_d^{Spin/String}$
8	$\longrightarrow$	$\mathbb{Z}/2 \oplus \mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}^2$	$\longrightarrow$	$\mathbb{Z}$
7	$\longrightarrow$	0	$\longrightarrow$	0	$\longrightarrow$	$\mathbb{Z}/2$
6	$\longrightarrow$	$\mathbb{Z}/2$	$\longrightarrow$	0	$\longrightarrow$	0
5	$\longrightarrow$	0	$\longrightarrow$	0	$\longrightarrow$	0
4	$\longrightarrow$	0	$\longrightarrow$	$\mathbb{Z}$ [K3]	$\xrightarrow{24\times}$	$\mathbb{Z}$ [D^4, $S^3 \cong SU(2)$]
3	$\longrightarrow$	$\mathbb{Z}/24$	$S^3 \cong SU(2)$ $\longrightarrow$	0	$\longrightarrow$	0
2	$\longrightarrow$	$\mathbb{Z}/2$	$\longrightarrow$	$\mathbb{Z}/2$	$\longrightarrow$	0
1	$\longrightarrow$	$\mathbb{Z}/2$	$\longrightarrow$	$\mathbb{Z}/2$	$\longrightarrow$	0
0	$\longrightarrow$	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}$	$\longrightarrow$	0

Relation with the Anderson duality of TMF

We have

$$\begin{tikzcd} MString \arrow[r] \arrow[d, "Wit"'] \arrow[dr, phantom, "\circlearrowright"] & MSpin \arrow[r] \arrow[d, "Wit"] \arrow[dr, phantom, "\circlearrowright"] & MSpin/MString \arrow[d, "Wit"] \\ TMF \arrow[r] & KO((q)) \arrow[r] & KO((q))/TMF \end{tikzcd}$$

Thm (TY'23)

Anderson duality in TMF!

$\exists$ isom $KO((q))/TMF \overset{\downarrow}{\simeq} \Sigma^{-20} I_{\mathbb{Z}} TMF$

and the dual of $\tilde{\alpha} : TMF \to \Sigma^{-20} I_{\mathbb{Z}} MSpin/MString$ factors as

$$\tilde{\alpha}^{\vee} : MSpin/MString \xrightarrow[Wit]{} KO((q))/TMF \simeq \Sigma^{-20} I_{\mathbb{Z}} TMF .$$

$\because$ the isom is essentially another version of

Fact (Stojanoska'13) $Tmf \simeq \Sigma^{-21} I_{\mathbb{Z}} Tmf$.

☺ $\tilde{\alpha}$ is very nontrivial!!

How to "see" $\tilde{\alpha} : TMF \to \Sigma^{-20} I_{\mathbb{Z}} MSpin/MString$?

… via Pairings induced by $\tilde{\alpha}$:

① Non-torsion pairings

$$\langle\, , \rangle_{\tilde{\alpha}} : \pi_{-d} TMF \otimes \Omega^{Spin/String}_{d-20} \to \mathbb{Z}.$$

② Torsion pairings

$$\langle\, , \rangle_{\tilde{\alpha}} : \pi_{-d} TMF_{tor} \otimes \left(\Omega^{Spin/String}_{d-21}\right)_{tor} \to \mathbb{Q}/\mathbb{Z}.$$

They are

- very nontrivial, and
- computable by differential-geometric ways!
  (e.g. characteristic forms / eta invariants)

$\tilde{\alpha} : TMF^{d} \to I_{\mathbb{Z}} MSpin/MString^{d-20}$ induces :

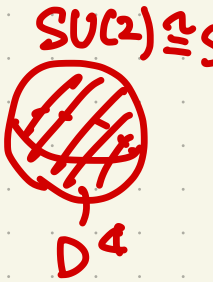
① Non-torsion pairings :

$$\langle , \rangle_{\tilde{\alpha}} : \pi_{-d} TMF \otimes \Omega^{Spin/String}_{d-20} \to \mathbb{Z}.$$

Example $d = 24$

$\pi_{-24} TMF \ni 24/\Delta$ ($\Delta \in MF_{12}$ modular discriminant)

$\Omega^{Spin/String}_{4} \cong \mathbb{Z} \ni [D^4, S^3 \cong SU(2)]$



Prop

$$\langle 24/\Delta , [D^4, S^3 \cong SU(2)] \rangle_{\tilde{\alpha}} = 1.$$

proof : computation of relative Witten genus.

