Revisiting compaction functions for PBH formation

Tomohiro Harada

Department of Physics, Rikkyo University

Focus week on PBH at Kavli IPMU 2023 @ Kavli IPMU,
13-17/11/2023

This talk is based on arXiv:2304.13284 with C.M. Yoo and Y. Koga.

Outline

Introduction

- Primordial black holes (PBHs) are black holes formed in the early Universe (Zeldovich & Novikov (1967), Hawking (1971)).
  - Fossils of the early Universe
  - Dark matter candidate
  - Hawking evaporation
  - High-energy physics
  - GW sources

THE HYPOTHESIS OF CORES RETARDED DURING EXPANSION AND THE HOT COSMOLOGICAL MODEL

Ya. B. Zel'dovich and I. D. Novikov

Translated from Astronomicheskii Zhurnal, Vol. 43, No. 4, pp. 758-760, July-August, 1966
Original article submitted March 14, 1966

The existence of bodies with dimensions less than $R_g = 2GM/c^2$ at the early stages of expansion of the cosmological model leads to a strong accretion of radiation by these bodies. If further calculations confirm that accretion is catastrophically high, the hypothesis on cores retarded during expansion [3, 4] will conflict with observational data.

GRAVITATIONALLY COLLAPSED OBJECTS OF VERY LOW MASS

Stephen Hawking

(Communicated by M. J. Rees)

(Received 1970 November 9)

SUMMARY

It is suggested that there may be a large number of gravitationally collapsed objects of mass 10^{-5} g upwards which were formed as a result of fluctuations in the early Universe. They could carry an electric charge of up to ± 30 electron units. Such objects would produce distinctive tracks in bubble chambers and could form atoms with orbiting electrons or protons. A mass of 10^{17} g of such objects could have accumulated at the centre of a star like the Sun. If such a star later became a neutron star there would be a steady accretion of matter by a central collapsed object which could eventually swallow up the whole star in about ten million years.

Observational constraints

- Observational constraints on the abundance of PBHs
  - Dark matter mass windows: $\sim \mathbf{10^{16} - 10^{23}}$ g for all CDM and $\sim \mathbf{10^{27} - 10^{28}}$ g and $\sim \mathbf{1 - 10^{3}} M_{\odot}$ for a large fraction of CDM

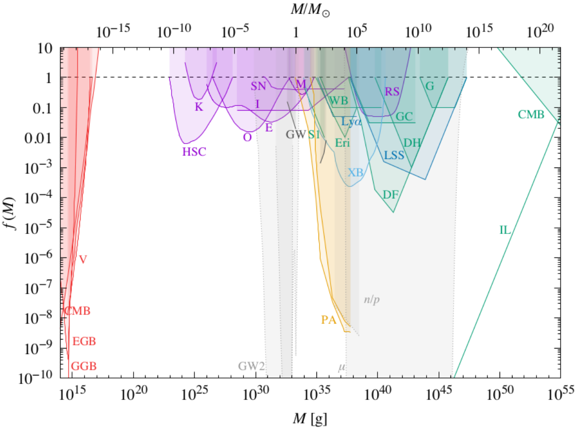


Figure: $f(M) = \Omega_{PBH}/\Omega_{CDM}$ (Carr et al. (2021))

GW Observations

- Many BBHs of $\sim \mathbf{30} M_{\odot}$ discovered by GW observation
  - Those BHs may be of cosmological origin. (Sasaki et al. (2016), Bird et al. (2016), Clesse & Garcia-Bellido (2017)).
  - Constraints on the spin parameter χ_{eff} of LIGO BBHs (Abbott et al. (2017))
  - Search for PBH population in LIGO-Virgo BBHs (Franciolini et al. (2022))
- Evidence or detection of nHz GWs by NANOGrav (Agazie et al. (2023), ...) and other PTAs
  - Maybe consistent with the secondary GWs of scalar perturbations that may have produced PBHs of solar mass or subsolar masses (Kohri & Terada (2021), Inomata et al. (2023), ...).

Outline

Outline

Mass of PBHs

- PBH mass is approximately equal to the mass enclosed within the cosmological horizon at the formation.

$$M \simeq M_H(t_f) \simeq \frac{c^3}{G} t_f \simeq 1 M_\odot \left(\frac{t_f}{10^{-5} \text{ s}} \right), \; R_g \simeq 1 \text{ km} \left(\frac{M}{M_\odot} \right)$$

- The mass accretion does not significantly affect the initial mass (Carr & Hawking (1974) ...).
- Hawking evaporation (Hawking (1974))

$$\begin{aligned} T_H &= \frac{\hbar c^3}{8\pi G M k} \simeq 100 \text{ MeV} \left(\frac{M}{10^{15} \text{g}} \right)^{-1}, \\ \frac{dM}{dt} &= -\frac{g_{\text{eff}} \hbar c^4}{15360 \pi G^2 M^2}, \quad t_{\text{ev}} \simeq \frac{G^2 M^3}{g_{\text{eff}} \hbar c^4} \simeq 10 \text{ Gyr} \left(\frac{M}{10^{15} \text{g}} \right)^3. \end{aligned}$$

Thus, they have dried up until now for $M \lesssim 10^{15}$ g.

