The LYZ equation and the CJY conjecture

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1. Definition.

- At the most fundamental level, mirror symmetry describes a framework for relating complex geometry to symplectic geometry on two Calabi-Yau manifolds.

- It defines a duality between the underlying structures on each manifold, and the development of this elegant theory has lead to progress in both physics and mathematics.

- One particular aspect of mirror symmetry we are interested in is the relationship between the derived category of coherent sheaves on one manifold, and Fukaya's category of Lagrangian submanifolds with local systems on the other.

- This duality can be approached from a differential geometric perspective.

- In the simple case of a torus, C. Leung, E. Zaslow and I gave an explicit formulation of an equation on a line bundle that corresponds to the special Lagrangian equation on the mirror.

- Specifically, let X and $\check{X}$ be dual torus fibrations over a base tori B. In this semi-flat setting, a connection A on a holomorphic line bundle L over X is dual to a Lagrangian section $\mathcal{L}$ of the torus fibration $\check{X} \to B$.

- If $\mathcal{L}$ is a special Lagrangian section, using the Fourier-Mukai Transform, Leung-Yau-Zaslow show that A must satisfy the equation:

$$\mathrm{Im}(\omega - F)^n = \tan\hat{\theta}\,\mathrm{Re}(\omega - F)^n, \tag{1}$$

where ω is the Kähler form on X, F is the curvature of the connection A, and $\hat{\theta}$ is the phase of the special Lagrangian.

- Hence equation (1) is called the LYZ equation instead of the deformed Hermitian Yang-Mills (dHYM) equation.

- Such an equation is also derived from supersymmetry by M. Marino, R. Minasian, G. Moore, and A. Strominger.

- Now let (X, ω) be a compact Kähler manifold of complex dimension n and χ a closed real $(1,1)$-form on X.

- The LYZ equation on (X, ω, χ) is

$$\text{Re}(\chi_u + \sqrt{-1}\omega)^n = \cot\hat{\theta}\ \text{Im}(\chi_u + \sqrt{-1}\omega)^n. \tag{2}$$

- Here

$$\chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u$$

for a real smooth function u on M and $\hat{\theta}$ is the argument of the complex number

$$\int_X (\chi + \sqrt{-1}\omega)^n.$$

- Let

$$\lambda = (\lambda_1, \ldots, \lambda_n)$$

be the eigenvalues of χ_u with respect to ω. Let

$$\theta_i = \mathrm{arccot}\lambda_i \quad \text{for } 1 \le i \le n.$$

- Then $\lambda_i = \cot\theta_i$ and we can compute as follows:

$$\begin{aligned}
&(\chi_u + \sqrt{-1}\omega)^n \\
=& \prod_{i=1}^{n} (\lambda_i + \sqrt{-1})\,\omega^n \\
=& \frac{\exp\left(\sqrt{-1}\sum_{i=1}^{n}\theta_i\right)}{\prod_{i=1}^{n}\sin\theta_i}\,\omega^n \\
=& \frac{\cos\left(\sum_{i=1}^{n}\theta_i\right)}{\prod_{i=1}^{n}\sin\theta_i}\omega^n + \sqrt{-1}\frac{\sin\left(\sum_{i=1}^{n}\theta_i\right)}{\prod_{i=1}^{n}\sin\theta_i}\omega^n.
\end{aligned}$$

- So the LYZ equation becomes

$$\cos\Big(\sum_{i=1}^{n}\theta_i\Big) = \cot\widehat{\theta}\sin\Big(\sum_{i=1}^{n}\theta_i\Big),$$

or

$$\cot\Big(\sum_{i=1}^{n}\theta_i\Big) = \cot\widehat{\theta}.$$

- If we define

$$\theta_\omega(\chi_u) := \sum_{i=1}^{n}\theta_i = \sum_{i=1}^{n}\operatorname{arccot}\lambda_i,$$

then the LYZ equation is simply written as

$$\theta_\omega(\chi_u) = \theta_0. \tag{3}$$

- We assume $\theta_0 = \widehat{\theta} \in (0, \pi)$.

- In this formulation it is clear that the equation is the complex version of the special Lagrangian equation for graphs, with θ_0 the analog of the Lagrangian angle and λ_j the analogs of the eigenvalues of the Hessian of the generating function of graph.

- The LYZ equation has been extensively studied by many mathematicians.

- In 2014, Jacob and I initiated to study the LYZ equation.