② Torsion pairings :

$$\langle\, , \,\rangle_{\tilde{\alpha}} : \pi_{-d} TMF_{tor} \otimes \left(\Omega^{Spin/String}_{d-21}\right)_{tor} \to \mathbb{Q}/\mathbb{Z}.$$

Example $d = 28$

chromatic height $= 2$!

$$\begin{array}{c} \pi_{-28} TMF_{tor} \cong \mathbb{Z}/2, \\ \rotatebox{90}{$\in$} \\ [E_7 \times E_7] \end{array} \qquad \begin{array}{c} \Omega^{Spin/String}_{7} \cong \mathbb{Z}/2. \\ \rotatebox{90}{$\in$} \\ [D^4 \times \underset{SU(2)}{S^3},\ \underset{SU(2)}{S^3} \times \underset{SU(2)}{S^3}] \end{array}$$

$\leftarrow$ VOA

Prop [TY23]

$$\left\langle [E_7 \times E_7],\ [D^4 \times S^3,\ S^3 \times S^3] \right\rangle_{\tilde{\alpha}} = \frac{1}{2}.$$

Proof : Compute **eta invariants**. We use $TMF^{\tau+\tau'}_{E_7 \times E_7}$

Applications

Math detect power operations in TMF via differential-geometric methods.

Phys extract "chromatic height 2 torsions" from SQFTs.

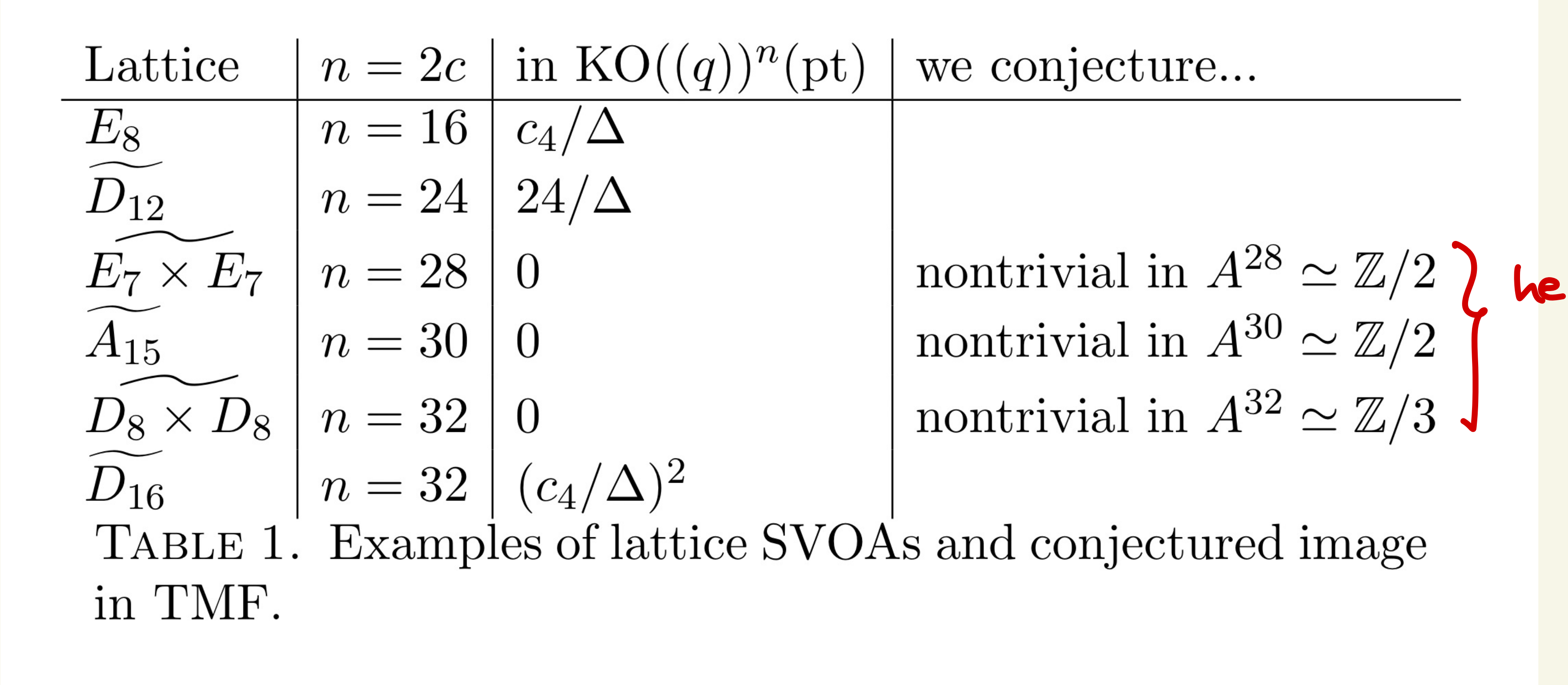
e.g.