Abundance of PBHs

- $\beta(M)$: The fraction of the Universe which goes into PBHs
- $f(M)$: The fraction of PBHs to all of the CDM at the present time

$$f(M) = \left. \frac{\Omega_{\rm PBH}}{\Omega_{\rm CDM}} \right|_{t=t_0} .$$

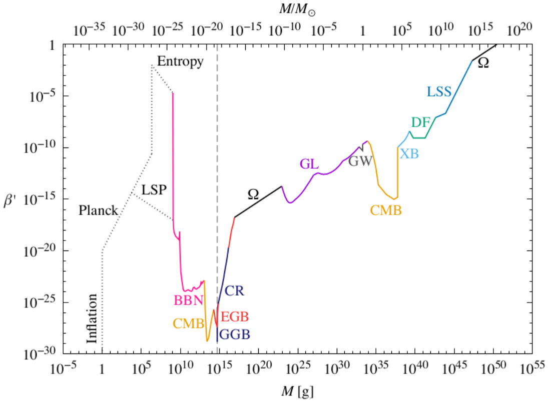
- PBHs are condensed during the radiation-dominated (RD) era because they behave as nonrelativistic particles, so

$$\beta(M) \simeq 2 \times 10^{-18} \left(\frac{M}{10^{15}\ {\rm g}} \right)^{1/2} f(M)$$

for $M > 10^{15}$ g if they are formed in the RD era.

Constraint on $\beta(M)$

- Constraint on $\beta(M)$ (Carr (1975), Carr et al. (2021))



Taken from Carr et al. (2021). The dotted lines below $\mathbf{10^9}$ g involve less secure assumptions.

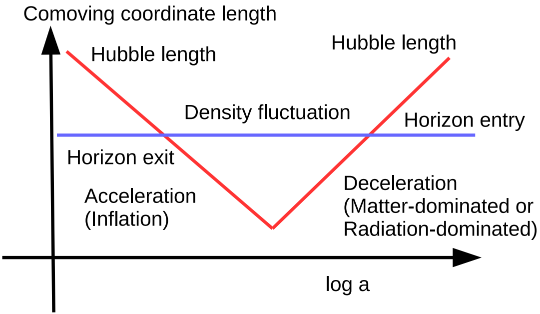
Outline

Formation studies of PBHs

- Central question: Can we predict $\beta(M)$ and other properties of PBHs for a given cosmological scenario?
- Possible mechanism
  - Conventional: growth of primordial fluctuations generated by inflation
  - New physics: collapse of domain walls, bubble nucleation, collision of bubbles, phase transitions, ...
- We here focus on the conventional scenario. Key ideas are
  - Primordial fluctuation generated by inflation
  - Cosmological long-wavelength solutions
  - Formation threshold
  - Critical behaviour
  - Abundance estimate

Fluctuation generated in inflation

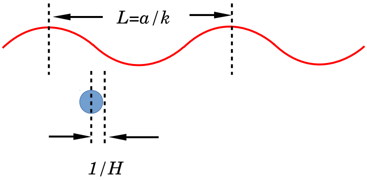
- The scales of perturbations of super-horizon scale generated in inflation enter the horizon in the decelerated expansion.



- Inflation gives the power spectrum $P_\zeta(k)$ and the statistics of curvature perturbations ζ and thereby the standard deviation $\sigma(k)$ and the statistics of density perturbation δ. See Sasaki et al. (2018).

Cosmological longwave-length solutions

- Initial data: long-wavelength solutions obtained by the gradient expansion in powers of $\epsilon = k/(aH) \ll 1$ (Shibata & Sasaki (1999), Polnarev & Musco (2007), Harada, Yoo, Nakama & Koga (2015))



- $3+1$ decomposition of spacetime

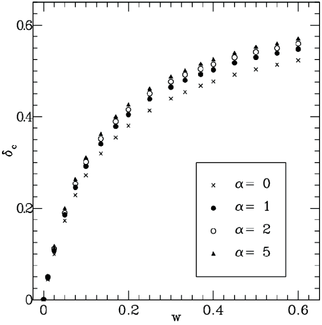
$$ds^2 = -\alpha^2 dt^2 + e^{2\zeta} a^2(t) \tilde{\gamma}_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

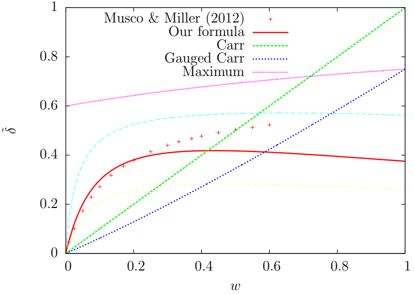
where $\tilde{\gamma}_{ij}$ is chosen so that $\det(\tilde{\gamma}_{ij}) = \det(\eta_{ij})$.

