Highly *Asymmetric Probability Distribution* from a finite-width upward step during inflation

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Collaboration with Tomohiro Fujita and Misao Sasaki

1. Introduction

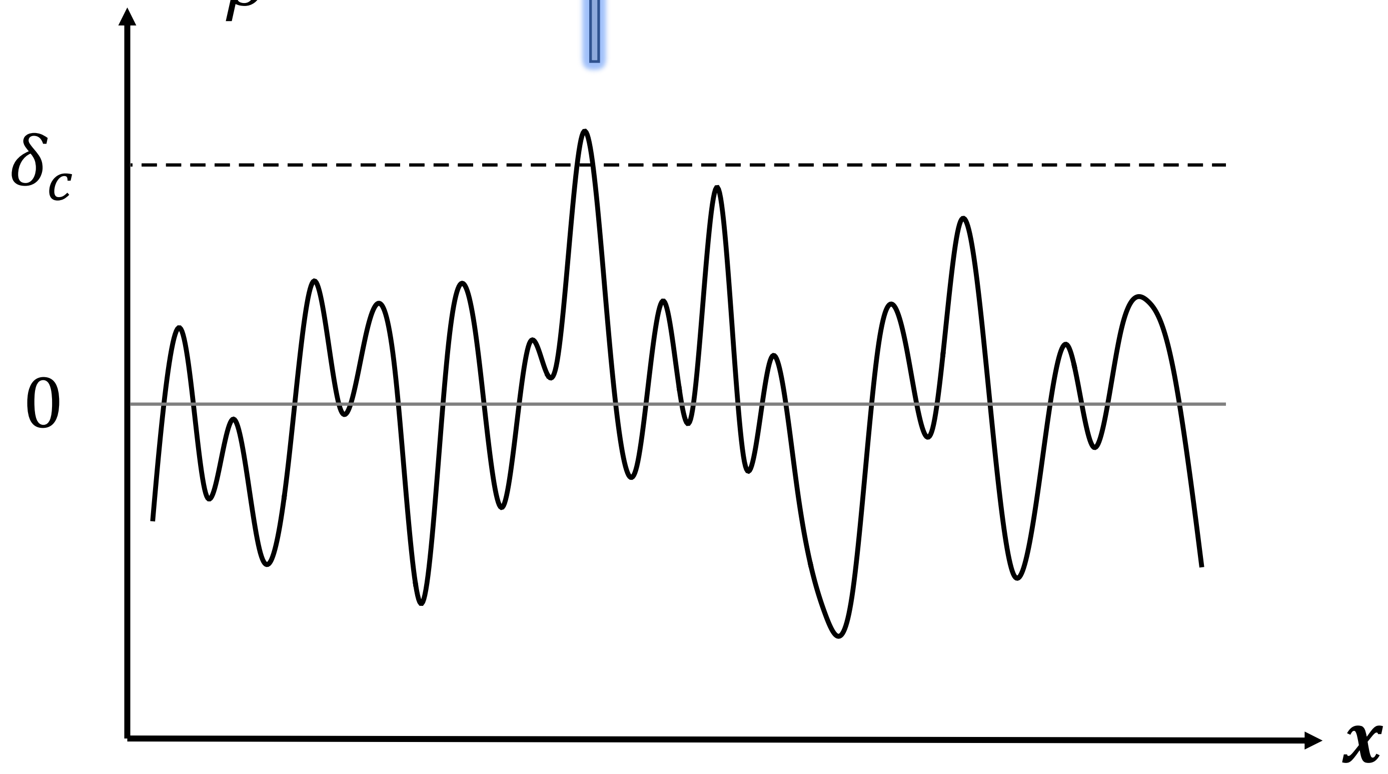
Primordial Black Hole (PBH)

Y.B.Zel'dovich and I.D.Novikov(1967)

S.Hawking(1971)

