

格子を用いて見える関数の性質 - Schur 関数とゼータ関数への利用-

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The world of Mathematical Sciences at IPMU

Introduction

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研究分野：解析数論, 表現論

ゼータ/Schur 多重ゼータ関数, Iwahori-Whittaker 関数



[Women In



Number theory Japan]

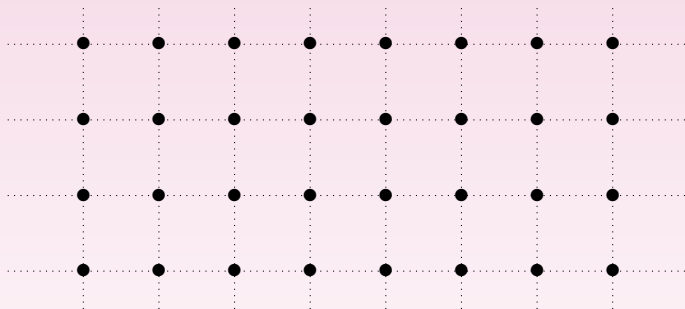
1 ゼータ/Schur 多重ゼータ関数：

リーマンゼータ関数を Schur 関数の構造を取り入れて組合せ論的に多変数化した Schur 多重ゼータ関数の性質の研究. 数論, 組合せ論, 表現論など, 色々な方向からのアプローチで研究に取り組んでいます.

2 Iwahori-Whittaker 関数：

p 進群上の Iwahori 部分群における作用で不変な主系列表現の性質についての研究. 特に, 絡作用素によって決まる基底に興味があります.

Lattice and Functions



Schur 多項式, リーマンゼータ関数

$\mathbf{x} = (x_1, x_2, \dots, x_n)$, $s \in \mathbb{C}$ に対し,

$$s_\lambda = s_\lambda(\mathbf{x}) = \frac{\det(x_j^{n-i+\lambda_i})}{\det(x_j^{n-i})}, \quad \zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$$

Schur polynomial

Schur 多項式

$\lambda = (\lambda_1, \lambda_2, \dots)$ s.t. $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, $\mathbf{x} = (x_1, x_2, \dots)$ に対し,

$$s_\lambda = s_\lambda(\mathbf{x}) = \frac{\det(x_j^{n-i+\lambda_i})}{\det(x_j^{n-i})},$$

例. $\lambda = (2, 1)$, $\mathbf{x} = (x_1, x_2, x_3)$ のとき. ($n = 3$)

$$s_\lambda = s_\lambda(\mathbf{x}) = \frac{\det(x_j^{3-i+\lambda_i})}{\det(x_j^{3-i})}$$

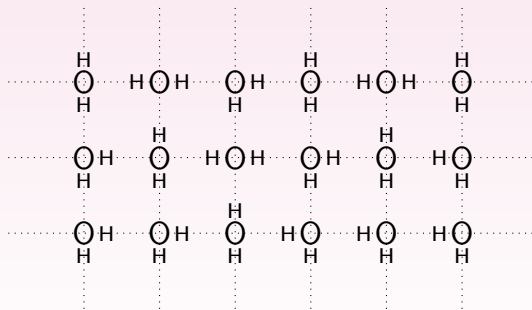
$$\text{分子} = \begin{vmatrix} x_1^{3-1+\lambda_1} & x_2^{3-1+\lambda_1} & x_3^{3-1+\lambda_1} \\ x_1^{3-2+\lambda_2} & x_2^{3-2+\lambda_2} & x_3^{3-2+\lambda_2} \\ x_1^{3-3+\lambda_3} & x_2^{3-3+\lambda_3} & x_3^{3-3+\lambda_3} \end{vmatrix}, \quad \text{分母} = \begin{vmatrix} x_1^{3-1} & x_2^{3-1} & x_3^{3-1} \\ x_1^{3-2} & x_2^{3-2} & x_3^{3-2} \\ x_1^{3-3} & x_2^{3-3} & x_3^{3-3} \end{vmatrix}$$

$$s_\lambda = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3$$

2-dimensional Crystal lattice

2次元結晶格子.

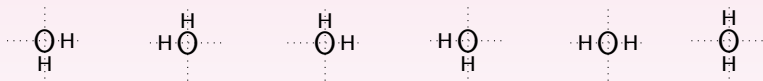
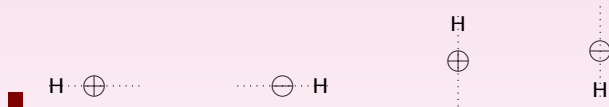
- 各頂点には1つの酸素原子(O)を配置
- 各辺には1つの水素原子(H)を配置
- 各酸素原子(O)には2つの水素原子(H)がつく



Pattern symbolization

パターンの記号化.