- The long-wavelength solutions have $\zeta = O(1)$ but $\delta = O(\epsilon^2)$.
- Only a nonliearly large amplitude of perturbation can form a PBH.

Formation threshold

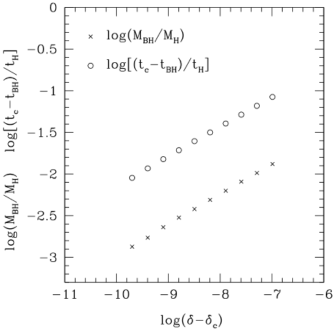
- PBH formation needs numerical relativity even in spherical symmetry.
- Threshold $\delta_{\rm th} \simeq 0.45$ for RD $p = \rho/3$ for the averaged density perturbation δ_H. Alternatively, $\mathcal{C} \simeq 0.4$ in terms of the compaction function $\mathcal{C}$. (Carr (1975), Shibata & Sasaki (1999), Musco et al. (2009), ...)
- EOS dependence ($p = w\rho$), implying enhancement for a soft EOS. The Jeans criterion works. (Musco & Miller (2013), Harada, Yoo & Kohri (2013))





Critical behaviour

- The overdensity collapses to a BH if $\delta_H > \delta_{\rm th}$, while it doesn't if $\delta_H < \delta_{\rm th}$. There appears critical behaviour with universality, self-similarity and power-law scaling laws. (Choptuik 1993, ...)
- PBH critical behaviour (Niemeyer & Jedamzik (1999), Musco & Miller (2013)) There appears mass scaling law.
$M_{BH} \approx k M_H (\delta - \delta_{\rm th})^\gamma, \quad \gamma \simeq 0.36$



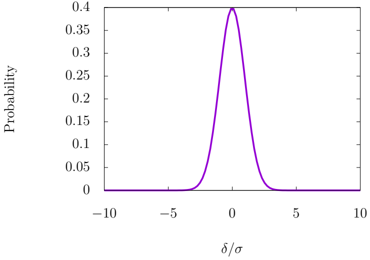
Abundance estimate

- Assuming δ_H obeys a Gaussian distibution, Carr (1975) obtained

$$\beta \simeq 2\frac{1}{\sqrt{2\pi}\sigma}\int_{\delta_{\rm th}}^{\delta_{\rm max}} d\delta e^{-\delta^2/(2\sigma^2)} \simeq \sqrt{\frac{2}{\pi}}\frac{\sigma}{\delta_{\rm th}}e^{-\delta_{\rm th}^2/(2\sigma^2)}.$$

Also called Press-Schechter. So, $\sigma \gtrsim 0.05$ or $P_\zeta \gtrsim 0.01$ is needed to have a cosmologically interesting amount.

- This means typically $\delta_{\rm th} \sim 8\sigma$. Only the tail of the distribution is responsible. The non-Gaussianity is crucial.



Outline

Outline

Inflationary cosmology

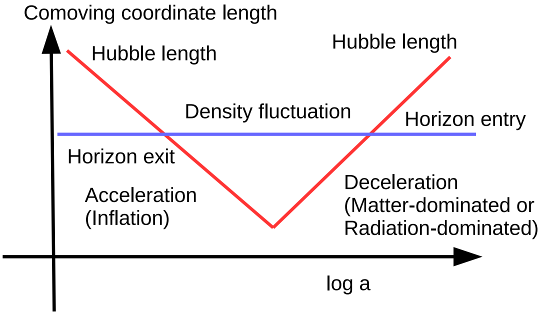


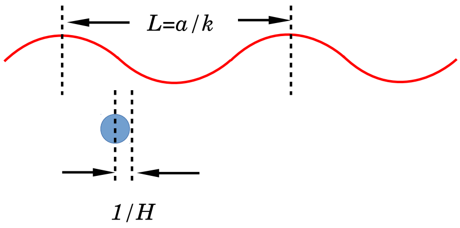
Figure: The time evolution of the Hubble length and the fluctuation scale

- LWL solutions as initial conditions
  - The fluctuations get stretched to super-horizon in an inflationary era.
  - After the inflation, the fluctuations are described by the LWL solutions.
  - Once they enter the Hubble horizon, the LWL scheme breaks down.

