One-loop correction in PBH formation from single-field inflation

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Based on 2211.03395 and 2303.00341 with Jun'ichi Yokoyama

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Canonical inflation

Action: $S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,\left[M_{\mathrm{pl}}^2 R - (\partial_\mu\phi)^2 - 2V(\phi)\right].$

Background: $\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\ \mathrm{d}\mathbf{x}^2 = a^2(\tau)(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2).$

Equation of motion:

Friedmann equation: $\dot{H} = -\frac{\dot{\phi}^2}{2M_{\mathrm{pl}}^2}$ and $H^2 = \frac{1}{3M_{\mathrm{pl}}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right).$

Klein-Gordon equation: $\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0.$

SR approximation: $\left|\frac{\ddot{\phi}}{\dot{\phi}H}\right| \ll 1$ and $\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_{\mathrm{pl}}^2 H^2} \ll 1.$

Cosmological perturbations

Small perturbations:

- Inflaton: $\phi(\mathbf{x}, t) = \bar{\phi}(t) + \delta\phi(\mathbf{x}, t)$
- Spacetime: $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)$

Gauge fixing condition: $\delta\phi(\mathbf{x}, t) = 0$ and $\gamma_{ij}(\mathbf{x}, t) = a^2(t)e^{2\zeta(\mathbf{x},t)}\delta_{ij}$

Some parameters ($\epsilon, \eta \ll 1$): $\epsilon = -\dfrac{\dot{H}}{H^2}$ and $\eta = \dfrac{\dot{\epsilon}}{\epsilon H}$.

Superhorizon evolution?

Two-point functions

Second-order action: $S^{(2)} = M_{\rm pl}^2 \int {\rm d}t\ {\rm d}^3x\ \epsilon a^3 \left[\dot{\zeta}^2 - \frac{1}{a^2}(\partial_i \zeta)^2\right]$.

Introducing Mukhanov-Sasaki (MS) variable $v = z\zeta M_{\rm pl}$ where $z = a\sqrt{2\epsilon}$, the second-order action becomes

$$S^{(2)} = \frac{1}{2}\int {\rm d}\tau\ {\rm d}^3x \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right].$$

Promoting MS variable as operator $v(\mathbf{k},\tau) \to \hat{v}(\mathbf{k},\tau) = v_k(\tau)\hat{a}_{\mathbf{k}} + v_k^*(\tau)\hat{a}^\dagger_{-\mathbf{k}}$ in momentum space with commutation relation $[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{-\mathbf{q}}] = (2\pi)^3\delta^3(\mathbf{p}+\mathbf{q})$, the equation of motion is

$$v_k'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right] v_k = 0, \text{ where } \nu = \frac{3}{2} + \epsilon + \frac{\eta}{2}$$

Two-point functions

In the limit $\tau \rightarrow -\infty$: $v_k'' + k^2 v_k = 0$.

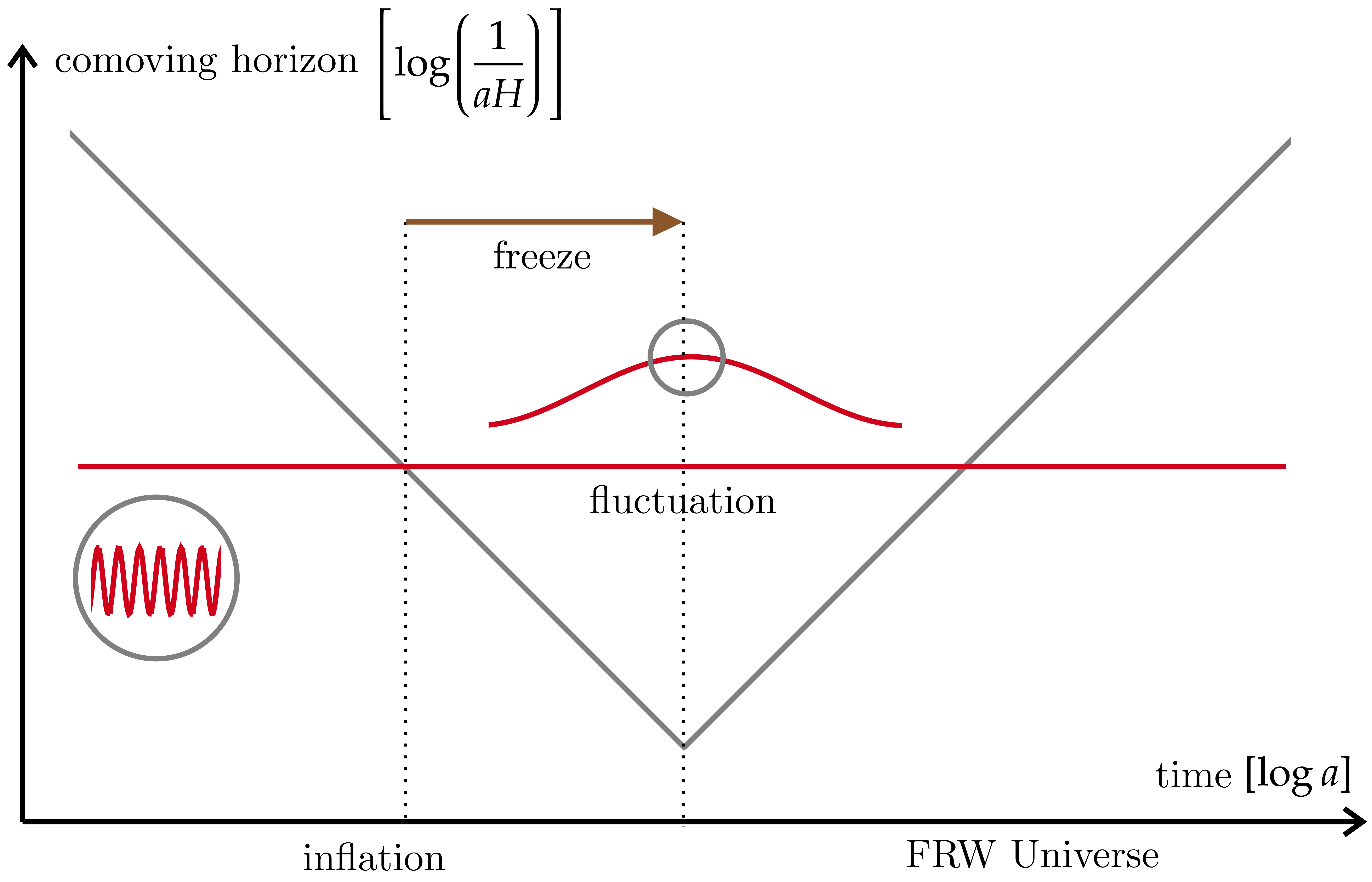
Early time solution: $v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}$ corresponds to Bunch-Davies initial vacuum $\hat{a}_{\mathbf{k}} \, |0\rangle = 0$.