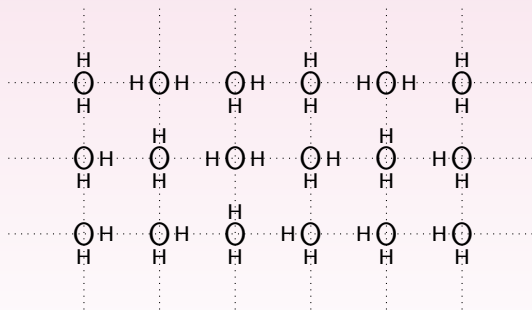
■ 各頂点の酸素原子 (O) を ● にする



2-dimensional Crystal lattice

2次元結晶格子

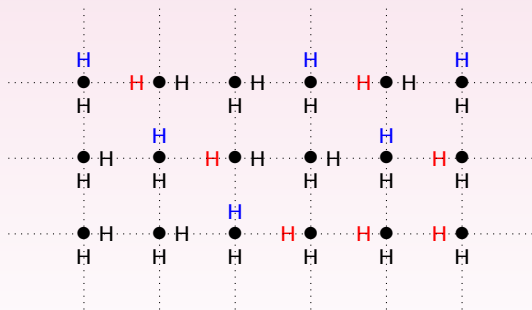
- 各頂点の酸素原子 (O) を ● にする
- 各辺にパターンに従って \oplus , \ominus をおく



2-dimensional Crystal lattice

2次元結晶格子

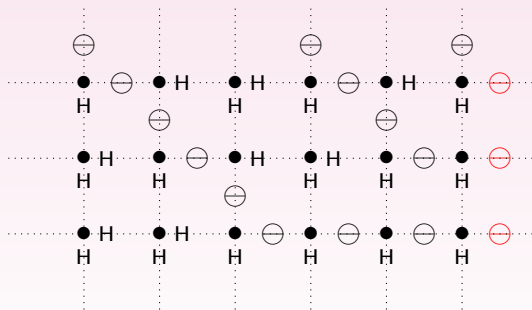
- 各頂点の酸素原子 (O) を ● にする
- 各辺にパターンに従って \oplus , \ominus をおく



2-dimensional Crystal lattice

2次元結晶格子

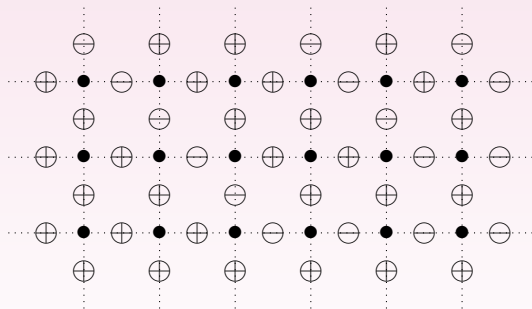
- 各頂点の酸素原子 (O) を ● にする
- 各辺にパターンに従って \oplus , \ominus をおく



2-dimensional Crystal lattice

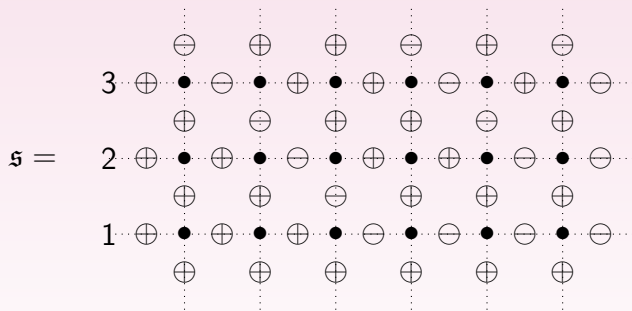
2次元結晶格子

- 各頂点の酸素原子 (O) を ● にする
- 各辺にパターンに従って \oplus , \ominus をおく



Boltzmann weight

型						
w	1	x_i	t_i	x_i	$x_i(t_i + 1)$	1

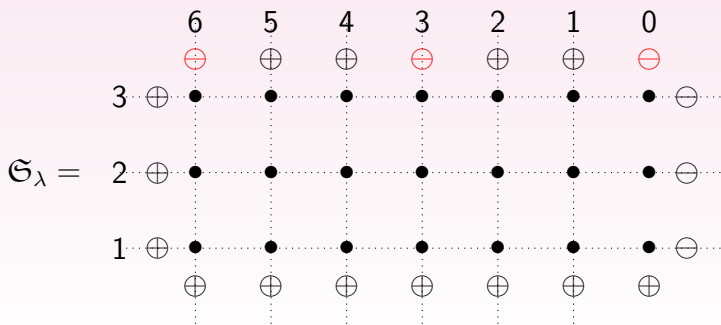


$$\begin{aligned}
 Z(\mathfrak{s}) = \prod_{v \in \mathfrak{s}} w(v, \mathfrak{s}) &= 1 \times x_3(t_3 + 1) \times 1 \times 1 \times x_3(t_3 + 1) \times 1 \\
 &\quad \times 1 \times 1 \times x_2(t_2 + 1) \times 1 \times 1 \times x_2 \\
 &\quad \times 1 \times 1 \times 1 \times 1 \times x_1 \times x_1
 \end{aligned}$$

Boundary condition

- 各頂点に \bullet をおく
- 左端の列のすべての辺には \oplus , 一番下の行のすべての辺には \oplus , 右端の列のすべての辺には \ominus をおく
- 一番上の行には, $\lambda_i + n - i$ 番目の列の辺には \ominus , それ以外の列の辺には \oplus をおく

例. $\lambda = (4, 2, 0)$, $\lambda + (3 - 1, 3 - 2, 3 - 3) = (6, 3, 0)$.