Long-wavelength limit

- A smoothing length is $L = a/k$, below which it is described by the FLRW, while the Hubble length is $\ell_H := H^{-1}$, where $H = \dot{a}/a$.
- Expansion parameter $\epsilon \ll 1$

$$\epsilon := \frac{\ell_H}{L} = \frac{k}{aH} \quad \text{with} \quad \frac{\partial_i \ln \Psi}{aH} = O(\epsilon)$$



- In the decelerated expansion, the limit $\epsilon \to 0$ realises as $t \to 0$.
- The term 'long-wavelength' is a bit misleading because we don't take the limit $k \to 0$.

Long-wavelength (LWL) solutions

- Flat FLRW solution:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

- Can we have a spacetime with a metric

$$ds^2 \approx -dt^2 + a^2(t)\Psi^4(x, y, z)(dx^2 + dy^2 + dz^2)$$

  as $t \to 0$? Yes, we can as a solution of the Einstein equation.
- In spherical symmetry, we have

$$ds^2 \approx -dt^2 + a^2(t)\Psi^4(r)(dr^2 + r^2 d\Omega^2)$$

  as $t \to 0$, where $d\Omega^2 := d\theta^2 + \sin^2\theta d\phi^2$.

Shibata & Sasaki (1999), Polnarev & Musco (2007)

Cosmological conformal 3+1 decomposition

- Metric

$$ds^2 = -\alpha^2 dt^2 + \psi^4 a^2(t)\tilde{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

where $\tilde{\gamma} = \eta$ with η_{ij} being the flat 3D metric.

- $\zeta := 2\ln\psi$ is called *curvature perturbation* in cosmology:
- Flat FLRW: $\alpha = 1$, $\beta^i = 0$, $\psi = 1$ and $\tilde{\gamma}_{ij} = \eta_{ij}$

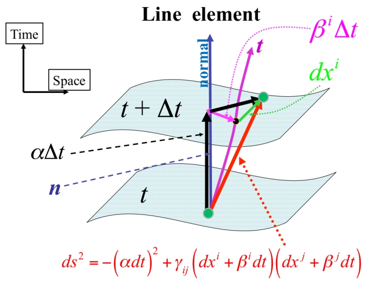


Figure: Slicing and threading with $\gamma_{ij} = \psi^4 a^2(t)\tilde{\gamma}_{ij}$

Long-wavelength solutions

- Additional assumption
  - $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$
  - $p = (\Gamma - 1)\rho$, where $\Gamma = 1 + w$
- We write $\psi = \Psi(\mathrm{x})(1 + \xi)$, $\alpha = 1 + \chi$ and $\tilde{\gamma}_{ij} = \eta_{ij} + h_{ij}$.
- ϵ expansion. The Einstein eqs imply the following:
  - Metric: ψ, α, $\tilde{\gamma}_{ij}$

$$\Psi(\mathrm{x}) = O(\epsilon^0), \quad \xi = O(\epsilon^2), \quad \beta^i = O(\epsilon),$$
$$\chi = O(\epsilon^2), \quad h_{ij} = O(\epsilon^2)$$

  - Matter: ρ, p, u^μ

$$\delta := \frac{\rho - \rho_b}{\rho_b} = O(\epsilon^2), \quad v^i := \frac{u^i}{u^t} = O(\epsilon)$$

  - Extrinsic curvature: $K_{ij} = A_{ij} + \gamma_{ij}K/3$

$$K = -3H(1 + \kappa), \ \kappa = O(\epsilon^2), \ \tilde{A}_{ij} = \psi^{-4}a^{-2}A_{ij} = O(\epsilon^2)$$

Shibata & Sasaki (1999), Lyth, Malik & Sasaki (2005)

Gauge issues

- We require the Einstein eqs and the EOM order by order in power of ϵ and solve them to obtain LWL solutions.
  Shibata & Sasaki (1999), Harada, Yoo, Nakama & Koga (2015)
- Gauge issues
  - Slicing: lapse function α
    - ⋆ Constant-Mean-Curvature slice: $\kappa = 0$
    - ⋆ Uniform-Density slice: $\delta = 0$
    - ⋆ Comoving slice: $u_i = 0$
    - ⋆ Geodesic slice: $\chi = 0$
  - Threading: shift vector β^i
    - ⋆ Comoving thread: $v^i = 0$
    - ⋆ Normal coordinates: $\beta^i = 0$
    - ⋆ Conformally Flat coordinates: $h_{ij} = 0$
  - Caveat: The conformal Newtonian gauge is inconsistent with the ansatz of the LWL solns.

