Complex Hyperbolic Structures for Moduli Spaces of Calabi-Yau Varieties

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Moduli spaces and locally symmetric spaces

Many locally symmetric spaces have modular interpretations. Let $\mathcal{M}$ be moduli of certain algebraic varieties, $\mathbb{D}$ Hermitian symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ discrete lattice acting on $\mathbb{D}$. The period map induces

$$\mathcal{M} \longleftrightarrow \Gamma\backslash\mathbb{D}$$

Famous examples include:

1. $\mathcal{M}$: moduli of polarized abelian varieties, $\mathbb{D}$: Siegel upper half space
2. $\mathcal{M}$: moduli of polarized $K3$ surfaces, $\mathbb{D}$: Type-IV domain

Complex hyperbolic balls

Let h be a Hermitian form on vector space V with signature $(1, n)$. The complex hyperbolic ball of dimension n is

$$\mathbb{B}^n = \{v \in \mathbb{P}(V) \mid h(v, v) > 0\}$$

Examples of moduli spaces with ball structure:

1. (Deligne-Mostow) Moduli of $(n+3)$-points on $\mathbb{P}^1$
2. (Allcock-Carlson-Toledo) Moduli of cubic surfaces and cubic threefolds
3. (Kondō) Moduli of genus 3 curves and genus 4 curves.

Cyclic cover of projective line

Euler, Riemann, Schwarz, Picard, $\cdots$, Shimura, Deligne-Mostow, Thurston: consider cyclic covers of $\mathbb{P}^1$

$$C_\mu : y^d = (x - x_1)^{a_1} \cdots (x - x_{n+3})^{a_{n+3}}$$

with $0 < \mu_i = \frac{a_i}{d} < 1$ and $\sum_i \mu_i \in \mathbb{Z}$. The cyclic group $\mathbb{Z}/d\mathbb{Z}$ acts on C_μ and decomposes $H^1(C_\mu)$ by characters.

- When $\sum \mu_i = 2$, $H^1_\chi(C_\mu)$ has hermitian form with sign $(1, n)$.
- Moduli of such C_μ is $\mathcal{M} = \mathrm{PSL}(2, \mathbb{C}) \backslash \big((\mathbb{P}^1)^{n+3} - \text{diagonals}\big)$
- Monodromy representation $\rho \colon \pi_1(\mathcal{M}) \to \mathrm{PU}(1, n)$, $\Gamma_\mu = \mathrm{Im}(\rho)$
- $\overline{\mathcal{M}} \cong \overline{\Gamma \backslash \mathbb{B}^n}^{BB}$

Connection with hypergeometric functions when $n = 1$

- Euler introduced hypergeometric functions

$$ {}_2F_1\left[\begin{matrix} a \ \ b \\ c \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \quad |x| < 1 $$

where $(a)_k = a(a-1)\cdots(a-k+1)$.

- It satisfies a 2^{nd}-order differential equation

$$ x(1-x)\frac{d^2y}{dx^2} + (c - (a+b+1)x)\frac{dy}{dx} - aby = 0 $$

- It has an integral representation

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a \ \ b \\ c \end{matrix}; x\right] &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 z^{a-1}(1-z)^{c-a-1}(1-zx)^{-b} dz \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^{\infty} z^{b-c}(z-1)^{c-a-1}(z-x)^{-b} dz \end{aligned} $$

Monodromy representation

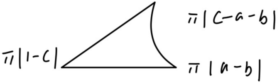
- The differential equation has three regular singular points $\{0, 1, \infty\}$ on the Riemann sphere $\mathbb{P}^1$.
- The analytic continuation of two linearly independent solutions F_1, F_2 gives rise to a representation

$$\rho\colon \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to \mathrm{PGL}(2, \mathbb{C}).$$

- Schwarz found that the two solutions are algebraic functions iff $\Gamma = \mathrm{Im}(\rho)$ is finite.
- Schwarz also showed that the inverse of $\frac{F_1}{F_2}$ being single-valued is equivalent to Γ being discrete.