Lattice and Schur polynomial

定義 (分配関数)

$$Z(\mathfrak{G}_\lambda) = \sum_{\mathfrak{s} \in \mathfrak{G}_\lambda} \prod_{v \in \mathfrak{s}} w(v, \mathfrak{s})$$

とする. $w(v, \mathfrak{s})$ は, 配置 \mathfrak{s} における点 v のウエイト

定理 (Brubaker-Bump-Friedberg, 2010)

分割 $\lambda = (\lambda_1, \dots, \lambda_n)$ に対し,

$$Z(\mathfrak{G}_\lambda) = \prod_{i < j} (x_i + t_i x_j) s_\lambda(x_1, \dots, x_n).$$

Yang-Baxter equation

型 I						
型 II						
w	a_1	a_2	b_1	b_2	c_1	c_2

定理 (Brubaker-Bump-Friedberg, 2010)

$$a_1(u) = a_1(v)a_2(w) + b_2(v)b_1(w),$$

$$a_2(u) = b_1(v)b_2(w) + a_2(v)a_1(w),$$

$$b_1(u) = b_1(v)a_2(w) - a_2(v)b_1(w),$$

$$b_2(u) = -a_1(v)b_2(w) + b_2(v)a_1(w), \quad c_1(u) = c_1(v)c_2(w),$$

$c_1(u) = c_1(v)c_2(w)$ を満たすとき、次が成り立つ。

Yang-Baxter equation

任意の符号 $\varepsilon_i \in \{\pm\}$ ($i = 1, 2, 3, 4, 5, 6$) に対して,

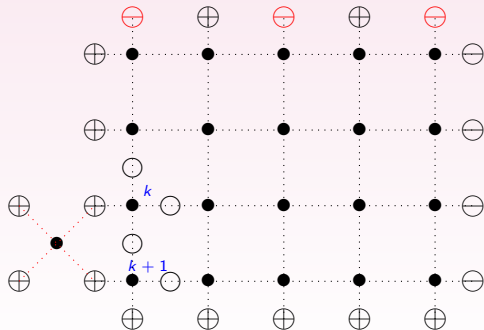
$$Z \left(\begin{array}{c} \varepsilon_3 \\ \text{---} v \text{---} v \text{---} \varepsilon_4 \\ \text{---} \gamma \text{---} \\ \text{---} w \text{---} \varepsilon_5 \\ \varepsilon_6 \end{array} \begin{array}{c} \varepsilon_2 \\ \text{---} v \\ \text{---} u \\ \text{---} \mu \\ \varepsilon_1 \end{array} \right) = Z \left(\begin{array}{c} \varepsilon_3 \\ \text{---} w \text{---} \psi \\ \text{---} \delta \text{---} \\ \text{---} v \text{---} \phi \\ \varepsilon_6 \end{array} \begin{array}{c} \varepsilon_4 \\ \text{---} u \\ \text{---} u \\ \text{---} u \\ \varepsilon_5 \end{array} \right)$$

型 I						
w	1	x_i	t_i	x_i	$x_i(t_i + 1)$	1
型 II						
w	$t_j x_i + x_j$	$t_i x_j + x_i$	$t_i x_j - t_j x_i$	$x_i - x_j$	$(t_i + 1)x_i$	$(t_j + 1)x_j$

Application of Yang-Baxter equation

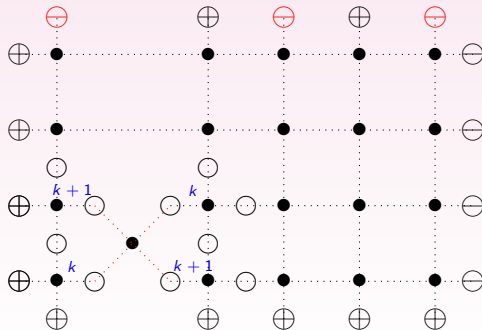
$$Z \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = Z \left(\begin{array}{c} \text{Diagram 2} \end{array} \right)$$

The equation shows the equality of two Z-operators. The left operator is a vertex with a central black dot labeled u . It has six external legs: ε_1 (bottom-left), ε_2 (top-left), v (top), ε_4 (top-right), μ (bottom), and ε_5 (bottom-right). Internal lines connect v to a dot labeled γ , and μ to a dot labeled w . The right operator is a vertex with a central black dot labeled u . It has six external legs: ε_1 (bottom-left), ε_2 (top-left), ε_3 (top), ε_4 (top-right), ε_5 (bottom-right), and ε_6 (bottom). Internal lines connect ε_2 to a dot labeled w , ε_3 to a dot labeled ψ , ε_4 to a dot labeled δ , and ε_5 to a dot labeled ϕ .



