Cluster algebras and 3D integrable systems

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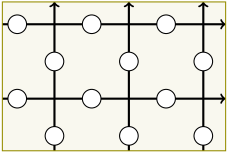
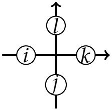
2D lattice spin models

Place spin variables $\bigcirc \in \{1, 2, \ldots, N\}$ on the edges of a lattice:



They interact at vertices, with interaction energy

E_{ij}^{kl} for 

.

Goal: compute the partition function

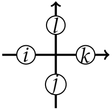
$$Z = \sum_{\text{spin configs}} e^{-E/k_B T}\,, \qquad E = \sum_{\text{vertices}} \text{local energies}\,.$$

Hard... but can be done for a special class of models!

2D integrable lattice models

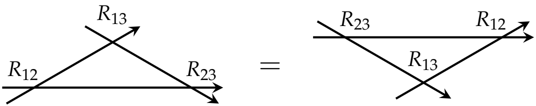
Assign an N-dim vector space V_a to the ath line. Introduce

$$R_{ab} \in \mathrm{End}(V_a \otimes V_b)\,, \qquad (R_{ab})_{ij}^{kl} := e^{-E_{ij}^{kl}/k_B T} =$$



If the R-matrix R satisfies the Yang–Baxter equation

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$$



then the model is integrable. Integrability helps you compute Z!

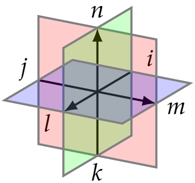
Examples: 8-vertex model ($\simeq$ XYZ spin chain), Ising model, …

Appears in gauge theories, string theory, quiver varieties, …

3D integrable lattice models

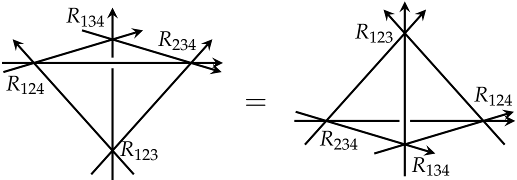
What about a spin model on a 3D lattice?

Assign V_{ab} to the intersection of the ath & bth planes. Introduce

$$R_{abc} \in \mathrm{End}(V_{bc} \otimes V_{ca} \otimes V_{ab}), \quad (R_{abc})_{ijk}^{lmn} =$$  .

The tetrahedron equation [Zamolodchikov '80] implies integrability:

$$R_{234}R_{134}R_{124}R_{123} = R_{123}R_{124}R_{134}R_{234} \,.$$



3D integrable lattice models

TE has a relatively long history.
(Before BPZ on 2D CFT!)

Far less developed than YBE.
(Only one book [Kuniba '22] on TE!)

Too difficult?
(Zamolodchikov wrote down first nontrivial solutions "by what appears to be an extraordinary feat of intuition" (Baxter).)

But it could be as rich.
(Interesting solutions found by Zamolodchikov, Baxter, Kapranov–Voevodsky, Bazhanov–Mangazeev–Sergeev, Kuniba–Matsuike–Yoneyama, ...)

And important: we live in 3D space!

Tetrahedron equation & cluster algebras

A new approach to TE via cluster algebras:

- Sun–Y '22: new solutions constructed from *triangle*, *square* and *butterfly quivers* and related to 3D gauge theories.
- Inoue–Kuniba–Terashima '23: known solutions and new ones from triangle and square quivers.
- IKSTY '24: construct two more solutions using *symmetric butterfly quiver*.

Gavrylenko–Semenyakin–Zenkevich '20: a classical L-operator from cluster integrable system.

Many known interesting solutions of TE can be obtained from the last solutions as special cases.

Relations to 3-manifolds, wall-crossing, 4D Chern–Simons, …

Wiring diagrams

The symmetric group S_n is generated by $s_1, \ldots, s_{n-1}$ satisfying

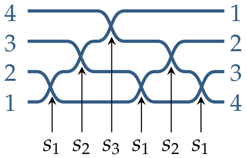
(1) $s_a^2 = 1$;

(2) $s_a s_b = s_b s_a$ for $|a-b| \geq 2$;

(3) $s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}$.

An expression $s_{a_1} \cdots s_{a_k}$ can be represented by a wiring diagram:

$s_1 s_2 s_3 s_1 s_2 s_1 =$



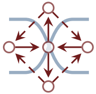
- The wires are numbered $1, 2, \ldots, n$.
- s_a "braids" the wires at the ath and $(a+1)$th positions.

Identify isotopic wiring diagrams (impose Relation (2)).