LWL solns: next-to-leading order

- $\Psi(\mathrm{x})$ generate the LWL solutions. The explicit expressions to $O(\epsilon^2)$ are obtained in different gauges (Harada, Yoo, Nakama, Koga (2015)).
- CMC slice

$$\delta_{\rm CMC} \approx f\left(\frac{1}{aH}\right)^2, \ \ u_{{\rm CMC}j} \approx \frac{2}{3\Gamma(3\Gamma+2)H}\delta_{{\rm CMC},j},$$

where

$$f = f(\mathrm{x}) := -\frac{4}{3}\frac{\bar{\Delta}\Psi}{\Psi^5}$$

with $\bar{\Delta}$ being the flat Laplacian.

- Comoving slice

$$\delta_{\rm com} \approx \frac{3\Gamma}{3\Gamma+2}f\left(\frac{1}{aH}\right)^2, \ \ u_{{\rm com}j} = 0.$$

- The above do not depend on the threading condition.

Outline

Quasi-local mass in spherical symmetry

- We focus on spherically symmetric spacetimes.

$$ds^2 = g_{AB}(x^C)dx^A dx^B + R^2(x^C)d\Omega^2,$$

where R is the areal radius and A, B and C run over 0 and 1.

- Misner-Sharp mass as total energy enclosed within a sphere of x^C

$$M := \frac{1}{2}R(1 - D_A R D^A R)$$

with D_A being the covariant derivative compatible with g_{AB}.

- The Misner-Sharp mass has an integral form (or Kodama mass):

$$M := -\int_\Sigma S^\mu d\Sigma_\mu,$$

where Σ is a 3-ball bounded by the 2-sphere of x^A.
  - Kodama current: $S^\mu := -T^\mu{}_\nu K^\nu$
  - Kodama vector: $K^\mu := -\epsilon^{AB}\partial_B R \left(\partial/\partial x^A\right)^\mu$.

Misner & Sharp (1964), Kodama (1980), Hayward (1996), Yoo (2022)

Outline

Shibata-Sasaki compaction function

- Shibata and Sasaki (1999) used the conformally flat coordinates

$$ds^2 = -\alpha^2 dt^2 + \psi^4 a^2[(dr + \beta r dt)^2 + r^2 d\Omega^2],$$

with the CMC slice in spherical symmetry.

- Gave the expressions of the mass excess and the compaction function

$$\begin{aligned} \delta M_{\rm SS} &:= 4\pi a^3 \rho_0 \int_0^r x^2 dx \delta_{\rm CMC} \cdot \psi^6 \left(1 + \frac{2x}{\psi}\frac{\partial \psi}{\partial x}\right) \\ \mathcal{C}_{\rm SS} &:= \frac{\delta M_{\rm SS}(t,r)}{R(t,r)} = \frac{\delta M_{\rm SS}(t,r)}{r\psi^2(t,r)a}. \end{aligned}$$

- $\mathcal{C}_{\rm SS}$ becomes time-independent in the limit $\epsilon \to 0$ or $t \to 0$, so that

$$\mathcal{C}_{\rm SS}(t,r) \approx \mathcal{C}_{\rm SS}(r) \approx \frac{\int d\Sigma \delta\rho}{R} \approx \frac{1}{2}\bar{\delta}_{{\rm CMC},H}(r),$$

where $\bar{\delta}_{{\rm CMC},H}$ is the density perturbation averaged over the horizon patch at the horizon entry $a\Psi^2 r = H^{-1}$ to the next-leading order of $O(\epsilon)$.

$\mathcal{C}_{\rm SS}$ in terms of Ψ

- $\delta_{\rm CMC}(t,r)$ and $\mathcal{C}_{\rm SS}(r)$

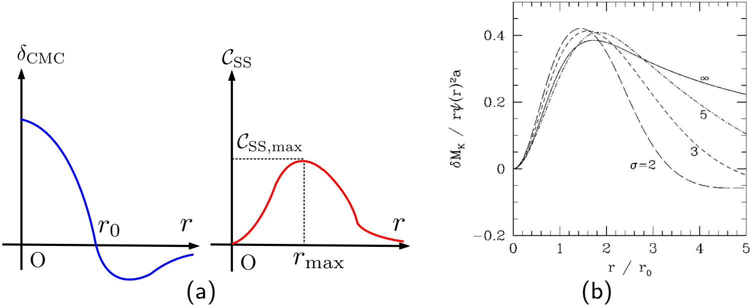


Figure: (a) $\delta_{\rm CMC}(t,r)$, $\mathcal{C}_{\rm SS}(r)$, (b) $\mathcal{C}_{\rm SS}$ for the critical cases

- Empirically, the maximum of $\mathcal{C}_{\rm SS}(r)$ (or its volume average) gives a good indicator for PBH formation. The threshold value is $\simeq 0.4$ for radiation $\Gamma = 4/3$.

