Crepant Resolution of Calabi-Yau singularity

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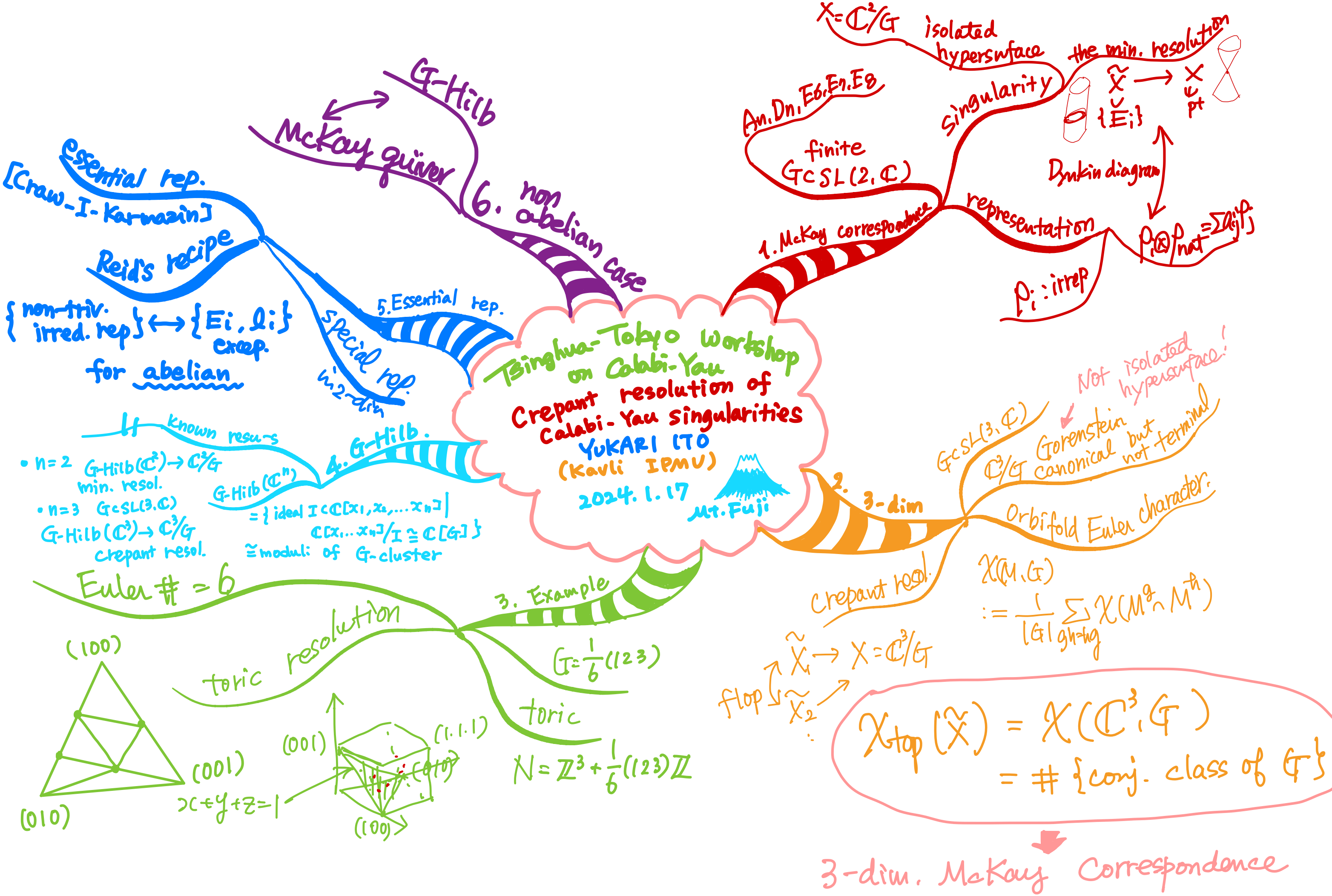