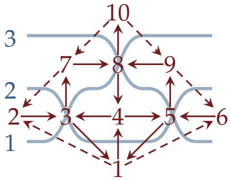
Wiring diagrams to quivers

To a wiring diagram we assign a quiver:

(1) Put vertices on the crossing and the chambers.
(2) Around each crossing, draw arrows as follows:



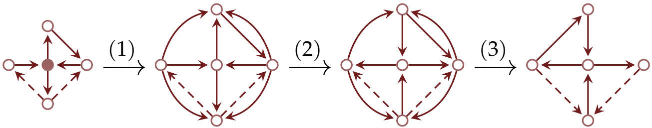
Solid arrows have weight 1, dashed arrow have weight $\frac{1}{2}$.
(3) Combine arrows between the same pair of vertices into a single arrow of the total signed weight.



Cluster transformations

The mutation μ_k at vertex k transforms a quiver to a new one:

(1) For each $i \to k \to j$, add an arrow $i \to j$.
(2) Reverse the orientation of all arrows connected to k.
(3) Combine arrows.



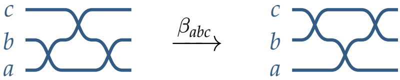
Mutations are involutive: $\mu_k \circ \mu_k = \mathrm{id}$.

An automorphism σ permutes vertex labels: $i \mapsto \sigma(i)$.

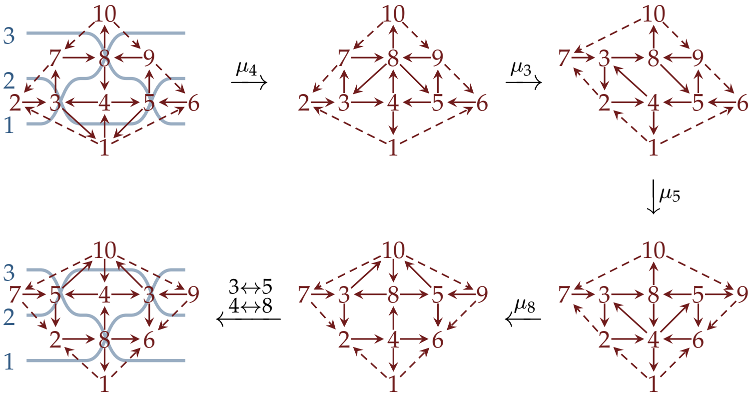
A cluster transformation is a composition of any sequence of mutations and automorphisms.

Braid moves are cluster transformations

The braid move



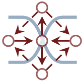
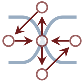
induces a cluster transformation on the assigned quiver:



Quivers to gauge theories

A quiver specifies a gauge theory with 4 supercharges:

- vertex i: gauge group $\mathrm{SU}(N)_i$
- arrow $i \to j$: matter in the rep $(\overline{\square}, \square)$ of $\mathrm{SU}(N)_i \times \mathrm{SU}(N)_j$

Don't like half arrows? Replace  $\to$  .

So we have

$$\text{wiring diagram} \rightsquigarrow \text{quiver} \rightsquigarrow \text{gauge theory}\,.$$

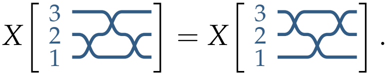
Mutations translates to infrared dualities.

Calculating a quantity X in dual theories T and $\mathsf{T}^\vee$ gives

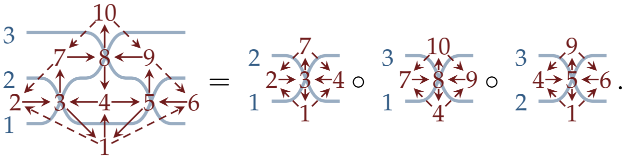
$$X[\mathsf{T}] = X[\mathsf{T}^\vee]\,.$$

Gauge/YBE correspondence

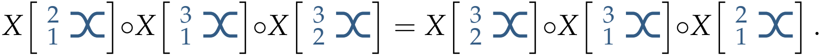
Since a braid move induces a cluster transformation/IR duality,



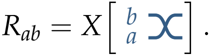
The theory on each side decomposes into 3 simpler theories:



If X is nice (e.g. SUSY index), we have the decomposition



Thus we obtain a solution of YBE [Yamazaki, Y, Yan, …]:

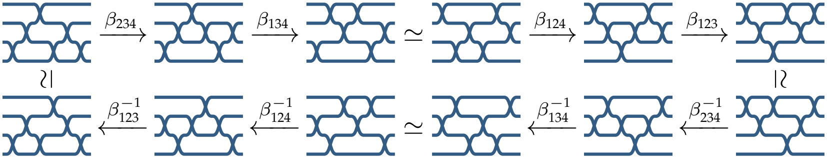


Lift to TE

Instead consider the space of states $\mathcal{H}$. We get an isomorphism

$R_{abc} : \mathcal{H}\Big[$ c,b,a  $\Big] \xrightarrow{\sim} \mathcal{H}\Big[$ c,b,a  $\Big]$.