Discreteness, arithmeticity and Commensurability

- Discreteness. When Γ is discrete, it is a triangle group, generated by even number of reflections about sides of a spherical, euclidean or hyperbolic triangle. (Schwarz, Mostow, Knapp, $\cdots$)



- Arithmeticity. Γ is arithmetic if the discreteness is provided by $\mathbb{Z}$ being discrete in $\mathbb{R}$. Takeuchi (1977) listed all the arithmetic triangle groups (85 examples).
- Commensurability. Γ_1, Γ_2 are commensurable in G, iff there is $g \in G$, such that $g\Gamma_1 g^{-1} \cap \Gamma_2$ has finite indices in $g\Gamma_1 g^{-1}, \Gamma_2$. This is commensurability of triangles.

Why ball quotients?

- Discreteness. Deligne-Mostow and Thurston's list: $94 + 10$ tuples of μ for $n \geq 2$.
- Arithmeticity. Nonarithmetic lattices in $PU(1,2)$ and one example in $PU(1,3)$.
- For other simple groups, there are either infinitely many commensurability classes of nonarithmetic lattices, ($O(1,n)$ by Gromov-Piatetski-Shapiro), or only arithmetic lattices (Margulis superrigidity, Corlette, Gromov-Schoen).
- Other constructions in $PU(1,2)$ by Barthel-Hirzebruch-Höfer via Bogomolov-Miyaoka-Yau inequality, and Yau's criterion for ball quotients.

Cyclic cover of $\mathbb{P}^1 \times \mathbb{P}^1$

Kondō considers nonhyperelliptic curves of genus 4, $D \subset \mathbb{P}^1 \times \mathbb{P}^1$. Taking triple covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branching along D, gives surfaces $S \to \mathbb{P}^1 \times \mathbb{P}^1$.

- S is $K3$.
- $H^2_\chi(S)$ has hermitian form with sign $(1, 9)$.
- Moduli of genus 4 curves $\mathcal{M}_4$ is birational to $\Gamma \backslash \mathbb{B}^9$.
- Nikulin theory on $K3$ lattice relates this ball quotient to Deligne-Mostow ball quotient with $\mu_1 = \cdots = \mu_{12} = \frac{1}{6}$ up to finite cover.

Moduli spaces of genus three curves, cubic surfaces and cubic threefolds are ball quotiens in a similar way.

Cyclic cover of $\mathbb{P}^3$

Sheng-Xu-Zuo studied cyclic cover $Y \to \mathbb{P}^3$ branching along 6 hyperplanes.

- Y is a Calabi-Yau orbifold admitting crepant resolution.
- $\mathbb{Z}/3\mathbb{Z}$ operation decomposes $H^3(Y)$ as follows.

	$H^{3,0}$	$H^{2,1}$	$H^{1,2}$	$H^{0,3}$
$H^3_\chi(Y)$	1	3	0	0
$H^3_{\bar\chi}(Y)$	0	0	3	1

- Period domain is $\mathbb{B}^3$.
- This is also related to Deligne-Mostow's example $C\colon y^3 = (x - x_1)\cdots(x - x_6)$.
- Sheng-Xu proved global Torelli theorem for this family.
- Sheng-Xu-Zuo classified such examples for cyclic covers of $\mathbb{P}^n$ branching along hyperplanes.

Classification for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Let $Y \xrightarrow{3:1} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be cyclic cover branching along simple normal crossing divisor $D = D_1 + \cdots + D_r \in |\mathcal{O}(3,3,3)|$. As D_i vary in $|L_i|$, we obtain a family of Calabi-Yau orbifolds.