Tsinghua-Tokyo Workshop on Calabi-Yau
Crepant resolution of Calabi-Yau singularities
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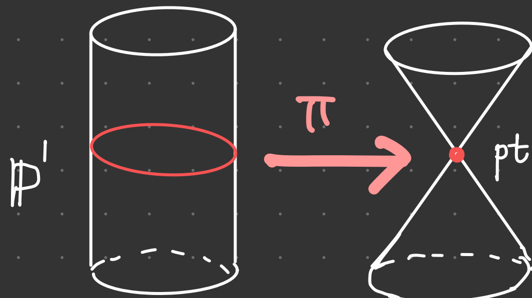
2024.1.17
Mt. Fuji
1. McKay correspondence
$X = \mathbb{C}^2/G$ isolated hypersurface
singularity
A_n, D_n, E_6, E_7, E_8
finite $G \subset SL(2,\mathbb{C})$
the min. resolution
$\tilde{X} \to X$
$\cup$ $\cup$
$\{E_i\}$ pt
Dynkin diagram
representation
$\rho_i \otimes \rho_{nat} = \sum a_{ij} \rho_j$
ρ_i : irrep
2. 3-dim
$G \subset SL(3,\mathbb{C})$
$\mathbb{C}^3/G$ Gorenstein canonical but not terminal
Not isolated hypersurface!
Orbifold Euler character.
$\chi(M,G) := \frac{1}{|G|} \sum_{gh=hg} \chi(M^g \cap M^h)$
crepant resol.
$\tilde{X}_1 \to X = \mathbb{C}^3/G$
flop
$\tilde{X}_2$
$\chi_{top}(\tilde{X}) = \chi(\mathbb{C}^3, G)$
$= \#\{\text{conj. class of } G\}$
3-dim. McKay Correspondence
3. Example
$G = \frac{1}{6}(1\,2\,3)$
toric
$N = \mathbb{Z}^3 + \frac{1}{6}(1\,2\,3)\mathbb{Z}$
toric resolution
Euler # = 6
(100)
(010)
(001)
$x+y+z=1$
(001)
(1.1.1)
(010)
(100)
4. G-Hilb.
known results
• n = 2 G-Hilb$(\mathbb{C}^2) \to \mathbb{C}^2/G$
min. resol.
• n = 3 $G \subset SL(3,\mathbb{C})$
G-Hilb$(\mathbb{C}^3) \to \mathbb{C}^3/G$
crepant resol.
G-Hilb$(\mathbb{C}^n)$
$= \{ \text{ideal } I \subset \mathbb{C}[x_1, x_2, \ldots x_n] \mid \mathbb{C}[x_1, \ldots x_n]/I \cong \mathbb{C}[G] \}$
$\cong$ moduli of G-cluster
5. Essential rep.
special rep.
in 2-dim
essential rep.
[Craw-I-Karmazin]
Reid's recipe
$\left\{ \begin{array}{l} \text{non-triv.} \\ \text{irred. rep} \end{array} \right\} \longleftrightarrow \{E_i, l_i\}$ excep.
for abelian
6. non abelian case
McKay quiver
G-Hilb

1. McKay correspondence



the minimal resolution

$$\pi : Y \longrightarrow X = \mathbb{C}^2/G$$

$$\begin{array}{ccc} Y & & X \\ \cup & & \Psi \\ \mathbb{P}^1 & & p \end{array}$$

↑ exceptional divisor

where

$$Y - \mathbb{P}^1 \cong X - \{p\}$$

A_n singularity is given by

$$G = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \,\middle|\, \varepsilon^{n+1} = 1 \right\rangle$$

$$\mathbb{C}^2/G : f(x, y, z) = x^2 + y^2 + z^{n+1} = 0$$

and the minimal resolution

$$\pi : Y \longrightarrow X = \mathbb{C}^2/G$$

$$\begin{array}{ccc} Y & & X \\ \cup & & \Psi \\ E & & p \\ \vdots & & \end{array}$$

Dynkin diagram of A_n type

McKay's Observation

$G \subset SL(2,\mathbb{C})$ finite

$\{\rho_i\}$: irreducible representations

ρ_0 : trivial representation

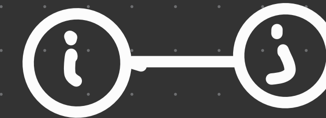
$\rho_{nat} : G \to SL(2,\mathbb{C})$
natural representation

tensor product

$$\rho_i \otimes \rho_{nat} \simeq \bigoplus a_{ij}\rho_j$$

$$\boxed{\underset{i,j \neq 0}{2E - (a_{ij})} = \text{Cartan matrix}}$$

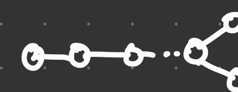
Let's try!