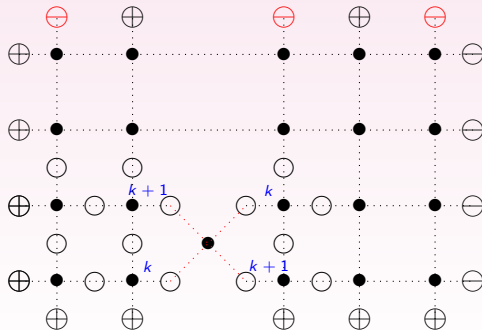
Application of Yang-Baxter equation

$$Z \left(\begin{array}{c} \textcircled{\varepsilon_3} \\ \textcircled{\varepsilon_2} \quad \textcircled{v} \quad \textcircled{v} \quad \textcircled{\varepsilon_4} \\ \bullet \\ \textcircled{\varepsilon_1} \quad \textcircled{\mu} \quad \textcircled{\gamma} \quad \textcircled{w} \quad \textcircled{\varepsilon_5} \\ \textcircled{\varepsilon_6} \end{array} \right) = Z \left(\begin{array}{c} \textcircled{\varepsilon_3} \\ \textcircled{\varepsilon_2} \quad \textcircled{w} \quad \textcircled{\psi} \quad \textcircled{\varepsilon_4} \\ \bullet \\ \textcircled{\varepsilon_1} \quad \textcircled{\delta} \quad \textcircled{\phi} \quad \textcircled{u} \quad \textcircled{\varepsilon_5} \\ \textcircled{\varepsilon_6} \end{array} \right)$$



Application of Yang-Baxter equation

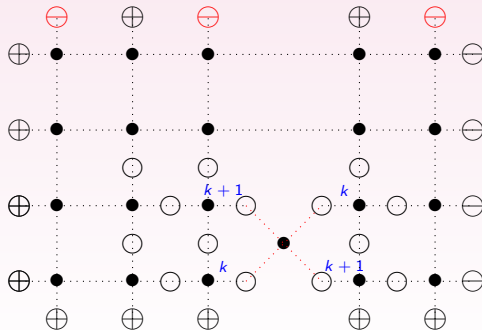
$$Z \left(\begin{array}{c} \textcircled{\varepsilon_3} \\ \textcircled{\varepsilon_2} \quad \textcircled{v} \quad \textcircled{v} \quad \textcircled{\varepsilon_4} \\ \bullet \\ \textcircled{\varepsilon_1} \quad \textcircled{\mu} \quad \textcircled{\gamma} \quad \textcircled{w} \quad \textcircled{\varepsilon_5} \\ \textcircled{\varepsilon_6} \end{array} \right) = Z \left(\begin{array}{c} \textcircled{\varepsilon_3} \\ \textcircled{\varepsilon_2} \quad \textcircled{w} \quad \textcircled{\psi} \quad \textcircled{\varepsilon_4} \\ \bullet \\ \textcircled{\varepsilon_1} \quad \textcircled{\delta} \quad \textcircled{\phi} \quad \textcircled{u} \quad \textcircled{\varepsilon_5} \\ \textcircled{\varepsilon_6} \end{array} \right)$$



Application of Yang-Baxter equation

$$Z \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = Z \left(\begin{array}{c} \text{Diagram 2} \end{array} \right)$$

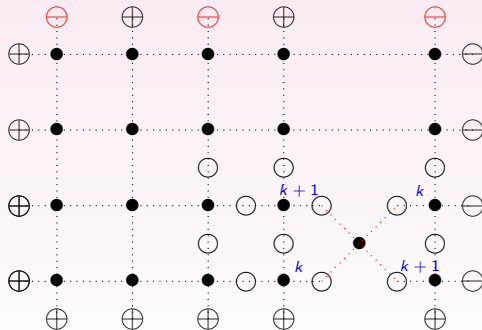
The diagram shows an equality between two Z-operators. Each operator is represented by a large pair of parentheses containing a graph. The graph on the left has a central vertex u connected to vertices $\epsilon_1, \epsilon_2, v, \mu$. Vertex v is connected to $\epsilon_3, \epsilon_4, \gamma$. Vertex μ is connected to ϵ_5, w . Vertex γ is connected to w . Vertex w is connected to ϵ_6 . The graph on the right has a central vertex u connected to vertices $\epsilon_4, \epsilon_5, \phi, \psi$. Vertex ϕ is connected to ϵ_1, v, δ . Vertex ψ is connected to ϵ_2, w, δ . Vertex v is connected to ϵ_6 . Vertex δ is connected to ϵ_3 .