Full solution: $v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right)$.

Curvature perturbation: $\zeta_k(\tau) = \left(\frac{iH}{2M_{\rm pl}\sqrt{\epsilon}}\right) \frac{e^{-ik\tau}}{k^{3/2}}(1 + ik\tau)$.

Two-point functions: $\langle \zeta(\mathbf{k}, \tau)\zeta(-\mathbf{k}, \tau)\rangle = \frac{H^2}{4M_{\rm pl}^2 \epsilon k^3}(1 + k^2\tau^2)$.

Two-point functions



Non-linearity?

Maldacena's cubic interaction

Performing field redefinition: $\zeta = \boldsymbol{\zeta} + \frac{\eta}{4}\boldsymbol{\zeta}^2 + \dots$

Cubic interaction of curvature perturbation ζ :

$$S_{\rm int} = M_{\rm pl}^2 \int {\rm d}t\ {\rm d}^3x\ a^3 \left[\underbrace{\epsilon^2 \dot{\boldsymbol{\zeta}}^2 \boldsymbol{\zeta} + \frac{1}{a^2}\epsilon^2 (\partial_i \boldsymbol{\zeta})^2 \boldsymbol{\zeta} - 2\epsilon \dot{\boldsymbol{\zeta}} \partial_i \boldsymbol{\zeta} \partial_i \boldsymbol{\chi}}_{\mathcal{O}(\epsilon^2)} \underbrace{- \frac{1}{2}\epsilon^3 \dot{\boldsymbol{\zeta}}^2 \boldsymbol{\zeta} + \frac{1}{2}\epsilon \boldsymbol{\zeta} (\partial_i \partial_j \boldsymbol{\chi})^2 + \frac{1}{2}\epsilon \dot{\eta} \dot{\boldsymbol{\zeta}} \boldsymbol{\zeta}^2}_{\mathcal{O}(\epsilon^3)} \right]$$

Quoted from Maldacena: It should be noted that the field redefinition does indeed matter for our computation since we are interested in computing expectation values of ζ and not of ζ_n ($\boldsymbol{\zeta}$). The reason is that ζ is the variable that stays constant outside the horizon while ζ_n does not. This last fact follows from the fact that ζ is constant where some of the coefficients of the quadratic terms are time dependent.

Maldacena (astro-ph/0210603)

SR bispectrum

We focus on $H_{\mathrm{int}}(\tau) = -\frac{1}{2}M_{\mathrm{pl}}^2 \int \mathrm{d}^3x\ \epsilon\dot{\eta}a^3\boldsymbol{\zeta}'\boldsymbol{\zeta}^2$.

Define third SR parameter: $\xi = \frac{\dot{\eta}}{\eta H}$

Bispectrum ($\tau_0 \to 0$): $\langle \boldsymbol{\zeta}_{\mathbf{k}_1}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_2}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_3}(\tau_0)\rangle = 2\int_{-\infty}^{\tau_0} \mathrm{d}\tau\ \mathrm{Im}\left\langle \boldsymbol{\zeta}_{\mathbf{k}_1}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_2}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_3}(\tau_0)H_{\mathrm{int}}(\tau)\right\rangle$.

Curvature perturbation: $\zeta_k(\tau) = \left(\frac{iH}{2M_{\mathrm{pl}}\sqrt{\epsilon}}\right)\frac{e^{-ik\tau}}{k^{3/2}}(1+ik\tau)$

SR bispectrum

Performing the time integral:

$$\langle \boldsymbol{\zeta}_{\mathbf{k}_1}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_2}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_3}(\tau_0)\rangle \to \frac{1}{2}\eta\xi\left(\frac{H^2}{4M_{\rm pl}^2\epsilon}\right)^2\frac{k_1^3+k_2^3+k_3^3}{(k_1k_2k_3)^3}\log(-\tau_0)$$

Clearly, bispectrum of $\boldsymbol{\zeta}$ does not constant outside the horizon. It grows logarithmically in terms of τ_0.

$$\langle \boldsymbol{\zeta}_{\mathbf{k}_1}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_2}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_3}(\tau_0)\rangle = \frac{1}{2}\eta\xi\log(-\tau_0)\left[|\zeta_{k_1}|^2|\zeta_{k_2}|^2+|\zeta_{k_1}|^2|\zeta_{k_3}|^2+|\zeta_{k_2}|^2|\zeta_{k_3}|^2\right]$$

SR bispectrum

Including field redefinition $\zeta = \boldsymbol{\zeta} + \frac{\eta}{4}\boldsymbol{\zeta}^2$

$$\langle \zeta_{\mathbf{k}_1}(\tau_0)\zeta_{\mathbf{k}_2}(\tau_0)\zeta_{\mathbf{k}_3}(\tau_0)\rangle = \langle \boldsymbol{\zeta}_{\mathbf{k}_1}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_2}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_3}(\tau_0)\rangle + \frac{\eta(\tau_0)}{4}2\left[\langle \boldsymbol{\zeta}_{\mathbf{k}_1}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_1}(\tau_0)\rangle\langle \boldsymbol{\zeta}_{\mathbf{k}_2}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_2}(\tau_0)\rangle + \mathrm{perm}\right]$$