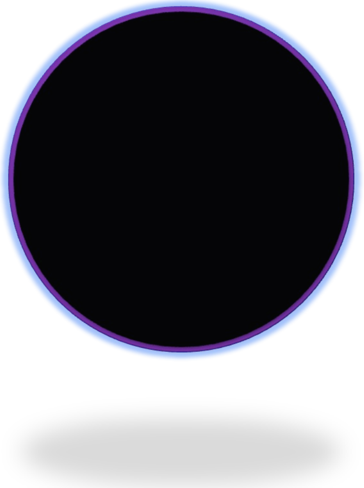
Density contrast

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}$$



Gravitational collapse

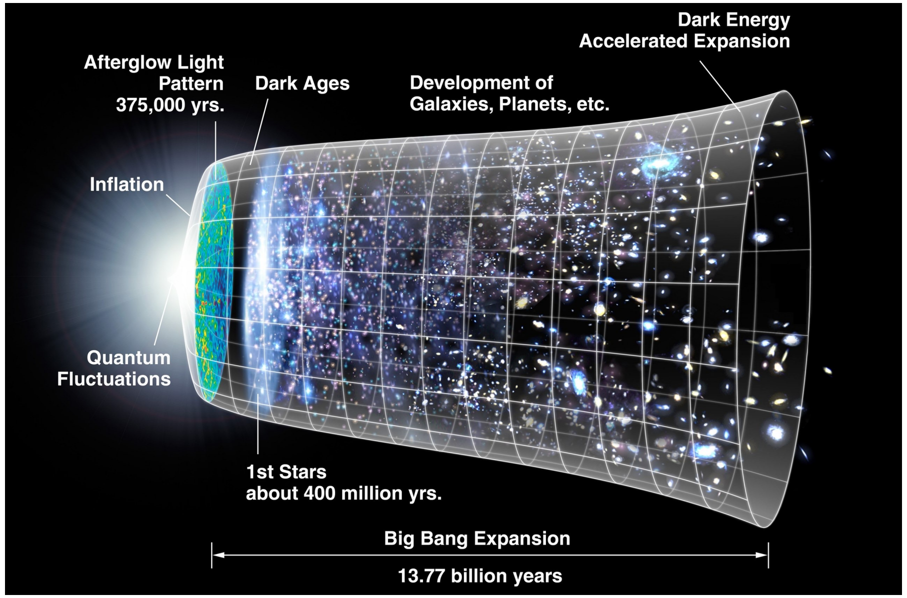
PBH



Candidate for
DM, BBH, SMBH ⋯

What is the origin of large density contrast ?

Cosmic Inflation



Quantum fluctuation

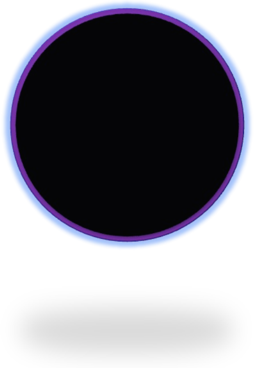
Origin of structure of universe

Inflation can be the origin of large density contrast

- The density contrast δ is related to the curvature perturbation $\mathcal{R}$

$$\delta_k = \frac{4}{9}\left(\frac{k}{aH}\right)^2 \mathcal{R}_k$$ in the linear perturbation theory

Large Curvature perturbation → **PBH**



amplified by some mechanism during inflation
e.g. Ultra Slow Roll (USR)