Theorem (Y.-Zheng)

The period map of this family factors through complex hyperbolic ball if and only if $L_1 \cdots L_r$ are

1. $L_1 = (3,3,0)$ *and* $L_2 = (0,0,3)$; *(Voisin, Borcea and Rohde)*
2. $L_1 = (3,2,0)$ *and* $L_2 = (0,1,3)$;
3. $L_1 = (2,2,0)$, $L_2 = (1,0,2)$ *and* $L_3 = (0,1,1)$;
4. $L_1 = (2,1,0)$, $L_2 = (1,0,2)$ *and* $L_3 = (0,2,1)$.

or their refinements. Moreover, the ball quotients in the 4 maximal cases have dimensions $9, 9, 7, 6$ *respectively.*

Refinement and half-twist

Let $Y \xrightarrow{d:1} X$ be cyclic cover branching along simple normal crossing divisor $D = D_1 + \cdots + D_r \in |-\frac{d}{d-1}K_X|$. Then D corresponds a partition of $-\frac{d}{d-1}K_X$.

- Refinements of D preserves the ball-type property.
- When $d = 3$, consider

  $$X' = X \times \mathbb{P}^1,\ D_i' = D_i \times \mathbb{P}^1,\ D_{r+1}' = X \times 3pts$$

  The corresponding Y' is called half-twist of Y.
- Half-twists generate the ball-type examples.
- When $X = (\mathbb{P}^1)^n$, all ball-type examples are generated by refinements and half-twists from the previous list together with one more example for $n = 4$.

Crepant resolution and completeness

The family of Calabi-Yau orbifolds Y admits crepant resolutions $\widetilde{Y}$ by Sheng-Xu-Zuo.

Theorem (Y-Zheng)

If the family of Calabi-Yau manifolds $\widetilde{Y}$ is of ball type and complete, then the divisor D is a refinement of the following 5 cases:

1. $L_1 = (3,1,0), L_2 = (0,2,1), L_3 = (0,0,2);$
2. $L_1 = (3,0,0), L_2 = (0,2,1), L_3 = (0,1,2);$
3. $L_1 = (2,1,0), L_2 = (1,0,2), L_3 = (0,2,1);$
4. $L_1 = (2,1,0), L_2 = (1,0,1), L_3 = (0,1,1), L_4 = (0,1,1);$
5. $L_1 = (2,1,0), L_2 = (1,1,0), L_3 = (0,1,1), L_4 = (0,0,2).$

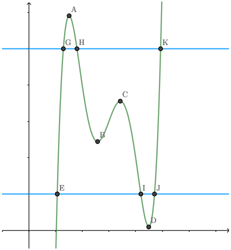
The dimensions of the balls for the five cases are $5,4,6,5,4$*.*
Higher dimensional families are generated by half-twists.

Ingredients in the Classification

- Local Torelli for equisingular deformation of cyclic covers.
- Stability and moduli dimension in GIT give the Hodge number.
- The classification method works for toric base or homogeneous variety.
- Refinements relation comes from a generalization of Clemens-Schmid long exact sequence by Kerr-Laza.
- The monodromy group is arithmetic subgroup in $\mathrm{PU}(1, n)$ by Borel extension.
- Half twist is $Y' = (Y \times E)/(\mathbb{Z}/3\mathbb{Z})$, where E is the elliptic curve with $j(E) = 0$.

Relation to Deligne-Mostow

Most of the examples are Deligne-Mostow ball quotients up to finite index. Consider $S \to \mathbb{P}^1 \times \mathbb{P}^1$ and $D_1 \in |\mathcal{O}(3,1)|$ and $D_2 \in |\mathcal{O}(0,2)|$. The branching divisor D is as follows.



- The fibration $S \to \mathbb{P}^1$ is isotrivial elliptic fibration with 6 singular fibers in both directions.
- The singular fibers in first projection gives rise to Deligne-Mostow tuple $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ by Kodaira.
- The second projection gives rise to Deligne-Mostow tuple $\nu = (\frac{2}{3}, \frac{2}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$.
- Corollary: the two Deligne-Mostow lattices are the same up to finite index (commensurable).
- In dimension 3, most examples are isotrivial fibrations of $K3$ surfaces. Singular fibers give the Deligne-Mostow data.

Commensurability relations

Theorem (Deligne-Mostow, Sauter (1980s))

Commensurability pairs $\Gamma_\mu \sim \Gamma_\nu$ in $PU(1,2)$ with explicit indices.