A. Jacob, S.-T. Yau. A special Lagrangian type equation for holomorphic line bundles. Math. Ann. 369 (2017), 869-898. arXiv:1411.7457.

2. Analytic condition: existence of a subsolution.

- When $n = 2$, Jacob and I proved that the solution exists if and only if there exists a function $\underline{u}$ such that

$$\chi_{\underline{u}} > \cot\theta_0\,\omega.$$

- Denote

$$\alpha = \chi_{\underline{u}} - \cot\theta_0\,\omega.$$

Then $\alpha > 0$ and it is a Kähler metric.

- Jacob and I then translated the LYZ equation into the complex Monge-Ampère equation

$$(\alpha + i\partial\bar{\partial}u)^2 = \frac{1}{\sin^2\theta_0}\omega^2$$

which was solved by myself in 1976.

- When $n \geq 3$, Collins, Jacob and I solved the LYZ equation for $\theta_0 \in (0, \pi)$ by assuming that the following two conditions hold.

(1) There exists a subsolution $\underline{u}$, which means that $\chi_{\underline{u}}$ satisfies the inequality

$$\max_{1 \leq j \leq n} \sum_{i \neq j} \operatorname{arccot} \lambda_i(\chi_{\underline{u}}) < \theta_0 (< \pi). \tag{4}$$

(2) The function $\underline{u}$ also satisfies an extra condition:

$$\theta_\omega(\chi_{\underline{u}}) \Big(= \sum_{i=1}^{n} \operatorname{arccot} \lambda_i(\chi_{\underline{u}}) \Big) < \pi. \tag{5}$$

T. Collins, A. Jacob, S.-T. Yau. (1,1) forms with specified Lagrangian phase: a priori estimates and algebraic obstructions. Camb. J. Math. 8 (2020), 407-452. arXiv:1508.01934.

- To be precise, Collins, Jacob and I proved the following

- **Theorem 1.** (Collins-Jacob-Yau) Let (X, ω) be a compact Kähler manifold of dimension n and χ a closed real $(1,1)$-form on X with $\theta_0 \in (0, \pi)$. Suppose there exists a subsolution $\underline{u}$ of LYZ equation (2) in the sense of (4) and $\underline{u}$ also satisfies inequality (5). Then there exists a unique smooth solution of the LYZ equation.

- The definition of subsolutions of fully nonlinear equations was introduced by B. Guan.

- G. Székelyhidi gave an equivalent version and Collins, Jacob and I used it to the LYZ equation which is equivalent to inequality (4).

B. Guan. Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. Duke Math. J. 163(2014), 1491-1524.

G. Székelyhidi. Fully non-linear elliptic equations on compact Hermitian manifolds. J. Differential Geom. 109(2018), 337-378.

- L. Huang-J. Zhang-X. Zhang also considered the solution on a compact almost Hermitian manifold for the case $\theta_0 \in (0, \frac{\pi}{2})$.

- C.-M. Lin generalized Collins-Jacob-Yau's result to the Hermitian manifold (X, ω) with $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$.

L. Huang, J. Zhang, X. Zhang. The deformed Hermitian-Yang-Mills equation on almost Hermitian manifolds. Sci. China Math. 65(2021), 127-152. arXiv:2011.14091.

C.-M. Lin. Deformed Hermitian-Yang-Mills equation on compact Hermitian manifolds. arXiv:2012.00487.

- Now the natural question is whether condition (5) is superfluous.

- When $n = 3$, Pingali can solve the LYZ equation without condition (5) by translating the LYZ equation into a mixed Monge-Ampère type equation.

- When $n = 3$ and $n = 4$, C.-M. Lin can solve the LYZ equation without condition (5).

V. P. Pingali. The deformed Hermitian Yang-Mills equation on three-folds. arXiv:1910.01870.

C.-M. Lin. The deformed Hermitian-Yang-Mills equation, the positivstellensatz, and the solvability. arXiv:2201.01438v2.

- Very recently, C.-M. Lin solved the LYZ equation without condition (5).

- **Theorem 2.** (Lin) Let (X, ω) be a compact Kähler manifold of dimension n and χ a closed real $(1,1)$-form on M with $\theta_0 \in (0, \pi)$. If there exists a subsolution $\underline{u}$ of LYZ equation (2) in the sense of (4), then there exists a unique smooth solution of the LYZ equation.