$a_{ij} \neq 0$ 

$a_{ij} = 0$ 

G :

cyclic  A_n

binary dihedral  D_n

binary tetrahedral  E_6

binary octahedral  E_7

binary icosahedral  E_8

Simple Lie algebra !!!
(Dynkin diagram)

McKay correspondence

(Gonzalez-Spinberg & Verdier)

$G \subset SL(2,\mathbb{C})$ finite

$$\tilde{X} \longrightarrow X := \mathbb{C}^2/G$$

the min. resolution

E_i

$\Updownarrow$ dual graph

i

Dynkin diagram

$$\{E_i\} \xleftrightarrow{1-1} \{\rho_i\}$$

exceptional curves

non-triv. irred. representations

(Wunram)

$\rightsquigarrow$ $G \subset GL(2,\mathbb{C})$ finite, small.

$$\tilde{X} \longrightarrow X = \mathbb{C}^2/G$$

the min. resolution

excep. curves

$\{E_i\}$

$\{$ non-trivial irred. rep. $\}$

$\cup$

$\updownarrow$ 1-1

$\{$ non-trivial irred. special rep. $\}$

2. 3-dimensional McKay correspondence

From String theory ~1985~

Orbifold Euler characteristic

$$\chi(M,G) = \frac{1}{|G|}\sum_{gh=hg} \chi^{\langle h,g\rangle}$$

$M=\mathbb{C}^3$, $G\subset SL(3,\mathbb{C})$

$\chi(\mathbb{C}^3,G) = \#\{\text{conjugacy class of } G\}$

Theorem (Markushevich[3], Roan[3], I[2])

$G\subset SL(3,\mathbb{C})$ finite

$\exists$ crepant resolution $Y\to X=\mathbb{C}^3/G$

s.t. $\chi_{top}(Y) = \#\{\text{conj. class of } G\}$

→ in Math

3-dim. McKay correspondence

[I.-Reid], [I-Nakajima].

[Bridgeland · King · Reid]

→ Higher dim.

Derived etc. ...

in Physics ~2020~

use crepant resol.

of non-abelian quotient.

[Tian – Wang] etc.

Crepant resolution

$G \subset SL(3, \mathbb{C})$ finite $\Rightarrow$ $^\exists$ crepant resolution

$$\widetilde{X} \longrightarrow X = \mathbb{C}^3/G$$

$$(K_{\widetilde{X}} \sim 0) \text{ "Calabi-Yau"}$$

singularity of X

- in general, non-isolated, non-hypersurface
- $^\exists$ crepant resolutions (not unique)

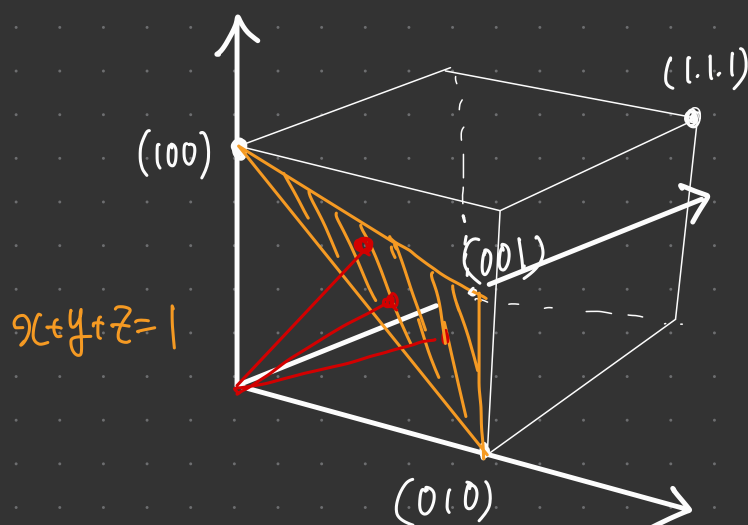
$$\text{flop} \updownarrow \quad \begin{matrix} \widetilde{X}_1 \longrightarrow \\ \widetilde{X}_2 \longrightarrow \\ \widetilde{X}_3 \nearrow \end{matrix} \quad X = \mathbb{C}^3/G$$