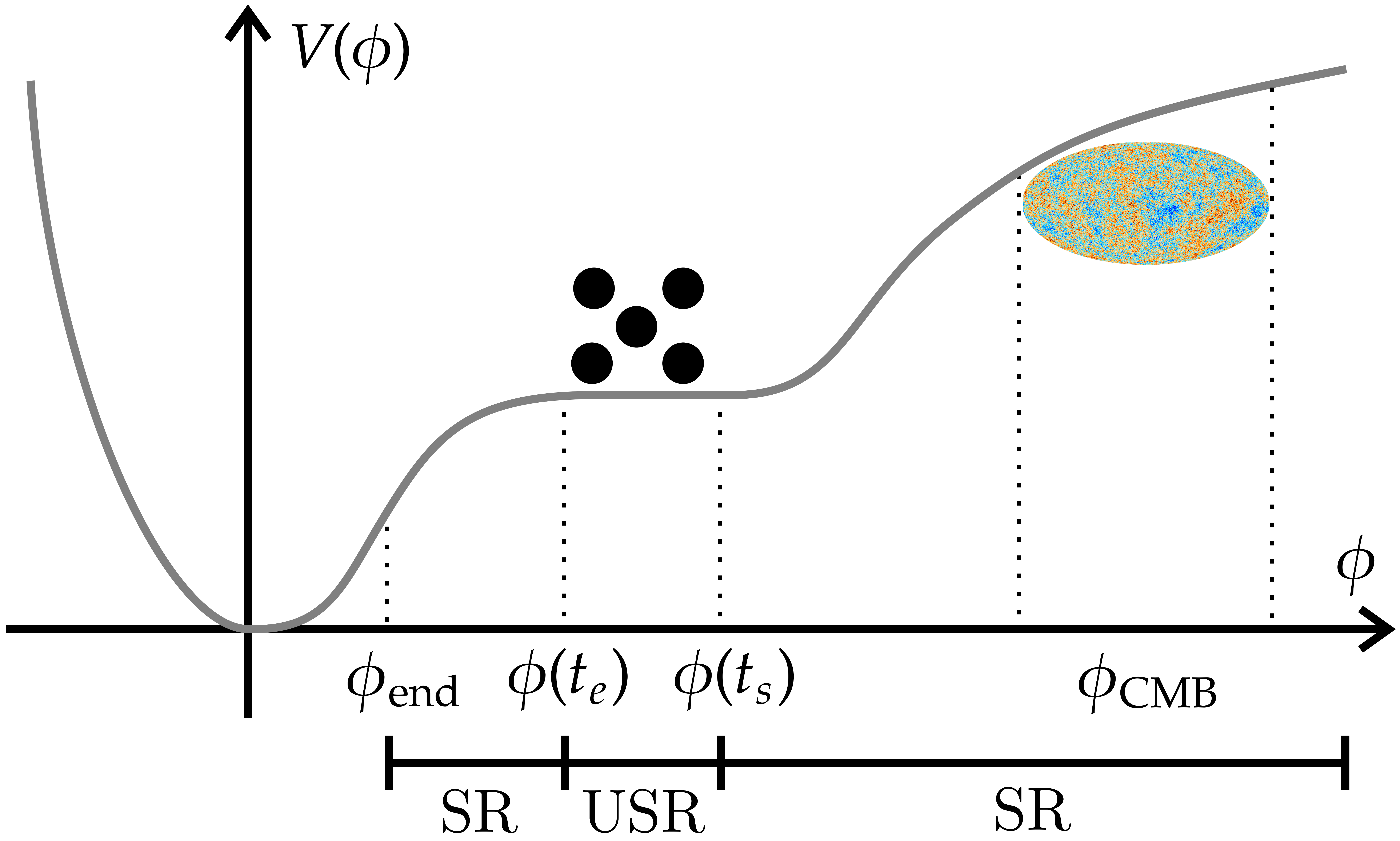
Considering time dependence: $\eta(\tau_0) = \eta_\star\left(\frac{\tau_0}{\tau_\star}\right)^{-\xi} \to \eta_\star\left(1 - \xi\log(-\tau_0)\right)$

$$\langle \zeta_{\mathbf{k}_1}(\tau_0)\zeta_{\mathbf{k}_2}(\tau_0)\zeta_{\mathbf{k}_3}(\tau_0)\rangle = \langle \boldsymbol{\zeta}_{\mathbf{k}_1}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_2}(\tau_0)\boldsymbol{\zeta}_{\mathbf{k}_3}(\tau_0)\rangle - \frac{\eta\xi}{2}\log(-\tau_0)\left[\,|\zeta_{k_1}|^2|\zeta_{k_2}|^2 + |\zeta_{k_1}|^2|\zeta_{k_3}|^2 + |\zeta_{k_2}|^2|\zeta_{k_3}|^2\right].$$

As Maldacena explained, we can see cancellation of $\log(-\tau_0)$ dependence.

How if SR approximation is violated?

Potential of the inflaton



Ivanov et. al. (PRD 1994)

Slow-roll and ultraslow-roll

SR approximation: $\left|\frac{\ddot{\phi}}{\dot{\phi}H}\right| \ll 1$ and $\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_{\mathrm{pl}}^2 H^2} \ll 1.$

USR condition: For an extremely flat potential $\mathrm{d}V/\mathrm{d}\phi \approx 0$, the Klein-Gordon equation becomes

$$\frac{\ddot{\phi}}{\dot{\phi}H} \approx -3 \text{ so } \dot{\phi} \propto a^{-3} \text{ and } \epsilon = \frac{\dot{\phi}^2}{2M_{\mathrm{pl}}^2 H^2} \propto a^{-6}.$$

Second SR parameter: $\eta \equiv \frac{\dot{\epsilon}}{\epsilon H} = 2\epsilon + 2\frac{\ddot{\phi}}{\dot{\phi}H}$

- SR: $|\eta| \ll 1$ and approximately constant.
- USR: $\eta \approx -6$.