Application of Yang-Baxter equation

$$Z \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = Z \left(\begin{array}{c} \text{Diagram 2} \end{array} \right)$$

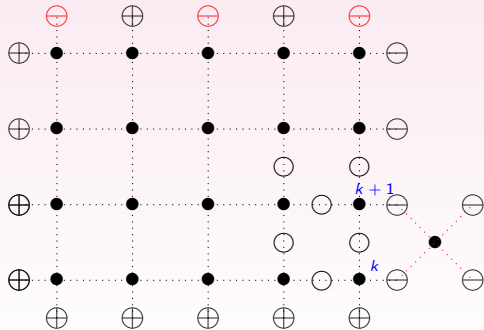
The equation shows the equality of two Z-factors. Diagram 1 (left) features a central vertex u connected to $\varepsilon_1, \varepsilon_2, v, \mu$. Vertex v is connected to $\varepsilon_3, \varepsilon_4, \gamma, w$. Vertex μ is connected to $\varepsilon_5, \varepsilon_6, w, \gamma$. Diagram 2 (right) features a central vertex u connected to $\varepsilon_4, \varepsilon_5, \psi, \phi$. Vertex ψ is connected to $\varepsilon_2, \varepsilon_3, \delta, w$. Vertex ϕ is connected to $\varepsilon_1, \varepsilon_6, v, \delta$.



Application of Yang-Baxter equation

$$Z \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = Z \left(\begin{array}{c} \text{Diagram 2} \end{array} \right)$$

The equation shows the equality of two Z-factors. The left Z-factor is a product of two R-matrices: $R(u, v)$ and $R(\mu, w)$. The right Z-factor is a product of two R-matrices: $R(\delta, v)$ and $R(\psi, u)$. The vertices are labeled with ε_1 through ε_6 .



Application to Schur polynomial

型 II						
w	$t_j x_i + x_j$	$t_i x_j + x_i$	$t_i x_j - t_j x_i$	$x_i - x_j$	$(t_i + 1)x_i$	$(t_j + 1)x_j$

$$Z(\mathfrak{S}_\lambda) = \prod_{i < j} (x_i + t_i x_j) s_\lambda(x_1, \dots, x_k, x_{k+1}, \dots, x_n)$$

$$(t_{k+1} x_k + x_{k+1}) Z(\mathfrak{S}_\lambda) = (t_k x_{k+1} + x_k) Z(\mathfrak{S}'_\lambda)$$

定理

s_λ は, x_1, \dots, x_n について対称で, t_i に独立な多項式である.

Zeta function

定義 (リーマンゼータ関数)

$s \in \mathbb{C}$ に対し,

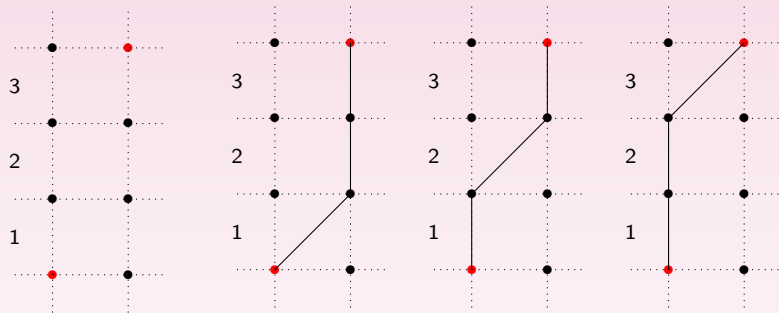
$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} \cdots \quad (\Re(s) > 1)$$

定義 (有限リーマンゼータ関数)

$s \in \mathbb{C}$ に対し,

$$\zeta^N(s) = \sum_{m=1}^N \frac{1}{m^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} \cdots \frac{1}{N^s}$$

Lattice and Riemann zeta function



型		
w	$1/i^s$	1

$$\sum \Pi w(\ell) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s}$$

Multiple zeta function

定義 (多重ゼータ関数)

$s_1, \dots, s_n \in \mathbb{C}$ に対し,

$$\zeta(s_1, \dots, s_n) = \sum_{m_1 < \dots < m_n} \frac{1}{m_1^{s_1} \dots m_n^{s_n}},$$

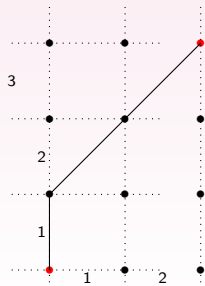
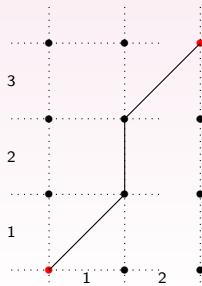
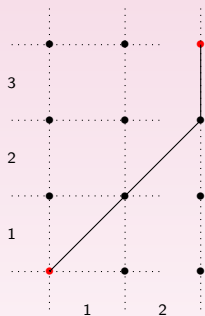
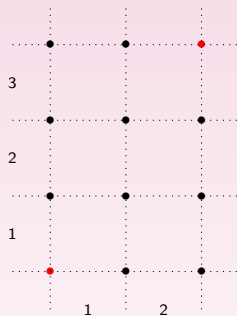
$(\Re(s_1), \dots, \Re(s_{n-1})) \geq 1 \text{ and } \Re(s_n) > 1$

定義 (有限多重ゼータ関数)

$s_1, \dots, s_n \in \mathbb{C}$ に対し,

$$\zeta^N(s_1, \dots, s_n) = \sum_{m_1 < \dots < m_n \leq N} \frac{1}{m_1^{s_1} \dots m_n^{s_n}},$$