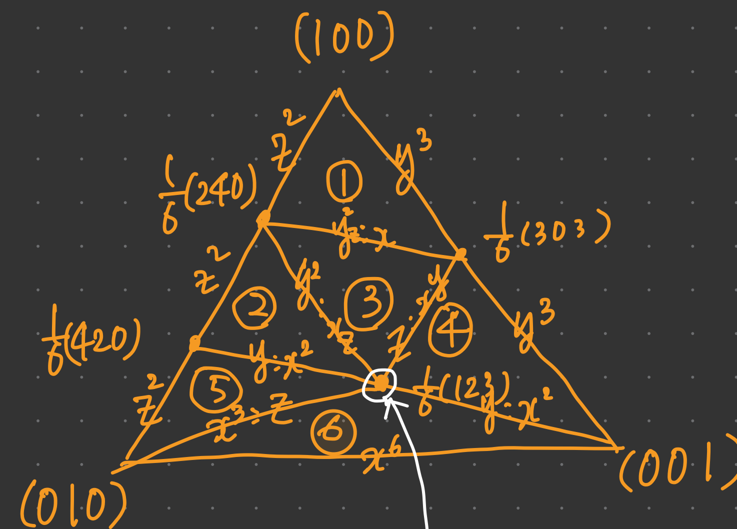
3. Example $\frac{1}{6}(1,2,3)$ … $G = \left\langle \begin{pmatrix} \varepsilon & & \\ & \varepsilon^2 & \\ & & \varepsilon^3 \end{pmatrix} \;\middle|\; \varepsilon^6 = 1 \right\rangle$

Toric geometry

$$N = \mathbb{Z}^3 + \frac{1}{6}(1\,2\,3)\,\mathbb{Z}$$

$$\sigma = \sum a_i e_i \quad (a_i \geq 0)$$





Euler # = 6

one exceptional divisor

4. G-Hilbert scheme.

$$G\text{-Hilb}(\mathbb{C}^n) = \{ \text{ideal } I \subset \mathbb{C}[x_1,..,x_n] \mid \mathbb{C}[x_1,...,x_n]/I \cong \mathbb{C}[G] \}$$

$$\cong \text{moduli space of } G\text{-clusters}$$

($\longleftrightarrow$ 0-generated quiver)

Known results

$n=2$: $G\text{-Hilb}(\mathbb{C}^2) \to \mathbb{C}^2/G$ the min. resolution.

$G \subset GL(2, \mathbb{C})$ (I-Nakamura, Kido. Ishii) (G: small)

$n=3$: $G\text{-Hilb}(\mathbb{C}^3) \to \mathbb{C}^3/G$ crepant resolution

$G \subset SL(3, \mathbb{C})$ [projective crepant resolution

$\cong$ moduli space of G-constellation

[Craw-Ishii], [Yamagishi]]

5. Essential representation

★ Special representation (Wunram)

$G\text{-Hilb}(\mathbb{C}^2) \to \mathbb{C}^2/G$: the min. resolution

Look at G-clusters

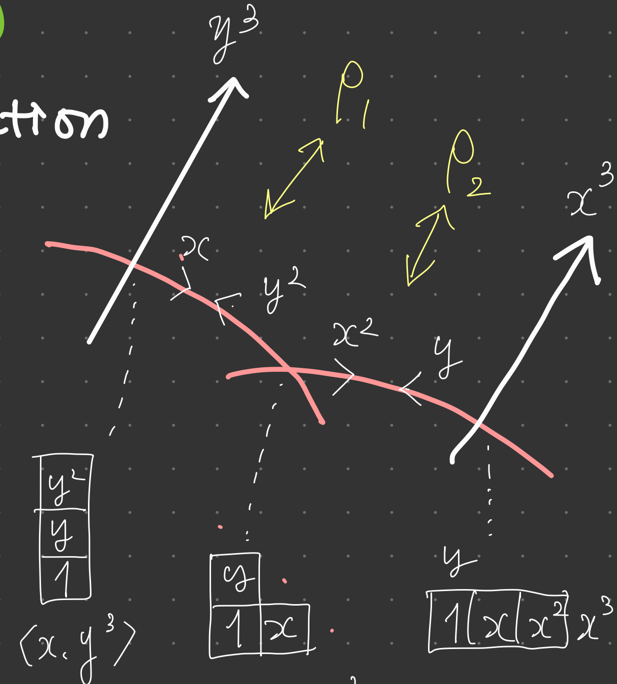
A_n case $n=2$

consider ideals in $G\text{-Hilb}(\mathbb{C}^2)$
I

G-cluster $\longleftrightarrow I^c$
(G-graph)

$G = \mathbb{Z}/3\mathbb{Z}$ $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \cdot \frac{1}{3}(1,2)$