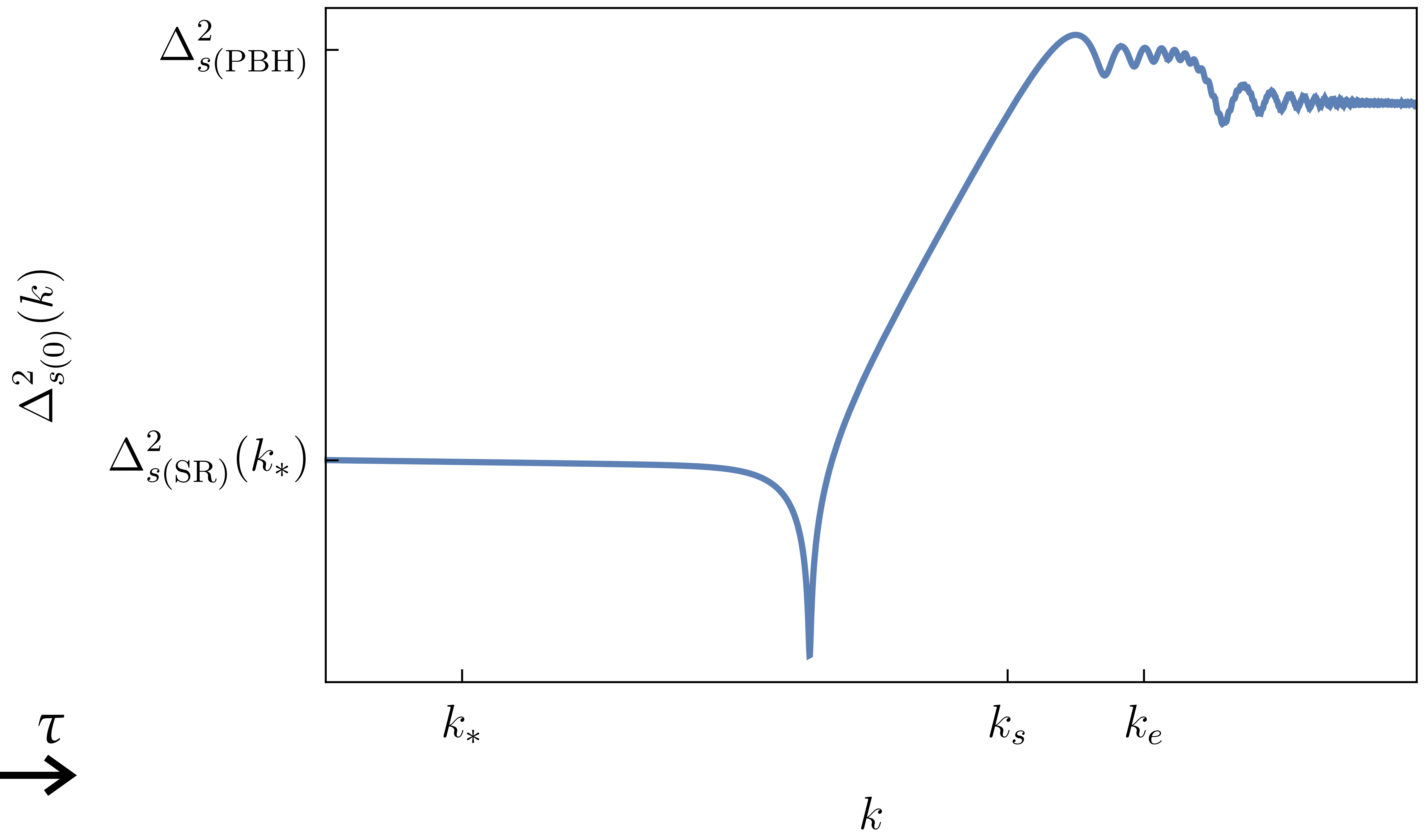
Kinney (gr-qc/0503017), Martin et. al. (1211.0083)

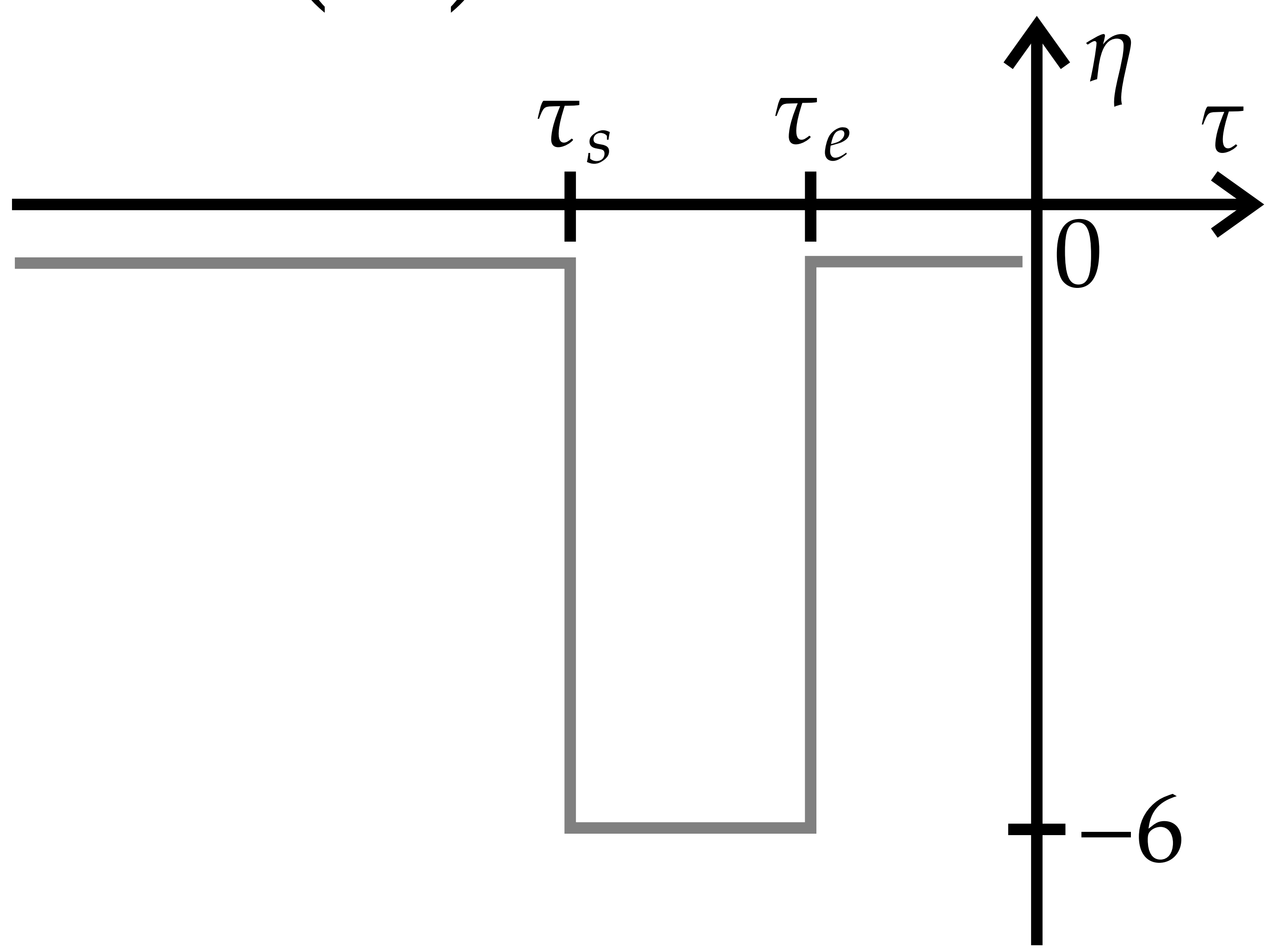
Power spectrum

Power spectrum: $\Delta^2_{s(0)}(k) = \dfrac{k^3}{2\pi^2}\,|\,\zeta_k(\tau \to 0)\,|^2.$