The loop of braid moves



shows

$R_{234}R_{134}R_{124}R_{123}R_{234}^{-1}R_{134}^{-1}R_{124}^{-1}R_{123}^{-1} \in \mathrm{End}\Big(\mathcal{H}\Big[$  $\Big]\Big)$.

Lift to TE

If this operator

$$R_{234}R_{134}R_{124}R_{123}R_{234}^{-1}R_{134}^{-1}R_{124}^{-1}R_{123}^{-1} = \mathrm{id}_{\mathrm{End}\left(\mathcal{H}\left[\quad\right]\right)}$$

then R_{abc} solves TE:

$$R_{234}R_{134}R_{124}R_{123} = R_{123}R_{124}R_{134}R_{234}\,.$$

Is this the case? I don't know.

But quantum cluster algebras allows us to construct quantum mechanical systems in which this is the case [SY].

Quantum cluster algebras

A quiver can be encoded in its exchange matrix

$$B = (b_{ij})_{i,j \in I}\,, \quad b_{ij} = -b_{ji}\,, \quad I = \{\text{vertices}\}\,.$$

In our case

$$b_{ij} = \begin{cases} 0 & \text{no arrow between } i \text{ and } j\,, \\ 1 & \text{solid arrow from } i \text{ to } j\,, \\ \frac{1}{2} & \text{dashed arrow from } i \text{ to } j\,. \end{cases}$$

The mutation μ_k transforms B to $B' = \mu_k(B)$ given by

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k\,, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise}\,. \end{cases}$$

Quantum cluster algebras

Let $\mathcal{Y}(B)$ be a skew field generated by quantum Y-variables:

$$Y_i Y_j = q^{2b_{ij}} Y_j Y_i\,, \quad i \in I\,.$$

Mutation $\mu_k\colon B \to B'$ induces $\mu_k^*\colon \mathcal{Y}(B') \xrightarrow{\sim} \mathcal{Y}(B)$:

$$\mu_k^* = \mathrm{Ad}(\Psi_q(Y_k)) \circ \tau_{k,+} = \mathrm{Ad}(\Psi_q(Y_k^{-1})^{-1}) \circ \tau_{k,-}\,,$$

where

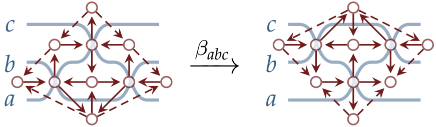
$$\Psi_q(x) := \prod_{n=0}^{\infty} (1 + x q^{2n+1})^{-1}$$

is the quantum dilogarithm and $\tau_{k,\epsilon}\colon \mathcal{Y}(B') \to \mathcal{Y}(B)$ is given by

$$\tau_{k,\pm}(Y'_i) = \begin{cases} Y_k^{-1} & i = k\,, \\ q^{-b_{ik}[\pm b_{ik}]_+} Y_i Y_k^{[\pm b_{ik}]_+} & i \neq k\,, \end{cases} \qquad [a]_+ := \max[0, a]\,.$$

Cluster transformation to TE

The braid moves



satisfy

$$\beta_{123}^{-1}\beta_{124}^{-1}\beta_{134}^{-1}\beta_{234}^{-1}\beta_{123}\beta_{124}\beta_{134}\beta_{234}$$ () =  .

It turns out that the induced transformation on the quantum Y-variables is also trivial (modulo relabeling):

$$(\beta_{123}^{-1}\beta_{124}^{-1}\beta_{134}^{-1}\beta_{234}^{-1}\beta_{123}\beta_{124}\beta_{134}\beta_{234})^*(Y_i) = Y_i\,, \quad Y_i \in \mathcal{Y}$$ () .