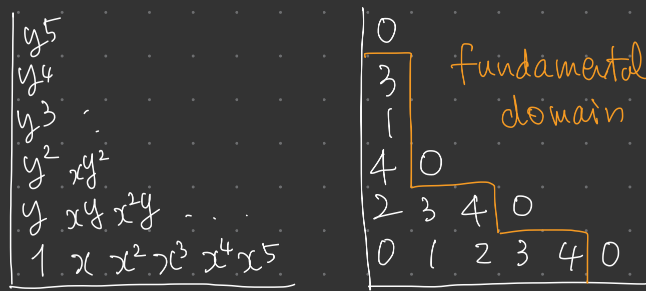
$$\begin{array}{l|llll} & \vdots & & & \\ y^2 & & & & \\ y & xy & & \cdots & \\ 1 & x & x^2 & x^3 & \\ \hline \end{array} \longleftrightarrow \begin{array}{l|llll} 0 & & & & \\ 1 & & & & \\ 2 & 0 & & & \\ 0 & 1 & 2 & 0 & \\ \hline \end{array}$$

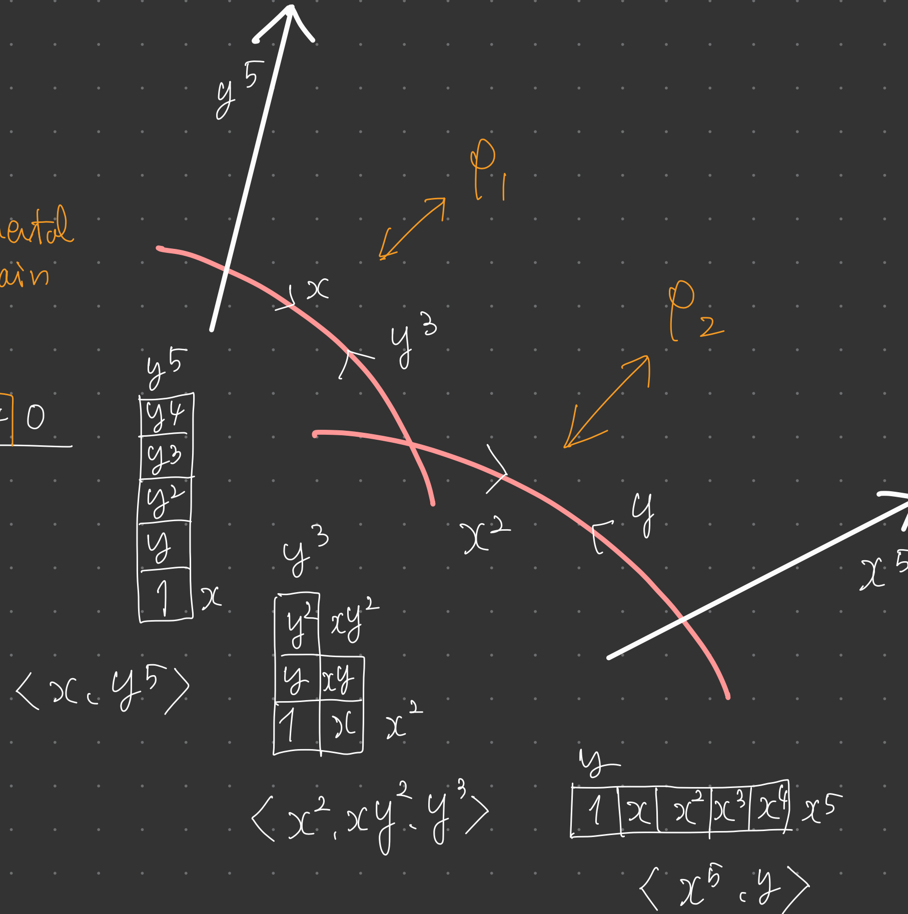


$\langle x^2, xy, y^2\rangle$ $\langle x^3, y\rangle$

$x = \alpha y^2$
$y^3 = \beta$
$xy = \alpha\beta$ $(\alpha:\beta)$

Cyclic Case $\frac{1}{5}(1,2)$



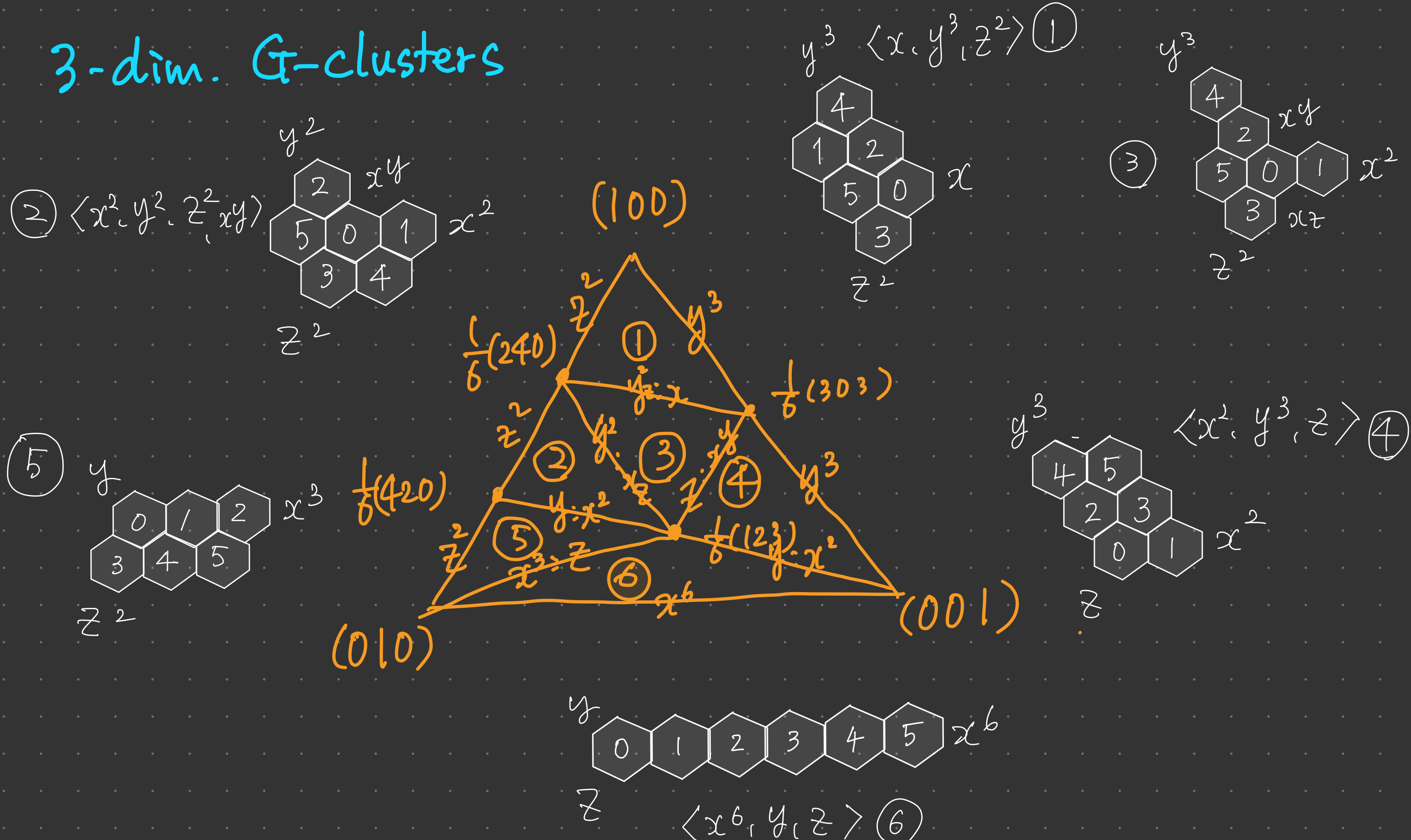


nontrivial

ρ_1, ρ_2 : special rep.