Shibata & Sasaki (1999), Escriva et al. (2019)

$\mathcal{C}_{\rm SS}$ in terms of Ψ

- LWL soln in the CMC slice in spherical symmetry

$$\delta_{\rm CMC} \approx f\left(\frac{1}{aH}\right)^2, \quad u_{{\rm CMC}r} \approx \frac{2}{3\Gamma(3\Gamma+2)H}\delta_{{\rm CMC},r},$$
$$\Psi = \Psi(r), \quad f = f(r) = -\frac{4}{3}\frac{1}{r^2\Psi^5}\left(r^2\Psi'\right)'$$

- $\mathcal{C}_{\rm SS}$ in the LWL soln can be rewritten as

$$\mathcal{C}_{\rm SS} \approx \frac{1}{2}\left[1-\left(1+2\frac{d\ln\Psi}{d\ln r}\right)^2\right]$$

This does not contain Ψ''.

Harada, Yoo, Nakama & Koga (2015)

Outline

Mass and mass excess

- Mass in the spatially conformally flat coordinates

$$
\begin{aligned}
M &= 4\pi \int_0^r x^2 dx a^3 \alpha \psi^6 T^t{}_\mu K^\mu \\
&= 4\pi a^3 \int_0^r dx (\psi^2 x)^2 \left\{ -[(\rho+p)u^t u_t + p](\psi^2 x)' \right. \\
&\quad \left. + (\rho+p)u^t u_r \frac{x}{a}(\psi^2 a)_{,t} \right\}.
\end{aligned}
$$

- Mass excess
  - The mass excess from the flat FLRW spacetime is naturally defined as

$$
\delta M(t,r) = M(t,r) - M_{\mathrm{FF}}(t, \psi^2(t,r) r),
$$

i.e., the difference between masses enclosed by two spheres of the same areal radius.

Mass excess in the CMC slice

- Mass excess

$$
\begin{aligned}
\delta M_{\rm CMC} &\approx 4\pi a^3 \rho_b \int_0^r dx (\Psi^2 x)^2 \left[\delta_{\rm CMC} (\Psi^2 x)' \right. \\
&\left. + \frac{2}{3(3\Gamma+2)} \delta'_{\rm CMC} (\Psi^2 x) \right] \\
&= 4\pi a^3 \rho_b \left[\frac{3\Gamma}{3\Gamma+2} \int_0^r dx (\Psi^2 x)^2 (\Psi^2 x)' \delta_{\rm CMC} \right. \\
&\left. + \frac{2}{3(3\Gamma+2)} \delta_{\rm CMC}(t,r) (\Psi^2(r) r)^3 \right],
\end{aligned}
$$

where integration by parts is implemented. This reduces to

$$
\mathcal{C}_{\rm CMC}(r) \approx \frac{3\Gamma}{3\Gamma+2} \mathcal{C}_{\rm SS}(r) + \frac{1}{3\Gamma+2} f(r) (\Psi^2(r) r)^2,
$$

where $\mathcal{C}_{\rm CMC} := \dfrac{\delta M_{\rm CMC}}{R}$ is the 'legitimate' compaction function.

$\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm CMC}$

- $\delta M_{\rm CMC}$ is different from $\delta M_{\rm SS}$ due to the nonvanishing $u_{{\rm CMC}\ r}$! There is no direct relation between $\mathcal{C}_{\rm CMC}$ and $\mathcal{C}_{\rm SS}$.
- $\mathcal{C}_{\rm SS}$ does not contain Ψ''. This is why $\mathcal{C}_{\rm SS}$ is empirically robust.
- If $\delta(t,r)$ has a spike like below, $\mathcal{C}_{\rm CMC}(r)$ has a large maximum of $O(\Delta^{-1/2})$ at r_1, whereas both $\mathcal{C}_{\rm SS}(r)$ and $\Psi(r)$ are kept small.

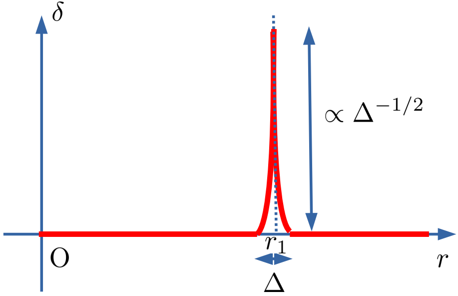


Figure: Density spike

$\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm com}$

- In the comoving slice, we have

$$\begin{aligned}\delta_{\rm com} &\approx \frac{3\Gamma}{3\Gamma+2} f \left(\frac{1}{aH}\right)^2, \quad u_{{\rm com}j}=0, \\ \delta M_{\rm com}(t,r) &\approx \frac{3\Gamma}{3\Gamma+2}\delta M_{\rm SS}(t,r)\end{aligned}$$

- The legitimate compaction function in the comoving slice is thus directly related to $\mathcal{C}_{\rm SS}$ as

$$\mathcal{C}_{\rm com}(r) := \frac{\delta M_{\rm com}}{R} \approx \frac{3\Gamma}{3\Gamma+2}\mathcal{C}_{\rm SS}(r).$$

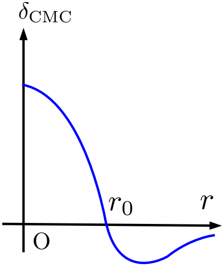
- The threshold value for the maximum of $\mathcal{C}_{\rm com}(r)$ is therefore $\simeq 0.27$ for $\Gamma = 4/3$.