Large scale: $\Delta^2_{s(\mathrm{SR})}(k) \equiv \Delta^2_{s(0)}(k \ll k_s) = \dfrac{H^2}{8\pi^2 M_{\mathrm{pl}}^2 \epsilon}.$

Small scale: $\Delta^2_{s(\mathrm{PBH})} \approx \Delta^2_{s(\mathrm{SR})}(k_s)\left(\dfrac{k_e}{k_s}\right)^6.$



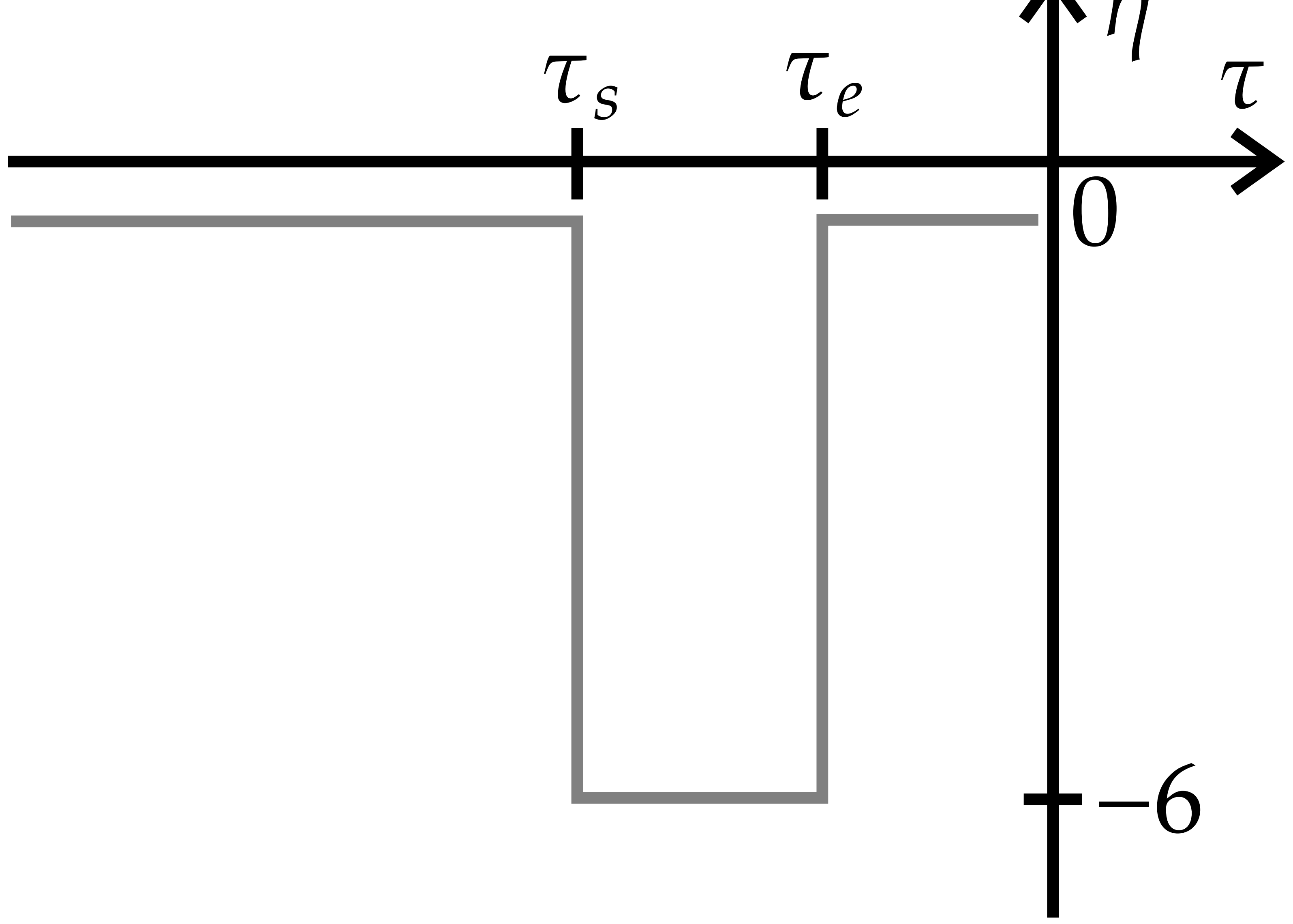


Third-order action

Cubic interaction for inflation **with** PBH formation:

$$S_{\rm int} = M_{\rm pl}^2 \int {\rm d}t\ {\rm d}^3x\ a^3 \left[\underbrace{\epsilon^2 \dot{\zeta}^2 \zeta + \frac{1}{a^2}\epsilon^2 (\partial_i \zeta)^2 \zeta - 2\epsilon \dot{\zeta} \partial_i \zeta \partial_i \chi}_{\mathcal{O}(\epsilon^2)} \underbrace{- \frac{1}{2}\epsilon^3 \dot{\zeta}^2 \zeta + \frac{1}{2}\epsilon \zeta (\partial_i \partial_j \chi)^2}_{\mathcal{O}(\epsilon^3)} \underbrace{+ \frac{1}{2}\epsilon \dot{\eta} \dot{\zeta} \zeta^2}_{\mathcal{O}(\epsilon)} \right],$$

because η can have $\mathcal{O}(1)$ transition from 0 to -6.



Cai et. al. (1712.09998), Davies et. al. (2110.08189)