C.-M. Lin. On the Solvability of General Inverse σ_k Equations. arXiv:2310.05339.

3. Geometric condition: A Nakai-Moishezon type criterion.

- Denote

$$\Omega_\omega(\chi, p) = \mathrm{Re}(\chi + \sqrt{-1}\omega)^p - \cot\theta_0 \mathrm{Im}(\chi + \sqrt{-1}\omega)^p$$

- Define

$$\mathcal{P}_\omega = \{[\chi] \in H^{1,1}_{\mathbb{R}}(X) \mid \int_Y \Omega_\omega(\chi, p) > 0 \text{ for any } p\text{-subvarity } Y,\ 0 < p < n\}$$

$$\mathcal{S}_\omega = \{[\chi] \in H^{1,1}_{\mathbb{R}}(X) \mid \text{the LYZ equation has a smooth solution}\}.$$

- **Conjecture.** (Collins-Jacob-Yau) For any compact Kähler manifold (X, ω),

$$\mathcal{P}_\omega = \mathcal{S}_\omega.$$

- G. Chen made an important progress on CJY conjecture.

- A smooth family χ_t, $t \in [0, \infty)$ of real closed $(1,1)$-forms is called a test family if and only if all the following conditions hold.

  (A) When $t = 0$, $\chi_0 = \chi$.

  (B) For all $s > t$, $\chi_s - \chi_t$ is positive definite.

  (C) There exists a large enough number $T \geq 0$ such that for all $t \geq T$, $\chi_t - \cot \frac{\theta_0}{n} \omega$ is positive definite.

G. Chen. The J-equation and the supercritical deformed Hermitian-Yang-Mills equation. Invent. Math. **225** (2021), 529-602.

- **Theorem 3.** (Chen) Let (X, ω) be a compact Kähler manifold of dimension n and χ a closed real $(1,1)$-form on X with $\theta_0 \in (0, \pi)$. Then the follwing statements are equivalent.

  (1) There exists a smooth solution of LYZ equation (3).

  (2) For any smooth test family χ_t, there exists a constant $\varepsilon_{1,1} > 0$ such that for any $t \geq 0$ and p-dimensional subvariety Y,

$$\int_Y \Omega_\omega(\chi_t, p) \geq (n-p)\varepsilon_{1,1} \int_Y \omega^p.$$

  (3) There exist a test family χ_t and a constant $\varepsilon_{1,1} > 0$ such that for any $t \geq 0$ and p-dimensional subvariety Y,

$$\int_Y \Omega_\omega(\chi_t, p) \geq (n-p)\varepsilon_{1,1} \int_Y \omega^p.$$

J.-P. Demailly, M. Paun. Numerical characterization of the Kähler cone. Ann. of Math. (2) 159 (2004), 1247-1274.

J. Song. Nakai-Moishezon criterions for complex Hessian equations. arxiv: 2012.07956.

- Motivated by Chen and Song, Chu-Lee-Takahashi established

- **Theorem 4.** (Chu-Lee-Takahashi) The LYZ equation on a compact Kähler manifold (X, ω) with complex dimension n is solvable for $\theta_0 \in (0, \pi)$ if and only if there exists a Kähler metric γ on X such that for any $1 \leq p \leq n$,

$$\int_X \Omega_\omega(\chi, p) \wedge \gamma^{n-p} \geq 0$$

and for any proper m-dimensional subvariety Y of X and $1 \leq p \leq m$,

$$\int_Y \Omega_\omega(\chi, p) \wedge \gamma^{m-p} > 0.$$

- Chu-Lee-Takahashi then confirmed the CJY conjecture for projective manifolds.

J. Chu, M.-C. Lee, R. Takahashi. A Nakai-Moishezon type criterion for supercritical deformed Hermitian-Yang-Mills equation. arxiv:2105.10725.

- Junsheng Zhang proved that $\mathcal{S}_\omega$ is a both open and closed subset of $\mathcal{P}_\omega$. He then disproved the CJY conjecture in the non-projective case. The counter-example is the blow up of $\mathbb{C}^3/\Lambda$ at a point. Here $\mathbb{C}^3/\Lambda$ is a three dimension torus without any positive dimensional proper analytic subvariety.

- The situation is very much like the work of Demailly-Paun, where they used my solution to the Calabi conjecture to give a differential geometric proof of the Nakai-Moishezon numerical criterion for ample line bundles in algebraic geometry.