$(\rho_3, \rho_4$: not special.$)$

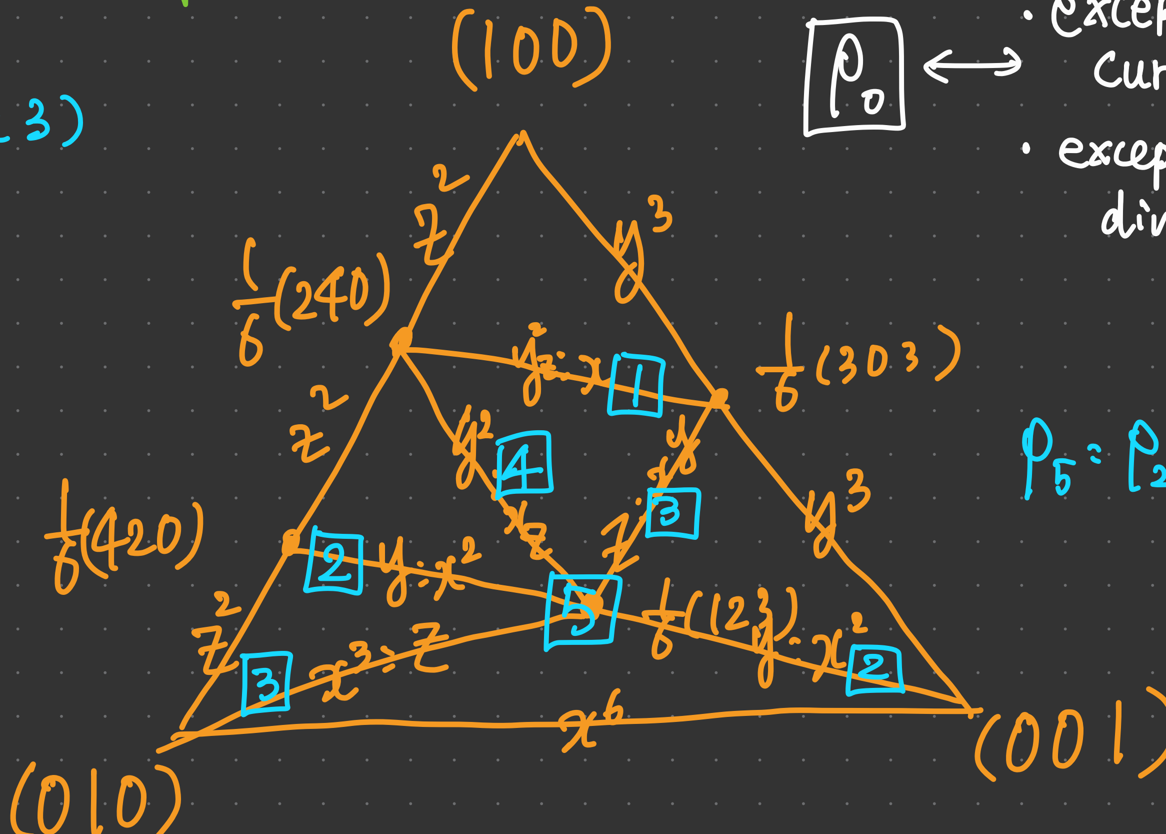
3-dim. G-clusters



Reid's recipe

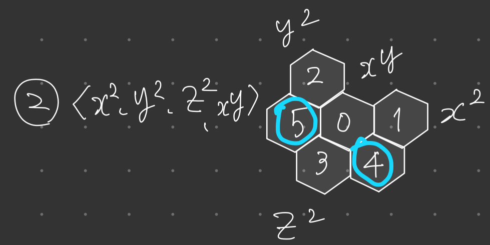
$\frac{1}{6}(1\,2\,3)$

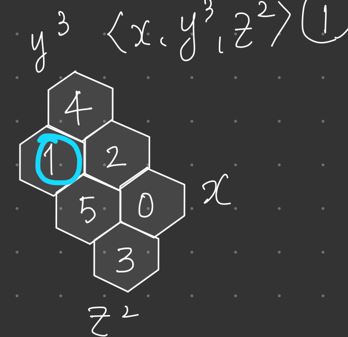
$\boxed{\rho_0} \longleftrightarrow$
- exceptional curve $\boxed{1} \sim \boxed{4}$
- exceptional divisor $\boxed{5}$