e.g. (d=31)

$$\begin{tikzcd}
TMF^{16} \arrow[rr, "\cup"] \arrow[dd, "\cong"] & & TMF^{32}(B\mathbb{Z}_2) \arrow[r, "\text{Möbius}"] & TMF^{32}(S^1) \arrow[r, "\int_{S^1}"] & TMF^{31} & \\
& {[E_8 \text{ level } 1]} \arrow[ul, phantom, "\in" sloped] \arrow[rrr, mapsto] & & & {[E_8 \text{ level } 2]} \arrow[u, phantom, "\in" sloped] & \times \\
MF[\Delta^{-1}]_{-8} & c_4/\Delta \arrow[l, phantom, "\ni"] \arrow[u, phantom, "=" sloped] & & & {[D^4\times S^3\times S^3,\ S^3\times S^3\times S^3]} & \Omega^{Spin/String}_{10} \arrow[l, phantom, "\ni"] \arrow[d, "\langle\,,\,\rangle_{\tilde{\alpha}}"] \\
& & & & & \text{nontrivial!}
\end{tikzcd}$$

⚠ Actually we need TMF^{τ}_{G} : equivariant twisted TMF.

This leads us to the explicit relation between VOAs and TMF_* :



Lattice	$n = 2c$	in $\mathrm{KO}((q))^n(\mathrm{pt})$	we conjecture...
E_8	$n = 16$	c_4/Δ	
$\widetilde{D}_{12}$	$n = 24$	$24/\Delta$	
$\widetilde{E_7 \times E_7}$	$n = 28$	0	nontrivial in $A^{28} \simeq \mathbb{Z}/2$
$\widetilde{A}_{15}$	$n = 30$	0	nontrivial in $A^{30} \simeq \mathbb{Z}/2$
$\widetilde{D_8 \times D_8}$	$n = 32$	0	nontrivial in $A^{32} \simeq \mathbb{Z}/3$
$\widetilde{D}_{16}$	$n = 32$	$(c_4/\Delta)^2$	

TABLE 1. Examples of lattice SVOAs and conjectured image in TMF.

height 2

$$A^n := \mathrm{Ker}(\mathrm{TMF}^n \to \mathrm{KO}^n((q)))$$

§3. Proof of 576-periodicity in SQFTs.

WITHOUT using TMF.

(j. w/ Theo Johnson-Freyd, in preparation.)

Math Fact : • TMF is exactly 576-periodic.