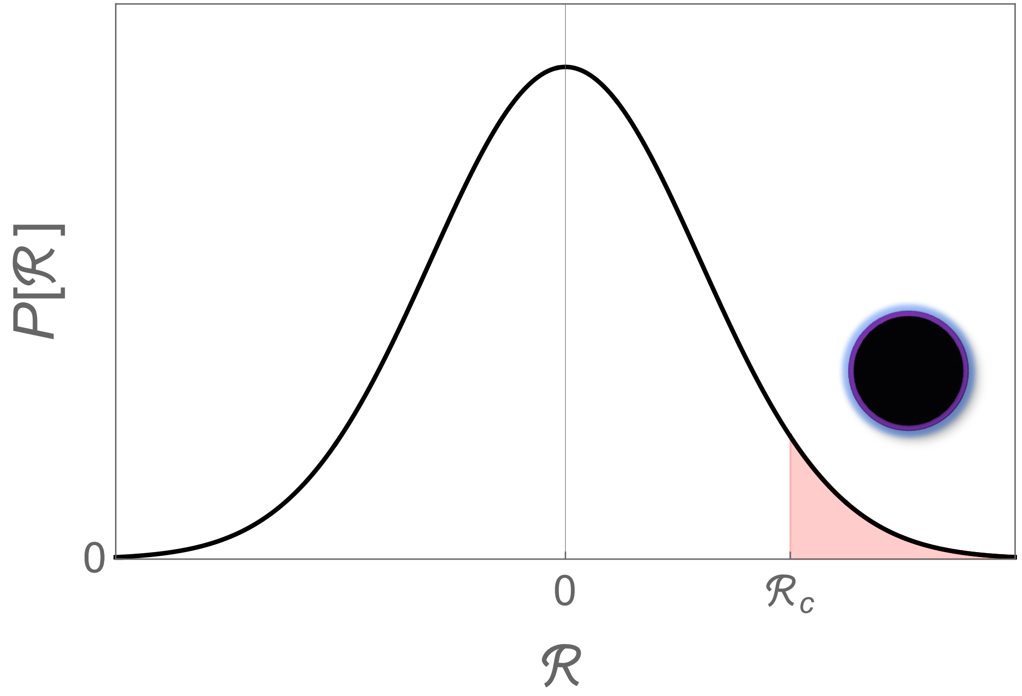
Garcia-Bellido et al (2017)

- As a candidate for DM, it is important to accurately estimate the abundance of PBHs.

- We can estimate it by calculating the complementary cumulative distribution function (**CCDF**) of curvature perturbation $\mathcal{R}$

$$\bar{F}[\mathcal{R}_c] = \int_{\mathcal{R}_c}^{\infty} P[\mathcal{R}] d\mathcal{R}$$

where $P[\mathcal{R}]$ is the probability distribution function (**PDF**).



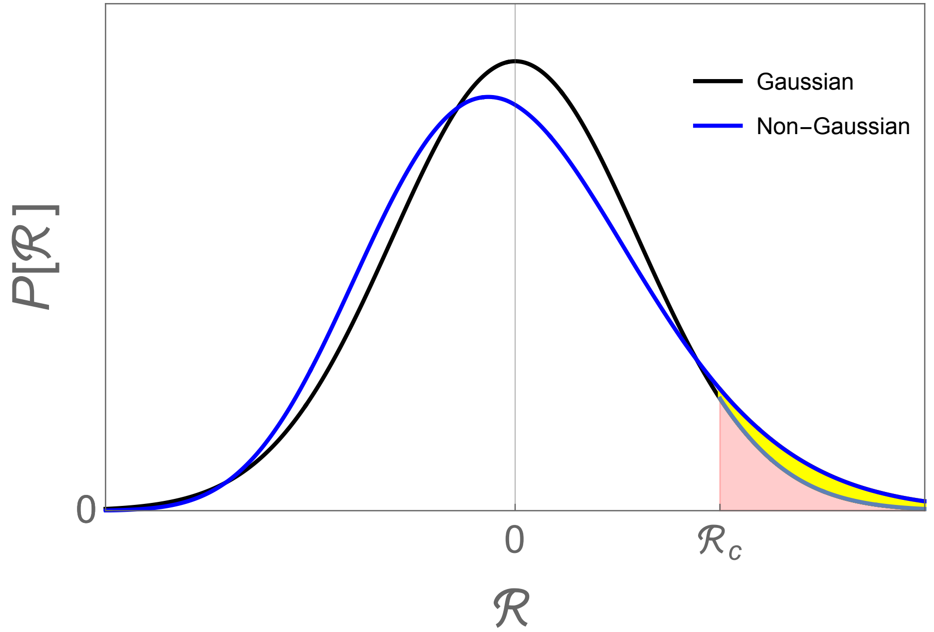
- Assuming the Gaussian distribution, CCDF is equal to the complementary error function.