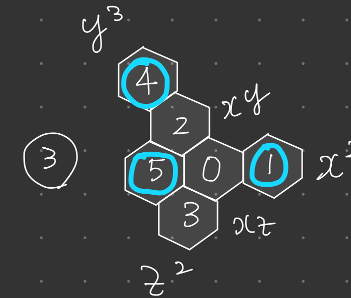


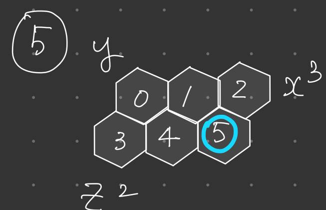
$\rho_5 = \rho_2 \otimes \rho_3$

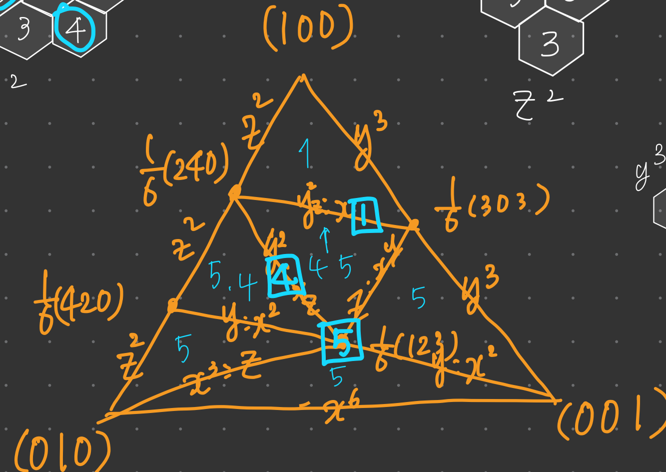
essential representation

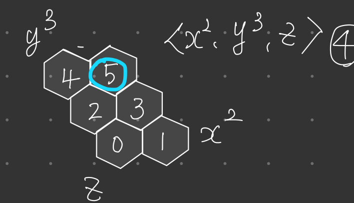


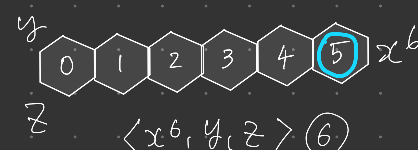












essential rep. was defined by [Craw-I-Karmazin]

→ moduli sp. with essential rep.

$\cong$ crepant resol. of $\mathbb{C}^3/G$

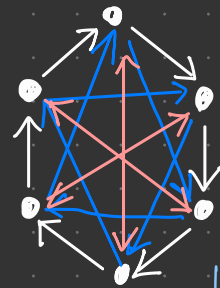
Theorem (I-Sato-Sato)

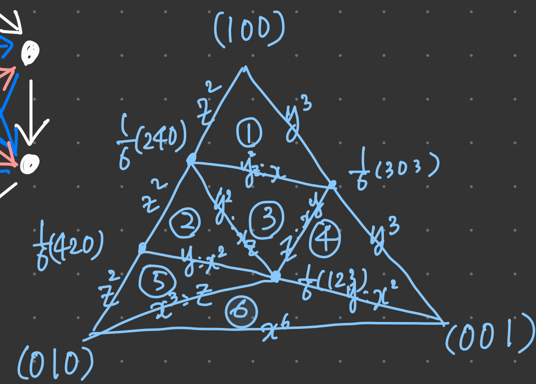
Essential rep

$\updownarrow$

excep. divisor & flopping curve

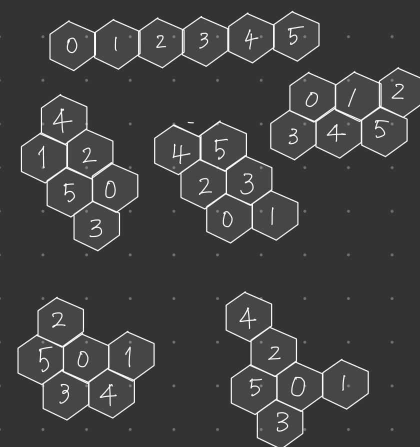
6. Non-abelian case





Want to see the geometric structure of G-Hilb$(\mathbb{C}^3)$!

G: abelian $\Rightarrow$ can use toric geometry



Non-abelian case

McKay quiver may be useful!

$$\rho_i \otimes \rho_{nat} = \sum a_{ij}\, \rho_j$$

$$a_{ij} \neq 0 \Rightarrow i \xrightarrow{a_{ij}} j$$

8. Higher dimensional case

Theorem (Batyrev)

If $G \subset SL(n, \mathbb{C})$: finite and $Y \to X = \mathbb{C}^n/G$: crepant resol, then $\chi_{top}(Y) = \#\{\text{conj. class}\}$

When $G \subset GL(n, \mathbb{C})$,

$$\chi(\mathbb{C}^n, G) = \#\{\text{conjugacy class}\}$$
$$\neq \chi_{top}(Y)$$

Batyrev defined Stringy Orbifold Euler number etc.

Q1 Existence of a crepant resolution

Q2 $G \subset SL(n, k)$ $\mathrm{char}(k) > 0$