J. Zhang. A note on the supercritical deformed Hermitian-Yang-Mills equation. arXiv:2302.06592.

4. Algebraic condition: Stability.

- Collins and I developed the mirror of the infinite dimensional GIT picture for the LYZ equation.

- We described a infinite dimensional symplectic manifold, admitting an action by a group of symplectomorphisms, together with a space $\mathcal{H} \subset C^{\infty}(X, \mathbb{R})$ and a Riemannian structure on $\mathcal{H}$, which can be thought of as analogous to G/K in the finite dimensional GIT.

- We computed the geodesic equation, and introduce a notion of ε-geodesics, which solve an approximate version of the geodesic equation.

T. Collins and S.-T. Yau. Moment maps, nonlinear PDE, and stability in mirror symmetry. arXiv:1811.04824.

- We also introduced the complexified Calabi-Yau functional, and extract from this functional analogues of the $\mathcal{C}$, and $\mathcal{J}$ functionals, as well as a $\mathbb{C}$-valued functional Z. The $\mathcal{J}$ functional is the Kempf-Ness functional for the GIT problem; it has critical points at solutions of the LYZ equation, and is convex along smooth geodesics.

- A fundamental issue in the analogy with finite dimensional GIT is that smooth geodesics need not exist.

- Thus, our main analytic contribution is to prove, in the hypercritical phase case, the existence of weak geodesics connecting points in $\mathcal{H}$, with $C^{1,\alpha}$ regularity. With this much regularity, we can show that the functionals $\mathcal{J}$, $\mathcal{C}$, Z are well-defined and we prove that they are convex/concave along these generalized geodesics.

- With these results in hand, we are in the setting of an infinite dimensional GIT problem with geodesics playing the role of one-parameter subgroups.

- Using algebraic geometry we constructed model infinite rays, analogous to one-parameter subgroups in the space $\mathcal{H}$, and evaluate the limit slope of the Calabi-Yau functional along these model curves in terms of algebraic data.

- Using the existence of regular geodesics these model curves give rise to algebro-geometric obstructions to the existence of solutions to LYZ.

- For example, we established the following theorem in the hypercritical phase case, i.e. $\theta_0 \in (0, \frac{\pi}{2})$.

- **Theorem 5.** (Collins-Yau) Let $\mathfrak{J}_1 \subset \mathfrak{J}_2 \subset \cdots \subset \mathfrak{J}_{r-1} \subset \mathfrak{J}_r = \mathcal{O}_X$ be a sequence of ideal sheaves, and define

$$\mathfrak{J} = \mathfrak{J}_0 + t \cdot \mathfrak{J}_1 + \cdots + t^{r-1} \cdot \mathfrak{J}_{r-1} + (t^r) \subset \mathcal{O}_X \otimes \mathbb{C}[t].$$

Let $\mu : \mathcal{X} \to X \times \triangle$ be a log resolution of $\mathfrak{J}$, so that $\mu^{-1}\mathfrak{J} = \mathcal{O}_{\mathcal{X}}(-E)$ for a simple normal crossing divisor E. If $[\chi]$ admits a solution of the LYZ equation then

$$E.\mathrm{Im}\left[\frac{(\mu^*[\omega] + \sqrt{-1}(\mu^*[\chi] - \delta E))^n}{(\omega + \sqrt{-1}\chi)^n.[X]}\right] \geq 0$$

5. Parabolic version.

- For the parabolic flow method, there are also several results.

- Jacob-Yau and Collins-Jacob-Yau proposed the line bundle mean curvature flow (LBMCF)

$$\begin{cases} u_t = \theta_0 - \theta_\omega(\chi_u) \\ u(0) = \underline{u} \end{cases} \tag{6}$$

  where $\underline{u}$ is a subsolution of the LYZ equation.

- They proved the existence and convergence of the long-time solution to the LBMCF for the case $\theta_0 \in (0, \frac{\pi}{2})$ and also $\theta_\omega(\chi_{\underline{u}}) \in (0, \frac{\pi}{2})$.

- Takahashi proposed the tangent Lagrangian phase flow (TLPF)

$$\begin{cases} u_t = \tan(\theta_0 - \theta_\omega(\chi_u)) \\ u(0) = \underline{u} \end{cases} \tag{7}$$

where $\underline{u}$ is a subsolution of the LYZ equation.