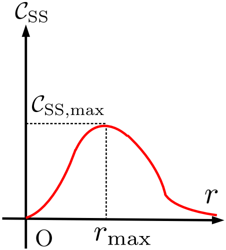
Gauge dependence of compaction functions

- $\mathcal{C}(r)$'s in different slices contain $\Psi''(r)$ except for $\mathcal{C}_{\rm com}(r)$.
- For $r = r_0$, where $f(r_0) = 0$, we find

$$\mathcal{C}_{\rm com}(r_0) \approx \mathcal{C}_{\rm CMC}(r_0) \approx \mathcal{C}_G(r_0) \approx \mathcal{C}_{\rm UD}(r_0) \approx \frac{3\Gamma}{3\Gamma+2}\mathcal{C}_{\rm SS}(r_0).$$

  Thus, $\mathcal{C}(r_0)$'s are gauge-independent except for $\mathcal{C}_{\rm SS}(r_0)$.
- However, $\mathcal{C}(r)$'s and their maxima are gauge-dependent.





Outline

Summary

- Despite the initial intention, $\mathcal{C}_{\rm SS}$ is not directly related to $\delta M/R$ in the CMC slice but happens to that in the comoving slice up to a constant factor.

- $\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm com}$ are unique for not containing Ψ''. This is why they are very robust to give a threshold for PBH formation.

Outline

Outline

Reconstruction to the next-to-leading order

- The Shibata-Sasaki gauge choice (the CMC slice and the Conformally Flat coordinates) gives the following LWL solutions:

$$
\begin{aligned}
\delta &\approx f\left(\frac{1}{aH}\right)^2, \quad u_j \approx \frac{2}{3\Gamma(3\Gamma+2)a^3H^2} f'(r)\delta^r_j, \\
\chi &\approx -\frac{3\Gamma-2}{3\Gamma} f \frac{1}{(aH)^2}, \\
\beta &\approx \left\{-\frac{6}{3\Gamma+2}\int_\infty^r d\tilde{r}\frac{1}{\tilde{r}^3}\left[\frac{1}{\Psi^4(\tilde{r})}\mathcal{C}_{\rm SS}(\tilde{r}) - \frac{1}{2}\tilde{r}^2 f(\tilde{r})\right]\right. \\
&\quad \left. +\tilde{\beta}_\infty\right\}\frac{1}{a^2H} =: \tilde{\beta}(r)\frac{1}{a^2H}, \\
\xi &\approx \frac{1}{2(3\Gamma-2)}\left\{-\frac{2}{3\Gamma+2}\frac{\mathcal{C}_{\rm SS}}{r^2\Psi^4} - \frac{9\Gamma^2-3\Gamma-4}{3\Gamma(3\Gamma+2)}f\right. \\
&\quad \left. +\left(1+\frac{2r\Psi'}{\Psi}\right)\tilde{\beta}(r)\right\}\frac{1}{a^2H^2} =: \tilde{\xi}(r)\frac{1}{a^2H^2},
\end{aligned}
$$

where $\tilde{\beta}_\infty$ is a constant of integration.

Mass in terms of the boundary

- The compaction function is originally given in terms of the spatial integral but can also be written in terms of the metric functions at the boundary.
- Shibata and Sasaki used the conformally flat coordinates

$$ds^2 = -\alpha^2 dt^2 + \psi^4 a^2[(dr + \beta r dt)^2 + r^2 d\Omega^2],$$

  with the CMC slice in spherical symmetry.
- We use the definition of the Misner-Sharp mass

$$\frac{2M}{R} = 1 - D_A R D^A R,$$

  where $R = \psi^2(t, r)a(t)r$. This is an expression in terms of the boundary.

Consistency check

- This expression implies

$$\frac{2\delta M_{\rm CMC}}{R} \approx 1-\left(1+\frac{2r\Psi'}{\Psi}\right)^2 + 2\left[2\dot{\xi}-H\chi-\beta\left(1+\frac{2r\Psi'}{\Psi}\right)\right](\Psi^2 r)^2(a^2 H),$$

  where we have used $2M_{\rm FF}/R = H^2R^2$.