Cluster transformation to TE

Thus $\widehat{R}_{abc} = \beta^*_{abc}$ solves TE:

$$\widehat{R}_{234}\widehat{R}_{134}\widehat{R}_{124}\widehat{R}_{123} = \widehat{R}_{123}\widehat{R}_{124}\widehat{R}_{134}\widehat{R}_{234} : \mathcal{Y}\Big($$

$$\Big) \to \mathcal{Y}\Big($$

$$\Big).$$

Can we find an operator

$$R_{abc} : \mathcal{H}\left[\begin{array}{c} c \\ b \\ a \end{array}\right.$$

$$\left.\right] \xrightarrow{\sim} \mathcal{H}\left[\begin{array}{c} c \\ b \\ a \end{array}\right.$$

$$\left.\right]$$

such that

$$\widehat{R}_{abc}(Y_i) = R_{abc} Y_i R^{-1}_{abc}\,, \quad Y_i \in \mathcal{Y}\left(\begin{array}{c} c \\ b \\ a \end{array}\right.$$

$$\left.\right)$$

and solves TE?

Cluster transformation to TE

$\beta_{123} = \sigma_{4,8}\sigma_{3,5}\mu_8\mu_5\mu_3\mu_4$ and $\mu_k^* = \mathrm{Ad}(\Psi_q(Y_k^\epsilon)^\epsilon)\tau_{k,\epsilon}$, $\epsilon = \pm$, so

$$\begin{aligned}\widehat{R}_{123} &= \mathrm{Ad}(\Psi_q(Y_4^{\epsilon_1})^{\epsilon_1})\tau_{4,\epsilon_1}\mathrm{Ad}(\Psi_q(Y_3^{\epsilon_2})^{\epsilon_2})\tau_{3,\epsilon_2} \\ &\qquad \times \mathrm{Ad}(\Psi_q(Y_5^{\epsilon_3})^{\epsilon_3})\tau_{5,\epsilon_3}\mathrm{Ad}(\Psi_q(Y_8^{\epsilon_4})^{\epsilon_4})\tau_{8,\epsilon_4}\sigma_{4,8}\sigma_{3,5} \\ &= \mathrm{Ad}(\dots)\tau_{4,\epsilon_1}\tau_{3,\epsilon_2}\tau_{5,\epsilon_3}\tau_{8,\epsilon_4}\sigma_{4,8}\sigma_{3,5}\,.\end{aligned}$$

We want to write

$$\tau_{4,\epsilon_1}\tau_{3,\epsilon_2}\tau_{5,\epsilon_3}\tau_{8,\epsilon_4}\sigma_{4,8}\sigma_{3,5} = \mathrm{Ad}(P_{123})$$

so that $\widehat{R}_{123} = \mathrm{Ad}(R_{123})$ with $R_{123} = (\dots)P_{123}$.

There are two ways:

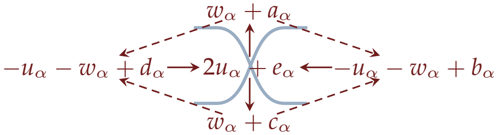
- Fock–Goncharov: realize $\mathcal{Y}(B)$ by position & momentum operators $(\hat{x}^i, \hat{p}_i)_{i\in I}$. This requires using the noncompact q-dilog Φ and doubling the number of q-dilogs [SY].
- Inoue–Kuniba–Terashima: *q*-Weyl realization [IKT, IKSTY]

q-Weyl realization

To the αth crossing of wires assign variables (u_α, w_α) satisfying

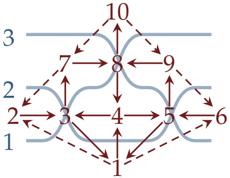
$$[u_\alpha, w_\alpha] = \hbar\,, \qquad q = e^{\hbar}\,.$$

$\log Y$-variables can be realized, locally around the crossing, by



$a_\alpha, b_\alpha, c_\alpha, d_\alpha, e_\alpha$ are parameters with $a_\alpha + b_\alpha + c_\alpha + d_\alpha + e_\alpha = 0$.

Globally, Y_i gets contributions from all surrounding crossings:



$$Y_1 \mapsto e^{w_1 + c_1 + w_3 + c_3}\,, \qquad Y_3 \mapsto e^{2u_3 + e_3}\,.$$

q-Weyl realization

There exists P_{123} such that $\tau_{4,\epsilon_1}\tau_{3,\epsilon_2}\tau_{5,\epsilon_3}\tau_{8,\epsilon_4}\sigma_{4,8}\sigma_{3,5} = \mathrm{Ad}(P_{123})$ iff $(\epsilon_1\epsilon_2\epsilon_3\epsilon_4) = (--++)$ or $(-+-+)$. Furthermore, R_{123} solves TE.