1. $\mu = (a, a, b, b, 1-2a-2b)$,
   $\nu = (1-b, 1-a, a+b-\frac{1}{2}, a+b-\frac{1}{2}, 1-a-b)$.
2. $\mu = (\frac{1}{2}-a, \frac{1}{2}-a, \frac{1}{2}-a, \frac{1}{6}+a, 2(\frac{1}{6}+a))$,
   $\nu = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}-a, \frac{2}{3}+a)$.

- The pairs were found by Mostow with computer investigation.
- Kappes-Möller (2012),McMullen (2013) proved that those pairs provide all commensurability classes for non-arithmetic Deligne-Mostow lattices.
- Commensurability invariants are adjoint trace fields and ratios of Chern numbers similar as Hirzebruch proportionality.

Theorem (Y.-Zheng)

Commensurabilities with explicit indices. (Not necessarily discrete)

1. $n = 2$, *reprove Deligne-Mostow, Sauter.*
2. $n = 3$, *two infinite series.*

$$\begin{aligned} \mu &= (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, 1-a, \tfrac{1}{3}+a) \\ \nu &= (a, a, a, \tfrac{2}{3}-a, \tfrac{2}{3}-a, \tfrac{2}{3}-a) \end{aligned}$$

and

$$\begin{aligned} \mu &= (\tfrac{1}{2}-a, \tfrac{1}{2}-a, \tfrac{1}{2}-a, \tfrac{1}{2}-a, 2a, 2a) \\ \nu &= (\tfrac{1}{2}-a, \tfrac{1}{2}-a, \tfrac{1}{2}-a, a, a, \tfrac{1}{2}+a) \end{aligned}$$

3. $n \geq 3$, *finite pairs.*

Cyclic covers of Calabi-Yau type

- The proof is based on moduli spaces $\mathcal{M}$ of Calabi-Yau type cyclic covers over $(\mathbb{P}^1)^m$.

$$Y : y^d = (f_1)^{a_1} \cdots (f_k)^{a_k}$$

- The Calabi-Yau condition $h_\chi^{m,0}(Y) = 1$, and ball-type conditions

$$\begin{aligned} \sum \deg f_i &= (3, \cdots, 3) \\ \sum \mu_i \deg f_i &= (1, \cdots, 1) \end{aligned}$$

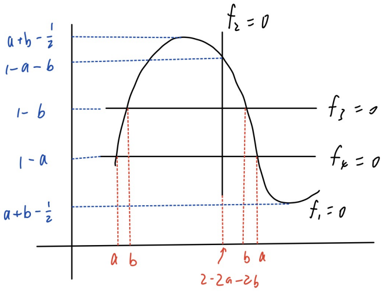
- Asymmetric solutions give rise to commensurability pairs.

Reprove Deligne-Mostow, Sauter

$$\deg f_1 = (2,1), \quad \deg f_2 = (1,0), \quad \deg f_3 = \deg f_4 = (0,1),$$

$$2a_1 + a_2 = a_1 + a_3 + a_4 = d$$

Then Y admits two fibrations with five singular fibres.



Commensurability invariants

- The arithmetic Deligne-Mostow lattices Γ_μ are related to $K = (\mathbb{Q}[\zeta_d] \cap \mathbb{Q})$-algebraic groups $PU(h_\mu)$.
- $\Gamma_\mu \sim \Gamma_\nu$ if and only if $PU(h_\mu) \cong PU(h_\nu)$ as K-algebraic groups.
- In Deligne-Mostow theory with $n \geq 2$, this is equivalent to $\mathbb{Q}[\zeta_d]$ being the same and h_μ conformal to h_ν, which can be determined by lattice invariants.
- When $n = 1$, this is no longer true. When $d_\mu = 4, d_\nu = 6$, there are $PU(h_\mu) \cong PU(h_\nu)$ as $\mathbb{Q}$-algebraic groups.
- This approach does not give commensurability indices.

Thank you!

Happy birthday, Professor Yau!