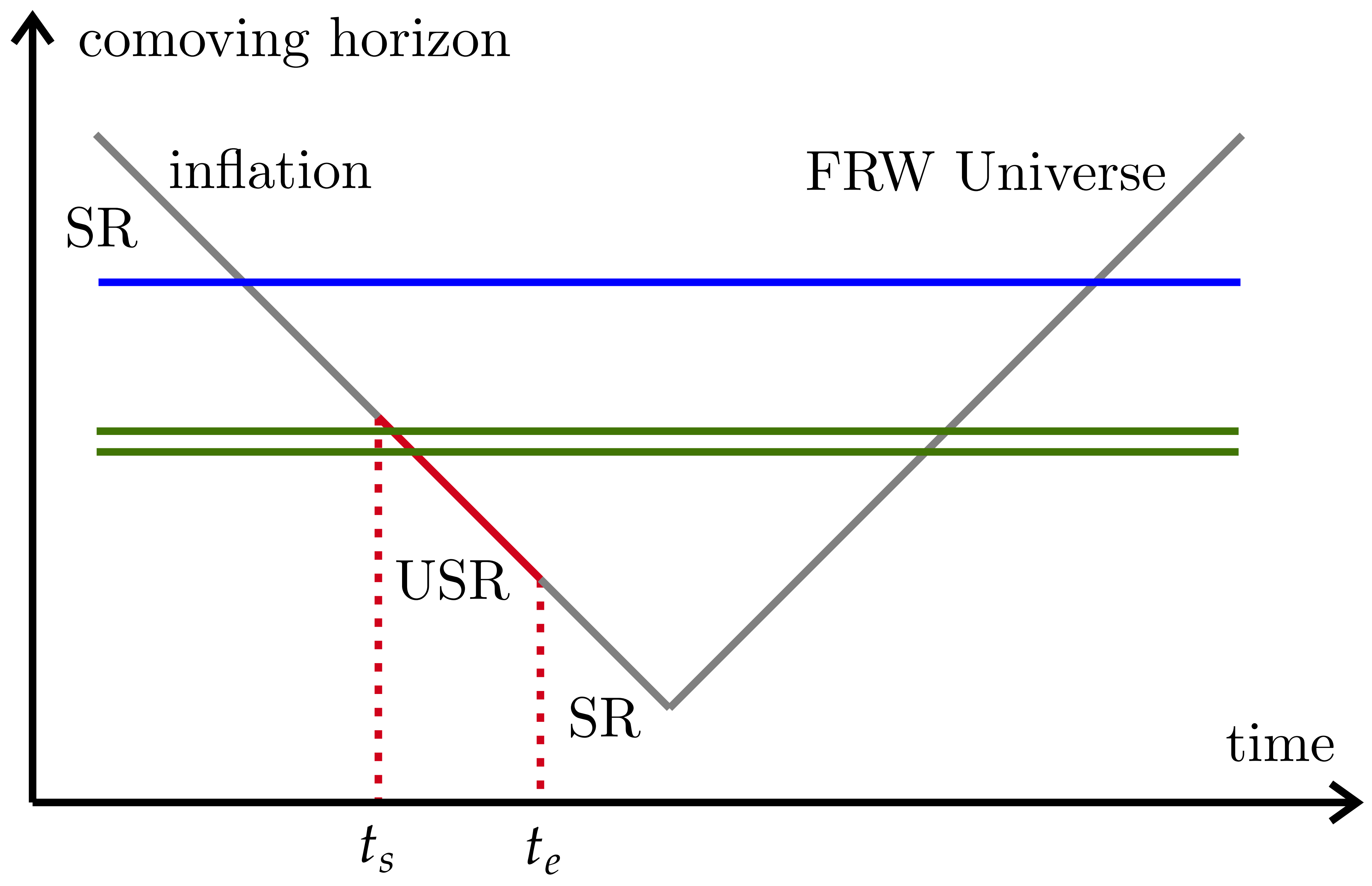
Squeezed bispectrum

Squeezed limit of the bispectrum:

$$\lim_{k_1 \to 0} \langle\langle \zeta_{\mathbf{k}_1}(\tau)\zeta_{\mathbf{k}_2}(\tau)\zeta_{-\mathbf{k}_2}(\tau)\rangle\rangle = -(n_s(k_2,\tau)-1)\langle\langle \zeta_{\mathbf{k}_2}(\tau)\zeta_{-\mathbf{k}_2}(\tau)\rangle\rangle\langle\langle \zeta_{\mathbf{k}_1}(\tau)\zeta_{-\mathbf{k}_1}(\tau)\rangle\rangle$$

$$\lim_{k_1 \to 0} \langle\langle \zeta_{\mathbf{k}_1}(\tau)\zeta_{\mathbf{k}_2}(\tau)\zeta_{-\mathbf{k}_2}(\tau)\rangle\rangle = -(n_s(k_2,\tau)-1)\,|\zeta_{k_2}(\tau)|^2\,|\zeta_{k_1}(\tau)|^2,$$

where $n_s(k,\tau)-1 = \dfrac{\mathrm{d}\log \Delta_s^2(k,\tau)}{\mathrm{d}\log k}$.



Comparison and conclusion

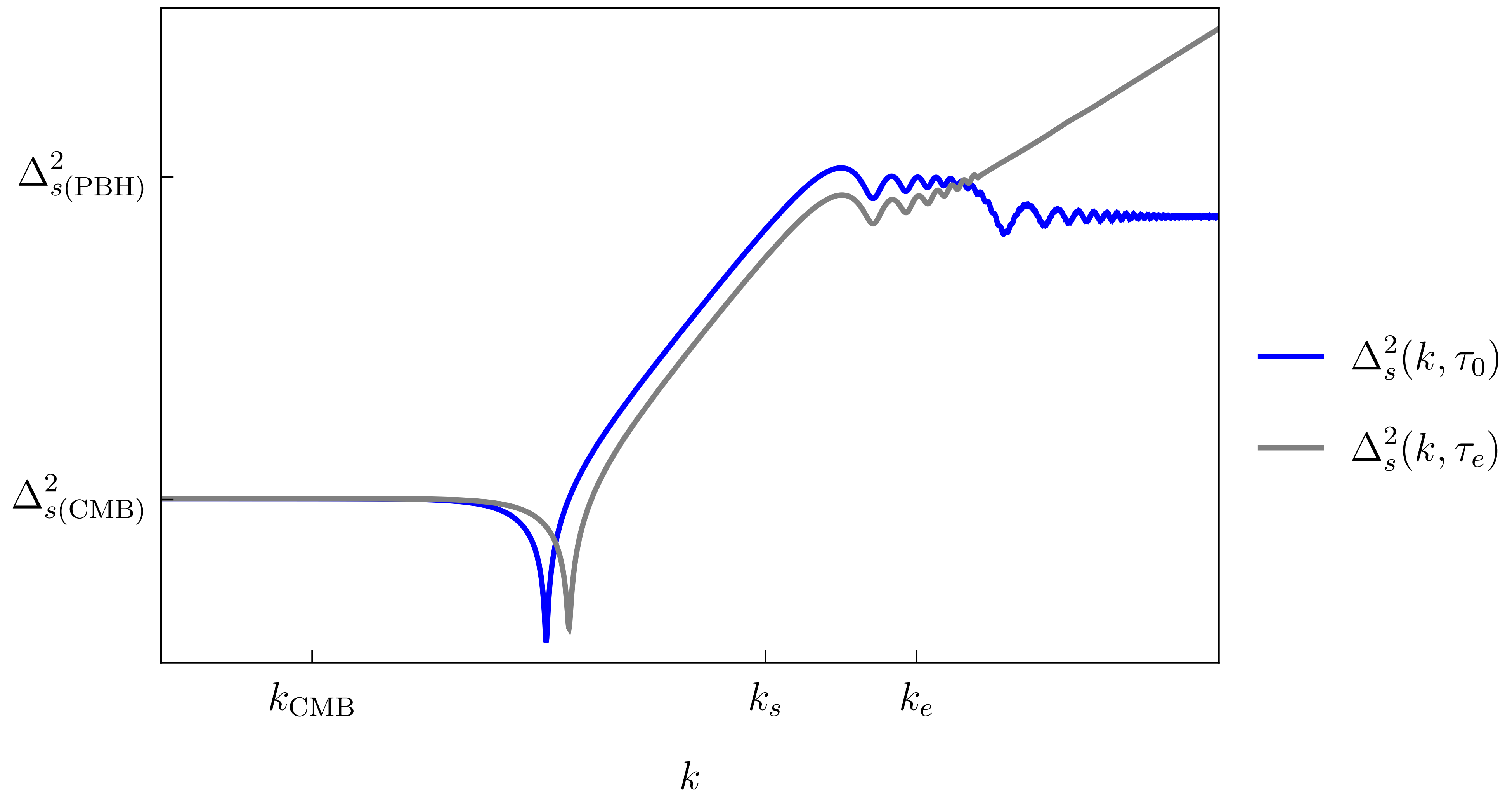
In SR case, bispectrum (of course also squeezed bispectrum) of ζ does not grow outside the horizon.