Explicitly, for $(--++)$,

$$R_{123} =$$
$$\Psi_q(e^{-d_1-c_2-b_3+u_1+u_3+w_1-w_2+w_3})^{-1}\Psi_q(e^{-d_1-c_2-b_3-e_3+u_1-u_3+w_1-w_2+w_3})^{-1}$$
$$\times P_{123}\Psi_q(e^{-b_1-a_2-d_3-e_3+u_1-u_3+w_1-w_2+w_3})\Psi_q(e^{-b_1-a_2-d_3+u_1+u_3+w_1-w_2+w_3})$$

with

$$P_{123} = e^{\frac{1}{\hbar}(u_3-u_2)w_1}e^{\frac{\lambda_0}{\hbar}(-w_1-w_2+w_3)}e^{\frac{1}{\hbar}(\lambda_1u_1+\lambda_2u_2+\lambda_3u_3)}\rho_{23}\,,$$

$$\lambda_0 = (e_2-e_3)/2\,, \qquad \lambda_1 = a_2-a_3+b_2-b_3+\lambda_0\,,$$
$$\lambda_2 = -a_1-b_2+b_3-\lambda_0\,, \quad \lambda_3 = c_1-c_2+c_3\,.$$

and $\mathrm{Ad}(\rho_{23})$ acts on u_α, w_α by the permutation $2\leftrightarrow 3$.

q-Weyl realization

Evaluating R_{123} in different reps gives different solutions of TE.

Let $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$ be spin variables on the lattice edges.

Coordinate rep

$$e^{u_k} |\mathbf{n}\rangle = i q^{n_k + \frac{1}{2}} |\mathbf{n}\rangle \,, \quad e^{w_k} |\mathbf{n}\rangle = |\mathbf{n} + \mathbf{e}_k\rangle$$

yields the solution Kapranov–Voevodsky '94 obtained using the quantized coordinate ring of SL_3 for $a_i = b_i = c_i = d_i = e_i = 0$.

Momentum rep

$$e^{u_k} |\mathbf{n}\rangle = |\mathbf{n} - \mathbf{e}_k\rangle \,, \quad e^{w_k} |\mathbf{n}\rangle = q^{n_k} |\mathbf{n}\rangle$$

yields a solution obtained by Kuniba–Matsuike–Yoneyama '22.

There is also a modular double version of R_{123}, which generalizes a solution of Bazhanov–Mangazeev–Sergeev '08.

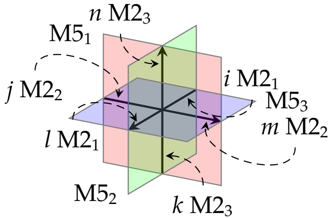
Brane construction

A Fock space representation is conjectured [Y '22] to arise from

	$\mathbb{R}_0$	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{R}_4$	$\mathbb{R}_5$	$\mathbb{R}_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	$\mathbb{R}_\natural$
$\mathrm{M5}_1$	—	$\cdot$	—	—	—	$\cdot$	$\cdot$	—	—	$\circ$	$\circ$
$\mathrm{M5}_2$	—	—	$\cdot$	—	$\cdot$	—	$\cdot$	—	—	$\circ$	$\circ$
$\mathrm{M5}_3$	—	—	—	$\cdot$	$\cdot$	$\cdot$	—	—	—	$\circ$	$\circ$

The partition function is the SUSY index of the brane system:

$$Z = \mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{\mathrm{i}\theta(J_{78}-J_{9\natural})} e^{-\beta H}\right), \quad R^{lmn}_{ijk} =$$



Related to 4D Chern–Simons by dualities.

3D gauge theories, 3-manifolds, wall-crossing

Fock–Goncharov representation leads to a solution with 8 noncompact q-dilogs Φ_b with $\hbar = b^2$.

The result coincides with an expression of the partition function of a 3D $\mathcal{N} = 2$ SUSY gauge theory on the squashed 3-sphere

$$S_b^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid b|z_1|^2 + b^{-1}|z_2|^2 = 1\}.$$

Similar to [Terashima–Yamazaki '13] but with twice as many q-dilogs.

This is a kind of theory that arises from compactification of 6D theory on a 3-manifold.

Is this is a domain wall in a 4D $\mathcal{N} = 2$ SUSY theory? Does TE hold at the level of domain walls?

Conclusions

TE is a 3D analog of YBE, important and needs more study.

Solutions can be constructed using quantum cluster algebras, via a loop of mutations on quivers assigned to wiring diagrams.

Solutions also arise from brane systems in M-theory and 3D $\mathcal{N} = 2$ SUSY gauge theories.

How are these approaches related?

Connections to 3-manifolds, wall-crossing, …?

What can we say about 3D statistical mechanics systems?