Lattice and Multiple zeta function



$$\sum \prod w(\ell)$$

$$= \frac{1}{1^{s_1} 2^{s_2}} + \frac{1}{1^{s_1} 3^{s_2}}$$

$$+ \frac{1}{2^{s_1} 3^{s_2}}$$

Multiple zeta-star function

定義 (多重ゼータスター関数)

$s_1, \dots, s_n \in \mathbb{C}$ に対し,

$$\zeta^*(s_1, \dots, s_n) = \sum_{m_1 \leq \dots \leq m_n} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}},$$

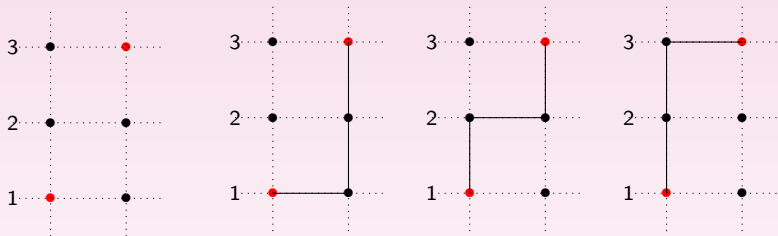
$(\Re(s_1), \dots, \Re(s_{n-1})) \geq 1 \text{ and } \Re(s_n) > 1$

定義 (有限多重ゼータスター関数)

$s_1, \dots, s_n \in \mathbb{C}$ に対し,

$$\zeta^{*,N}(s_1, \dots, s_n) = \sum_{m_1 \leq \dots \leq m_n \leq N} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}},$$

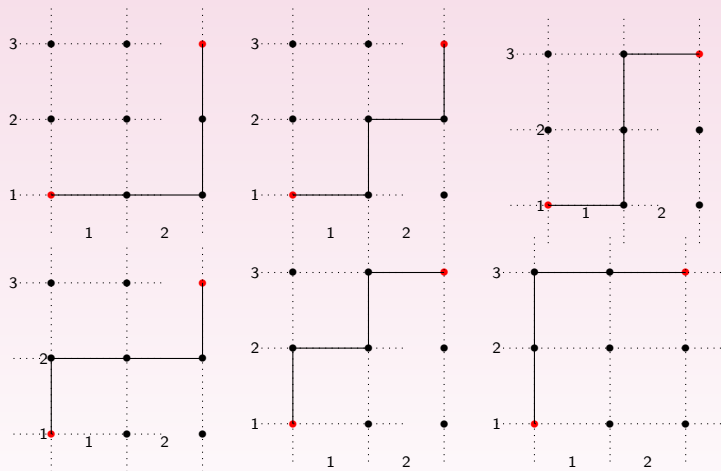
Lattice and Riemann zeta function 2



型		
w	$1/i^s$	1

$$\sum \prod w(\ell) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s}$$

Lattice and Multiple zeta-star function



$$\sum \prod w(\ell) = \frac{1}{1^{s_1} 1^{s_2}} + \frac{1}{1^{s_1} 2^{s_2}} + \frac{1}{1^{s_1} 3^{s_2}} + \frac{1}{2^{s_1} 2^{s_2}} + \frac{1}{2^{s_1} 3^{s_2}} + \frac{1}{3^{s_1} 3^{s_2}}$$

Relation between Multiple zeta and zeta-star functions

多重ゼータ関数は多重ゼータスター関数の線形結合で表される。(逆も同様)

例.

- $\zeta^*(s_1, s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2),$
- $\zeta(s_1, s_2) = \zeta^*(s_1, s_2) - \zeta^*(s_1 + s_2),$
- $\zeta^*(s_1, s_2, s_3) = \zeta(s_1, s_2, s_3) + \zeta(s_1 + s_2, s_3) + \zeta(s_1, s_2 + s_3) + \zeta(s_1 + s_2 + s_3),$
- $\zeta(s_1, s_2, s_3) = \zeta^*(s_1, s_2, s_3) - \zeta^*(s_1 + s_2, s_3) - \zeta^*(s_1, s_2 + s_3) + \zeta^*(s_1 + s_2 + s_3).$

Algebraic relations 1

その他の関係式の例.

- $\zeta^*(a, b) = \zeta(a)\zeta(b) - \zeta(b, a)$.
- $\zeta(a, b) = \zeta^*(a)\zeta^*(b) - \zeta^*(b, a)$.

定理

$s_1, \dots, s_n \in \mathbb{C}$ ($\Re(s_1), \dots, \Re(s_n) > 1$) に対し, 次が成り立つ.

$$(1) \zeta^*(s_1, \dots, s_n) = \begin{vmatrix} \zeta(s_1) & \zeta(s_2, s_1) & \cdots & \cdots & \zeta(s_n, \dots, s_2, s_1) \\ 1 & \zeta(s_2) & \cdots & \cdots & \zeta(s_n, \dots, s_2) \\ & 1 & \ddots & & \vdots \\ & & \ddots & 1 & \zeta(s_{n-1}) & \zeta(s_n, s_{n-1}) \\ 0 & & & & 1 & \zeta(s_n) \end{vmatrix}.$$