- He proved the existence and convergence of the long-time solution to the TLPF for the case $\theta_0 \in (0, \frac{\pi}{2})$ and also $\theta_\omega(\chi_{\underline{u}}) - \theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

R. Takahashi. Tan-concavity property for Lagrangian phase operators and applications to the tangent Lagrangian phase flow. Internat. J. Math. 31 (2020), 26 pp.

- Motivated by the concavity of $\cot\theta_\omega(\chi_u)$ by G. Chen, Fu, Zhang and I considered a new flow:

$$\begin{cases} u_t = \cot\theta_\omega(\chi_u) - \cot\theta_0, \\ u(x,0) = u_0(x) \end{cases} \tag{8}$$

with

$$\theta_\omega(\chi_{u_0}) < \pi.$$

G. Chen. The J-equation and the supercritical deformed Hermitian-Yang-Mills equation. Invent. Math. **225** (2021), 529-602.

J. Fu, S.-T. Yau and Dekai Zhang. A deformed Hermitian Yang-Mills flow. Accepted for publication in JDG. arXiv:2105.13576v3.

- We first proved the existence theorem of the long-time solution of flow (8).

- **Theorem 6.** (Fu-Yau-Zhang) Let (X, ω) be a compact Kähler manifold and χ a closed real $(1,1)$-form with $\theta_0 \in (0, \pi)$. If u_0 satisfies the inequality

$$\theta_\omega(\chi_{u_0}) < \pi,$$

then flow (8) has a unique smooth long-time solution u.

- Then we considered the convergence of the long-time solution of flow (8).

- **Theorem 7.** (Fu-Yau-Zhang) Let (X, ω) be a compact Kähler manifold and χ a closed real $(1,1)$ form with $\theta_0 \in (0, \pi)$. Suppose that exists a subsolution $\underline{u}$ of LYZ equation (3) in the sense of (4) which also satisfies (5). Then there exists a long-time solution $u(x,t)$ of flow (8) with $u_0 = \underline{u}$ and it converges to a smooth solution u^∞ to the LYZ equation:

$$\theta_\omega(\chi_{u^\infty}) = \theta_0.$$

- Hence we reproved the Collins-Jacob-Yau's existence theorem. Our parabolic version seems rather natural and the proof is simpler.

- The advantage of the new flow is that the imaginary part of the Calabi-Yau functional is constant along the flow, which is the key to do the C^0 estimate.

- We in fact identified which assumption is needed for the long-time existence part and which one is needed for the convergence part.

- The second motivation of our paper is to use flow (8) to study the equation under the existence of a semi-subsolution:

$$\max_{1\le j\le n}\sum_{i\ne j}\operatorname{arccot}\lambda_i(\chi_{\underline{u}})\le\theta_0(<\pi).$$

- We can solve the two dimensional case. In this case, a smooth function $\underline{u}$ is called a semi-subsolution of the LYZ equation if

$$\chi_{\underline{u}}\ge\cot\theta_0\,\omega. \tag{9}$$

- We hope our flow is also useful to solve the higher dimensional case. This is analog to the semi-stable case of the HYM equation. The HYM flow proposed by Donaldson is very useful in study of the HYM equation in the semi-stable case.

6. Further Discussions.

- If for some reason, we can solve the LYZ equation, then we can derive important consequences for the algebraic manifolds. Let me give examples.

- Five years ago, T. Collins and I observed that if the projective manifold is covered by a homogenous manifold, we can solve the LYZ equation by using homogeneity to reduce to pointwise calculations. (Good examples are abelian varieties or Shimura varieties.)

- Once it is done, the invariant Kähler forms and the invariant (1,1)-forms together give rise to a range of Chern number inequalities over algebraic subvarieties. These are not trivial inequalities.

- It would be nice to find out the cases when those inequalities become equalities. If a closed (p, p)-form can be represented by an effective algebraic subvariety, those inequalities should give non trivial information about the Hodge conjecture.

- On the other hand, for the complexified Kähler cone of X, we consider its open subset

$$\mathcal{C} = \{\omega + i\chi \mid \omega \in \mathcal{K} \text{ and } [\chi] \in \mathcal{S}_\omega\}.$$

- We propose the following equations:

  – Is the open set $\mathcal{C}$ is pseudo-convex (Stein)?

  – If not, what is the automorphism group of the pseudo-convex hull of $\mathcal{C}$? In general, is it discrete?

Thank you!