Gaussian?

Non-Gaussianity

$$\bar{F}[\mathcal{R}_c] = \int_{\mathcal{R}_c}^{\infty} P[\mathcal{R}]d\mathcal{R}$$

- CCDF is **sensitive to the non-Gaussianity** of curvature perturbation

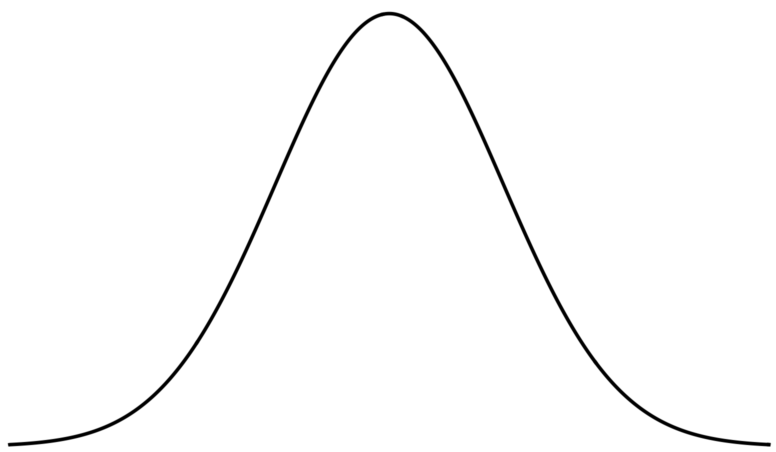


- The large and rare fluctuations that cause PBH formation cannot be precisely assessed by perturbative methods.

→ **δN formalism**

A.A. Starobinsky (1985)
M.Sasaki and E.D.Stewart (1996)

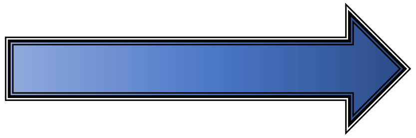
- Using the **δN formalism** and **Probability conservation**, we can obtain Probability distribution function (PDF) $P[\mathcal{R}]$.