Algebraic relations 2

定理

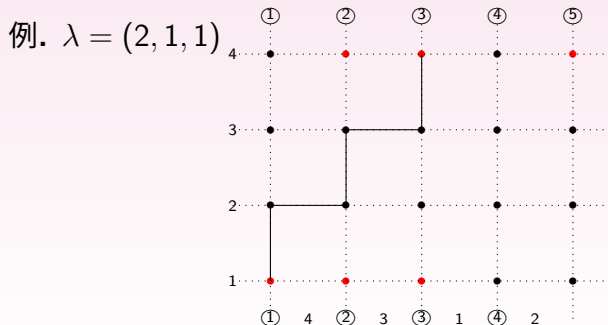
$s_1, \dots, s_n \in \mathbb{C}$ ($\Re(s_1), \dots, \Re(s_n) > 1$) に対し, 以下が成り立つ.

$$(2) \zeta(s_1, \dots, s_n) = \begin{vmatrix} \zeta^*(s_1) & \zeta^*(s_2, s_1) & \cdots & \cdots & \zeta^*(s_n, \dots, s_2, s_1) \\ 1 & \zeta^*(s_2) & \cdots & \cdots & \zeta^*(s_n, \dots, s_2) \\ & 1 & \ddots & & \vdots \\ & & \ddots & 1 & \zeta^*(s_n, s_{n-1}) \\ 0 & & & & 1 & \zeta^*(s_n) \end{vmatrix}$$

Application to Multiple zeta function

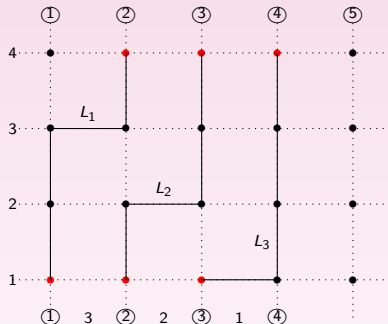
$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, ある N に対し, 以下のように定める.

- 頂点の座標は, 左下を $(0, 0)$ とし, 各列は上に, 各行は右に増加する
- 経路の始点は $(1, 1), (2, 1), \dots, (n, 1)$ とする
- 経路の終点は $(\lambda_i + n - i + 1, N)$ とする



Application to Multiple zeta function

例. $\lambda = (1, 1, 1)$

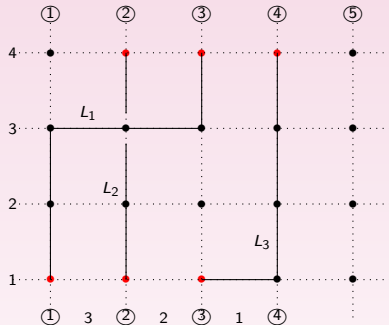


各 path L_i に対する分割関数の積 :

$$\prod_{L_i} w(L_i) = \zeta^{*,4}(s_3) \zeta^{*,4}(s_2) \zeta^{*,4}(s_1) = \zeta^*(s_3) \zeta^*(s_2) \zeta^*(s_1)$$

Application to Multiple zeta function

例. $\lambda = (1, 1, 1)$
 $\sigma = (1, 2)$



補題

$$\sum_{\sigma \in S'_3} \prod_{L_i} \varepsilon_{\sigma} w(L_i) = \begin{vmatrix} \zeta^*(s_1) & \zeta^*(s_2, s_1) & \zeta^*(s_3, s_2, s_1) \\ 1 & \zeta^*(s_2) & \zeta^*(s_3, s_2) \\ 0 & 1 & \zeta^*(s_3) \end{vmatrix}$$

Property of Multiple zeta function

補題

分割 $\lambda = (1, 1, 1)$, $\mathbf{s} = (s_1, s_2, s_3)$ に対し, 次が成り立つ.

$$\zeta(\mathbf{s}) = \zeta(s_1, s_2, s_3) = \sum_{\sigma \in S'_n} \prod_{L_i} \varepsilon_{\sigma} w(L_i)$$

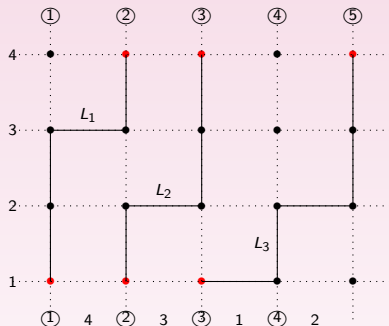
定理

分割 $\lambda = (1, 1, 1)$, $\mathbf{s} = (s_1, s_2, s_3)$ に対し, 次が成り立つ.

$$\zeta(\mathbf{s}) = \zeta(s_1, s_2, s_3) = \begin{vmatrix} \zeta^*(s_1) & \zeta^*(s_2, s_1) & \zeta^*(s_3, s_2, s_1) \\ 1 & \zeta^*(s_2) & \zeta^*(s_3, s_2) \\ 0 & 1 & \zeta^*(s_3) \end{vmatrix}$$