- Now that we have the full set of the LWL solution, let us check consistency about the compaction function.
- Using the obtained solution for χ, β and ξ, we can show

$$\mathcal{C}_{\rm CMC}(r) \approx \frac{3\Gamma}{3\Gamma+2}\mathcal{C}_{\rm SS}(r)+\frac{1}{3\Gamma+2}f(r)(\Psi^2(r)r)^2.$$

  This coincides with the expression obtained using the spatial integral.

Outline

7 Backup

- LWL solns in the Shibata-Sasaki gauge condition
- Calculations
- Alternative choice of the background mass

Calculation 1

- For the LWL soln, this reduces to

$$\frac{2M}{R} \approx H^2R^2 + 1 - \left(1 + \frac{2r\Psi'}{\Psi}\right)^2 + 2\left[2\dot{\xi} - H\chi - \beta\left(1 + \frac{2r\Psi'}{\Psi}\right)\right](\Psi^2 r)^2(a^2 H),$$

where we have used $\alpha = 1 + \chi$ and $\psi = \Psi(r)(1 + \xi)$.

- $2\delta M/R$ is then written as

$$\frac{2\delta M}{R} \approx 1 - \left(1 + \frac{2r\Psi'}{\Psi}\right)^2 + 2\left[2\dot{\xi} - H\chi - \beta\left(1 + \frac{2r\Psi'}{\Psi}\right)\right](\Psi^2 r)^2(a^2 H)$$

because

$$\frac{2M_{\mathrm{FF}}}{R} = H^2R^2$$

for the corresponding FLRW solns.

Calculation 2

- The evolution equations for ψ and h_{ij} in Einstein equations imply

$$2\dot{\xi} - H\chi - \beta\left(1 + \frac{2r\Psi'}{\Psi}\right) - \frac{1}{3}r\beta' \approx 0, \qquad (1)$$

$$r\beta' \approx -\frac{3}{r^2}\tilde{A}_{22}, \qquad (2)$$

where

$$\tilde{A}_{ij} := \psi^{-4}a^{-2}\left(K_{ij} - \frac{\gamma_{ij}}{3}K\right),$$

and we have imposed $\kappa = 0$ and $h_{ij} = 0$.

Calculation 3

- Using (1), we can eliminate $\dot{\xi}$, χ and β, while $\tilde{A}_{ij}$ is given in the LWL soln by

$$\tilde{A}_{22} \approx \left\{ \frac{1}{2\Psi^4} \left[1 - \left(1 + \frac{2r\Psi'}{\Psi} \right)^2 \right] + \frac{2}{3} r^2 \frac{\bar{\Delta}\Psi}{\Psi^5} \right\} \frac{2}{3\Gamma + 2} \frac{1}{a^2 H}.$$

- Therefore, eliminating β' using (2), we obtain

$$\frac{\delta M}{R} \approx \frac{3\Gamma}{2(3\Gamma+2)} \left[1 - \left(1 + \frac{2r\Psi'}{\Psi} \right)^2 \right] + \frac{1}{3\Gamma+2} f (\Psi^2 r)^2,$$

or

$$\mathcal{C}(r) \approx \frac{3\Gamma}{3\Gamma+2} \mathcal{C}_{\rm SS}(r) + \frac{1}{3\Gamma+2} f(r) (\Psi^2 r)^2,$$

where we have used

$$f = -\frac{4}{3} \frac{1}{r^2 \Psi^5} \left(r^2 \Psi' \right)'.$$

Outline

7. Backup
   - LWL solns in the Shibata-Sasaki gauge condition
   - Calculations
   - Alternative choice of the background mass

Alternative choice of the background mass

- We can alternatively choose $M_{\mathbf{FF}}(t, \Psi^2(r)r)$ rather than $M_{\mathbf{FF}}(t, \psi^2(t,r)r)$ for the background mass.
- This implies an alternative mass excess and a compaction function:

$$\begin{aligned} \Delta M(t,r) &:= M(t,r) - M_{\mathbf{FF}}(t, \Psi^2(r)r), \\ \mathscr{C}_{\mathrm{CMC}}(t,r) &:= \frac{\Delta M(t,r)}{R(t,r)} \approx \mathscr{C}_{\mathrm{CMC}}(r). \end{aligned}$$

- This definition implies

$$\mathscr{C}_{\mathrm{CMC}} - \mathcal{C}_{\mathrm{SS}} = \left(3H\xi + \frac{1}{3} r\beta'\right)(\Psi^2 r)^2 a^2 H$$

in the Shibata-Sasaki gauge conditions.

- In general, $\mathcal{C}_{\mathrm{SS}}$ does not coincide with $\mathscr{C}_{\mathrm{CMC}}$ either.