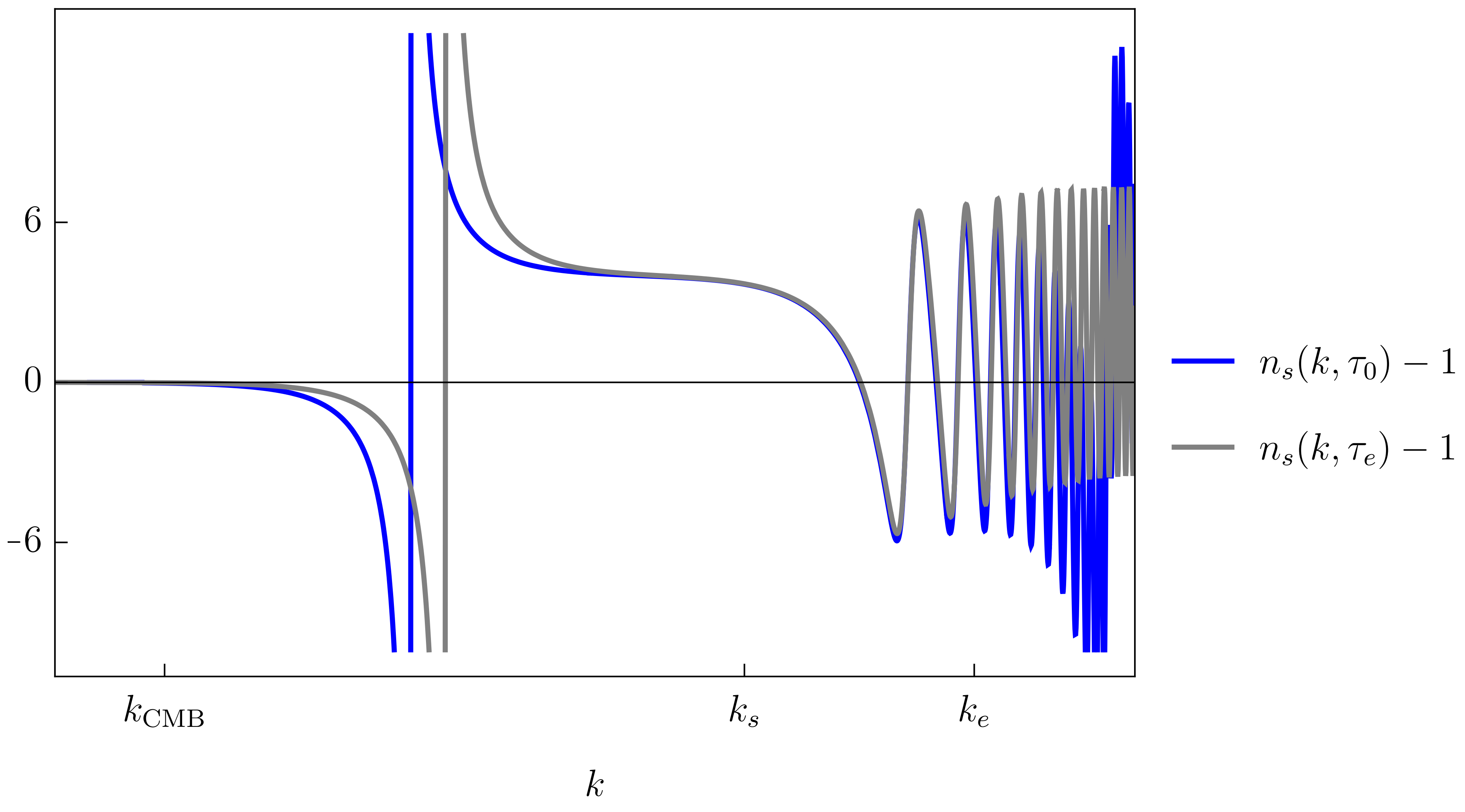
In transient USR case, the squeezed bispectrum satisfies

$$\lim_{k_1 \to 0} \langle\langle \zeta_{\mathbf{k}_1}(\tau)\zeta_{\mathbf{k}_2}(\tau)\zeta_{-\mathbf{k}_2}(\tau) \rangle\rangle = -\left(n_s(k_2,\tau)-1\right)|\zeta_{k_2}(\tau)|^2 |\zeta_{k_1}(\tau)|^2 .$$

We can see that the squeezed bispectrum evolves outside the horizon.

The squeezed bispectrum satisfies Maldacena's theorem but evolves outside the horizon.