Application to Multiple zeta function 2

例. $\lambda = (2, 1, 1)$

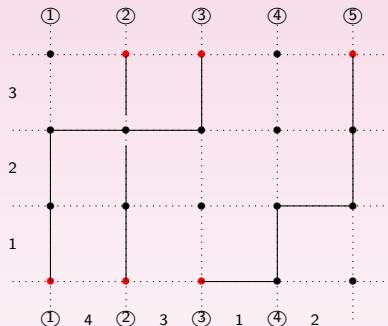


各 path L_i の始点と終点の横方向の差がそれぞれ, 1, 1, 2 なので

$$\prod_{L_i} w(L_i) = \zeta^*(s_4) \zeta^*(s_3) \zeta^*(s_1, s_2).$$

Application to Multiple zeta function 2

例. $\lambda = (2, 1, 1)$
 $\sigma = (1, 2)$



補題

$$\sum_{\sigma \in S'_3} \prod_{L_i} \varepsilon_{\sigma} w(L_i) = \begin{vmatrix} \zeta^*(s_1, s_2) & \zeta^*(s_3, s_1, s_2) & \zeta^*(s_4, s_3, s_1, 2) \\ 1 & \zeta^*(s_3) & \zeta^*(s_4, s_3) \\ 0 & 1 & \zeta^*(s_4) \end{vmatrix}$$

New kind of Multiple zeta function

補題

分割 $\lambda = (2, 1, 1)$, $\mathbf{s} = (s_1, s_2, s_3, s_4)$ に対し, 次が成り立つ.

$$\sum_{\substack{m_{11} \leq m_{12} \\ \wedge \\ m_{21} \\ \wedge \\ m_{31}}} \frac{1}{m_{11}^{s_1} m_{12}^{s_2} m_{21}^{s_3} m_{31}^{s_4}} = \sum_{\sigma \in S'_3} \prod_{L_i} \varepsilon_{\sigma} w(L_i)$$

Schur multiple zeta function

- 分割 $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ に対し,
 $M = (m_{ij}) \in \mathbb{N}^{|\lambda|}$, $\mathbf{s} = (s_{ij}) \in \mathbb{C}^{|\lambda|}$ を次のように書く.

$$\lambda = (3, 3, 2)$$

$$M = \begin{array}{|c|c|c|} \hline m_{11} & m_{12} & m_{13} \\ \hline m_{21} & m_{22} & m_{23} \\ \hline m_{31} & m_{32} & \\ \hline \end{array}$$

$$\mathbf{s} = \begin{array}{|c|c|c|} \hline s_{11} & s_{12} & s_{13} \\ \hline s_{21} & s_{22} & s_{23} \\ \hline s_{31} & s_{32} & \\ \hline \end{array}$$

Schur 多重ゼータ関数

$$\zeta_{\lambda}(\mathbf{s}) = \sum_M \prod_{(i,j)} \frac{1}{m_{ij}^{s_{ij}}}$$

で定義する. ここで, $M = (m_{ij})_{(i,j)}$ は, $m_{ij} \leq m_{i,j+1}$,
 $m_{ij} < m_{i+1,j}$ を満たす.

Schur polynomial

- 分割 $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, 変数 $\mathbf{x} = (x_1, x_2, \dots)$ に対し,
 $M = (m_{ij}) \in \mathbb{N}^{|\lambda|}$ を次のように書く。

$$\lambda = (3, 3, 2)$$

$$M = \begin{array}{|c|c|c|} \hline m_{11} & m_{12} & m_{13} \\ \hline m_{21} & m_{22} & m_{23} \\ \hline m_{31} & m_{32} & \\ \hline \end{array}$$

命題 (Schur 多項式)

$$s_\lambda = s_\lambda(\mathbf{x}) = \sum_M \prod_{(i,j)} x_{m_{ij}},$$

で表される。ここで, $M = (m_{ij})_{(i,j)}$ は, $m_{ij} \leq m_{i,j+1}$,
 $m_{ij} < m_{i+1,j}$ を満たす。

Jacobi-Trudi type formula for ζ_λ

定理. (N.–Phuksuwan–Yamasaki, 2018)

$\mathbf{s} = (s_{ij}) \in W_\lambda^{\text{diag}}$ (対角成分の変数が同じであるという条件を満たす収束領域) . $s_{ij} = a_{j-i}$ とする. このとき次が成り立つ.

(1) $\Re(s_{i,\lambda'_i}) > 1$ ($1 \leq i < \lambda'_1$) に対し,

$$\zeta_\lambda(\mathbf{s}) = \det [\zeta^*(a_{-j+1}, a_{-j+2}, \dots, a_{-j+(\lambda_i-i+j)})]_{1 \leq i, j \leq \lambda'_1} .$$

(2) $\Re(s_{i,\lambda'_i}) > 1$ ($1 \leq i < \lambda_1$) に対し,

$$\zeta_\lambda(\mathbf{s}) = \det [\zeta(a_{j-1}, a_{j-2}, \dots, a_{j-(\lambda'_i-i+j)})]_{1 \leq i, j \leq \lambda_1} .$$

ただし, λ' は λ の共役を表す.

Thank you very much

ご清聴ありがとうございました.