($TMF^n \simeq TMF^{n+576}$)

•
$$\begin{tikzcd} \text{periodicity element} \arrow[r, mapsto] \arrow[d, phantom, sloped, "\ni"] & \Delta^8 . \arrow[d, phantom, sloped, "\ni"] \\ \pi_{576} TMF \arrow[r] & MF[\Delta^{-1}]_{24\times 12} \end{tikzcd}$$

⇝ If you believe in SST proposal,

SQFT should be exactly 576-periodic.

However, there had been NO physical explanation

SQFT should be exactly 576-periodic.

To prove, we should do two things:

- give upper bound 576.

$\Leftrightarrow$ Verify

$$\begin{tikzcd} \pi_{-576}\mathrm{SQFT} \arrow[r] & \mathrm{MF}[\Delta^{-1}]_{24\times 12} \\ \exists J \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \Delta^{24} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

"Existence" result. Done by Gaiotto et al.

★ give lower bound 576.

"Non-Existence" result. Difficult!

e.g. need to show

$$\begin{tikzcd} \pi_{-576/2}\mathrm{SQFT} \arrow[r] & \mathrm{MF}[\Delta^{-1}]_{12\times 12} \\ \nexists \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \Delta^{12} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

$(\exists T_{288} \longmapsto 2\Delta^{12})$

We settle this!

Formulation of the problem.

Recall: Segal-Stolz-Teichner proposal ('04, '11)

> (1) The "Space"
>
> $\{$2-dim, $\mathcal{N}=(0,1)$ SUSY unitary QFTs$\}$
>
> forms a spectrum "SQFT".
>
> (2) We have SQFT $\simeq$ TMF.

In this work, we only use (1) of the Proposal

We do NOT use TMF. To do:

- Start from a spectrum SQFT
- Put assumptions for SQFT implied by physics.
- Show periodicity of SQFT ≥ 576.

(Mathematical) assumptions for SQFT (roughly)

- $$\begin{tikzcd} MString \arrow[r, "\exists", "\sigma\text{-model}"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & SQFT \arrow[r, "\exists", "ev_{T^2}"'] \arrow[d, "ev_{S^1}"] \arrow[dr, phantom, "\circlearrowleft"] & MF[\Delta^{-1}] \arrow[d] \\ MSpin \arrow[r, "Wit"'] & KO((q)) \arrow[r] & \mathbb{Z}((q)) \end{tikzcd}$$

- $\pi_{-21}\,SQFT = 0$ $\rightsquigarrow$ Heterotic anomaly vanishing.

Thm (JF-Y, in preparation)

- $k\Delta^{-12} \in Im(SQFT \to MF[\Delta^{-1}]) \Rightarrow 2 \mid k.$
- $k\Delta^{-16} \in Im(SQFT \to MF[\Delta^{-1}]) \Rightarrow 3 \mid k.$

Cor periodicity of SQFT ≥ 576.

Strategy : use **Anderson duality pairings.**

Assumptions allows us to do **analogy of** §2. [TY23]

we get $\mathbb{Z}$- valued pairing

integrality is the Key!

$$\langle , \rangle : \pi_{-d}\, SQFT \otimes \Omega^{Spin/String}_{d-20} \rightarrow \mathbb{Z}.$$

with explicit formula.

e.g. we can find $[$  $] \in \Omega^{Spin/String}_{24\times 12-20}$

such that

$$\langle KΔ^{-12}, [N,M] \rangle = \frac{3k}{2}$$

$\Rightarrow$ $2|k$, as desired.

Open questions

- Construct mathematicaly,

$$\begin{tikzcd} \{\mathcal{N}=(0,1)\ \text{SCFT}\} \arrow[r] & \text{TMF} \\ \{\mathcal{N}=(0,1)\ \text{SQFT}\} \arrow[u, phantom, "\cap"] \arrow[ur, dashed, "\text{Segal-Stolz-Teichner}"'] & \end{tikzcd}$$

↑ can be formulated via SVOAs

- Develop equivariant refinements of SST.

take Symmetries of SQFTs into account.

need Modular-Tensor-Category equivariant TMF

which is Not mathematically established...

- Find more physical applications!