One-loop correction

One-loop correction generated by cubic-order action is computed using in-in perturbation theory:

$$\langle \mathcal{O}(\tau) \rangle = \langle \mathcal{O}(\tau) \rangle^{\dagger}_{(0,2)} + \langle \mathcal{O}(\tau) \rangle_{(1,1)} + \langle \mathcal{O}(\tau) \rangle_{(0,2)}$$

$$\langle \mathcal{O}(\tau) \rangle_{(1,1)} = \int_{-\infty}^{\tau} \mathrm{d}\tau_1 \int_{-\infty}^{\tau} \mathrm{d}\tau_2 \left\langle H_{\mathrm{int}}(\tau_1) \hat{\mathcal{O}}(\tau) H_{\mathrm{int}}(\tau_2) \right\rangle$$

$$\langle \mathcal{O}(\tau) \rangle_{(0,2)} = - \int_{-\infty}^{\tau} \mathrm{d}\tau_1 \int_{-\infty}^{\tau_1} \mathrm{d}\tau_2 \left\langle \hat{\mathcal{O}}(\tau) H_{\mathrm{int}}(\tau_1) H_{\mathrm{int}}(\tau_2) \right\rangle$$

Operator: $\mathcal{O}(\tau_0) = \boldsymbol{\zeta}_{\mathbf{p}}(\tau_0) \boldsymbol{\zeta}_{-\mathbf{p}}(\tau_0)$ where $\tau_0 \to 0$.

Leading interaction: $H_{\mathrm{int}}(\tau) = -\dfrac{1}{2} M_{\mathrm{pl}}^2 \displaystyle\int \mathrm{d}^3x \; \epsilon \eta' a^2 \zeta' \zeta^2 \, .$

One-loop correction

For large loop momentum $k \gg p$:

$$\langle \zeta_{\mathbf{p}}(\tau_0)\zeta_{-\mathbf{p}}(\tau_0)\rangle_{(1)} = \frac{1}{4}M_{\rm pl}^4\epsilon^2(\tau_e)a^4(\tau_e)(\Delta\eta)^2|\zeta_p(\tau_0)|^2\int\frac{{\rm d}^3k}{(2\pi)^3}16\left[|\zeta_k|^2\ {\rm Im}(\zeta_p'\zeta_p^*)\ {\rm Im}(\zeta_k'\zeta_k^*)\right]_{\tau=\tau_e}.$$

Substituting $[{\rm Im}(\zeta_k'\zeta_k^*)]_{\tau=\tau_e} = -\dfrac{1}{4M_{\rm pl}^2\epsilon(\tau_e)a^2(\tau_e)}$, we obtain

$$\Delta^2_{s(1)}(p) = \frac{1}{4}(\Delta\eta)^2\Delta^2_{s(0)}(p)\int_{k_s}^{k_e}\frac{{\rm d}k}{k}\Delta^2_{s(0)}(k)\ ,$$

where we focus on finite effect of USR period from k_s to k_e .

Perturbativity bound

Ratio between one-loop correction and tree-level contribution:

$$R(p) \equiv \frac{\Delta^2_{s(1)}(p)}{\Delta^2_{s(0)}(p)} = \frac{1}{4}(\Delta\eta)^2 \int_{k_s}^{k_e} \frac{\mathrm{d}k}{k} \Delta^2_{s(0)}(k) = \frac{1}{4}(\Delta\eta)^2 \left(1.1 + \log\frac{k_e}{k_s} \right) \Delta^2_{s(\mathrm{PBH})}.$$

Perturbativity bound: $\frac{1}{4}(\Delta\eta)^2 \left(1.1 + \log\frac{k_e}{k_s} \right) \Delta^2_{s(\mathrm{PBH})} \ll 1.$

Upper bound on small-scale power spectrum: $\Delta^2_{s(\mathrm{PBH})} \ll \frac{1}{(\Delta\eta)^2} \approx 0.03.$

UV divergence

Including UV divergence

$$\int_{k_{\mathrm{IR}}}^{k_{\mathrm{UV}}} \frac{\mathrm{d}k}{k} \Delta^2_{s(0)}(k) = \left(\int_{k_s}^{k_e} + \int_{k_e}^{\Lambda a(\tau_e)} \right) \frac{\mathrm{d}k}{k} \Delta^2_{s(0)}(k).$$

Total power spectrum:

$$\Delta^2_s(p) = \Delta^2_{s(0)}(p_*) \left(\frac{p}{p_*} \right)^{n_s - 1} \left\{ 1 + \frac{1}{4} (\Delta\eta)^2 \Delta^2_{s(\mathrm{PBH})} \left(1.1 + \log \frac{k_e}{k_s} + \log \tilde{\Lambda} + \frac{\tilde{\Lambda}^2 - 1}{2} \right) + \mathcal{O}\left[\left((\Delta\eta)^2 \Delta^2_{s(\mathrm{PBH})} \right)^2 \right] \right\}$$

where $\tilde{\Lambda} = \Lambda / H$.

Renormalization

Renormalizing the tree-level power spectrum:

$$\Delta^2_{s(0)}(p_*) \equiv \Delta^2_{s(0)}(p_*, \tilde{\mu}) \left\{ 1 + \frac{1}{4}(\Delta\eta)^2 \Delta^2_{s(0)}(p_*, \tilde{\mu}) \left(\frac{k_e}{k_s} \right)^6 \left(-1.1 - \log\frac{k_e}{k_s} + \log\frac{\tilde{\mu}}{\tilde{\Lambda}} + \frac{\tilde{\mu}^2 - \tilde{\Lambda}^2}{2} \right) + \mathcal{O}\left[\left((\Delta\eta)^2 \Delta^2_{s(\mathrm{PBH})} \right)^2 \right] \right\}$$

where $\tilde{\mu} = \mu/H$.

Renormalized power spectrum:

$$\Delta^2_s(p) = \Delta^2_{s(0)}(p_*, \tilde{\mu}) \left\{ 1 + \frac{1}{4}(\Delta\eta)^2 \Delta^2_{s(0)}(p_*, \tilde{\mu}) \left(\frac{k_e}{k_s} \right)^6 \left(\log\tilde{\mu} + \frac{\tilde{\mu}^2 - 1}{2} \right) + \mathcal{O}\left[\left((\Delta\eta)^2 \Delta^2_{s(\mathrm{PBH})} \right)^2 \right] \right\}.$$

Renormalization

At renormalization scale $\mu = H$:

$$\Delta_s^2(p) = \Delta_{s(0)}^2(p_*, \mu = H)\left(\frac{p}{p_*}\right)^{n_s-1}\left\{1 + \mathcal{O}\left[\left((\Delta\eta)^2\Delta_{s(\mathrm{PBH})}^2\right)^2\right]\right\}$$

Requirement to renormalize loop correction order by order: $(\Delta\eta)^2\Delta_{s(\mathrm{PBH})}^2 \ll 1$