$P[\delta\varphi]$ (Gaussian)
Quantum fluctuation

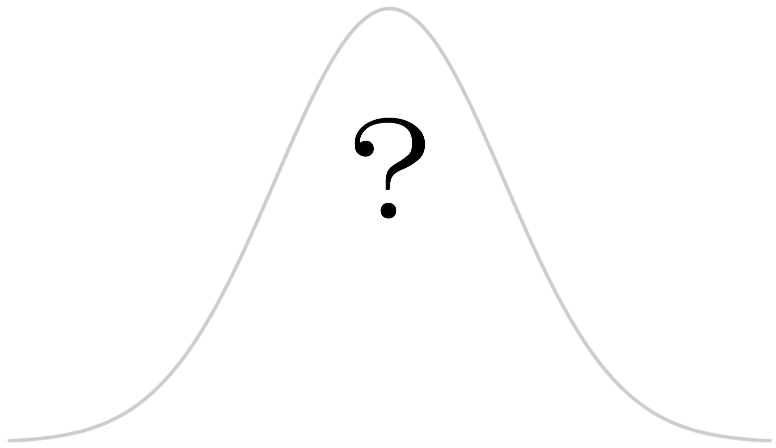
δN formalism

$$\mathcal{R} = \delta N(\delta\varphi)$$



Probability conservation

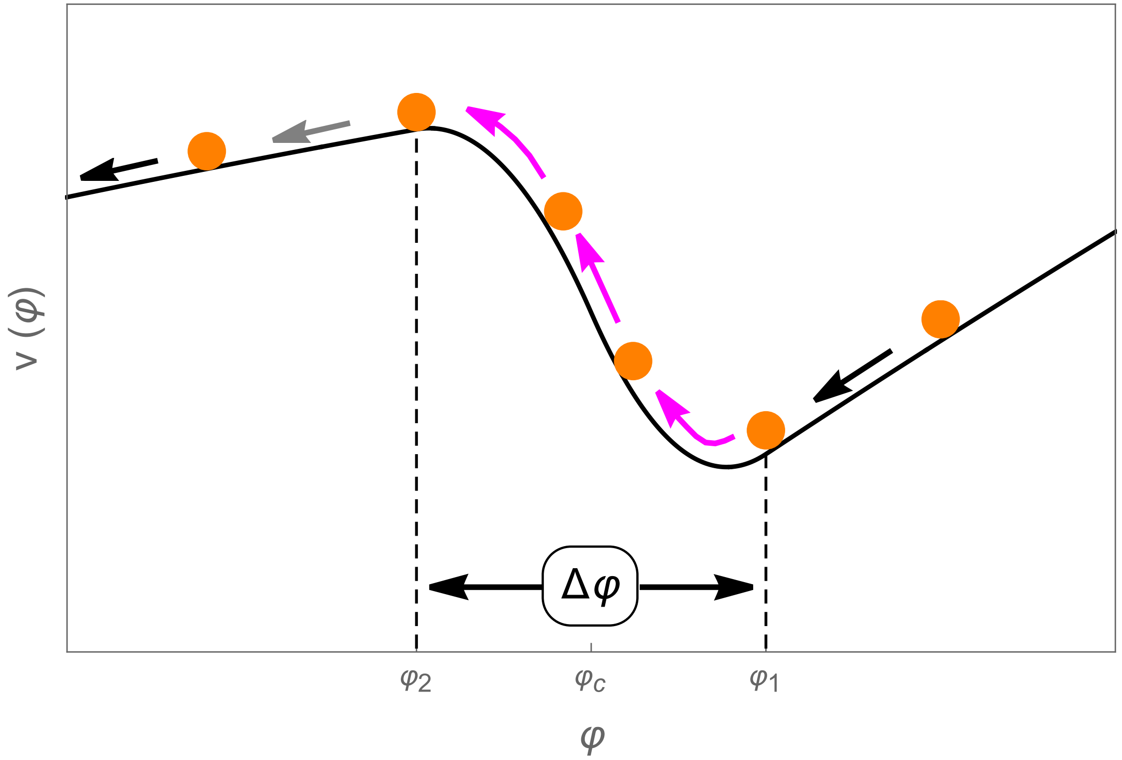
$$P[\mathcal{R}] = P[\delta\varphi]\left|\frac{d\delta\varphi}{d\mathcal{R}}\right|$$



$P[\mathcal{R}]$

2. Step Model

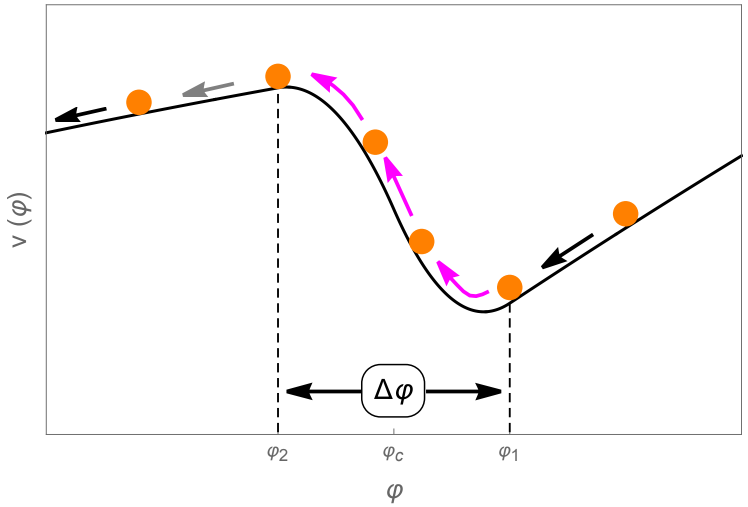
Step Model

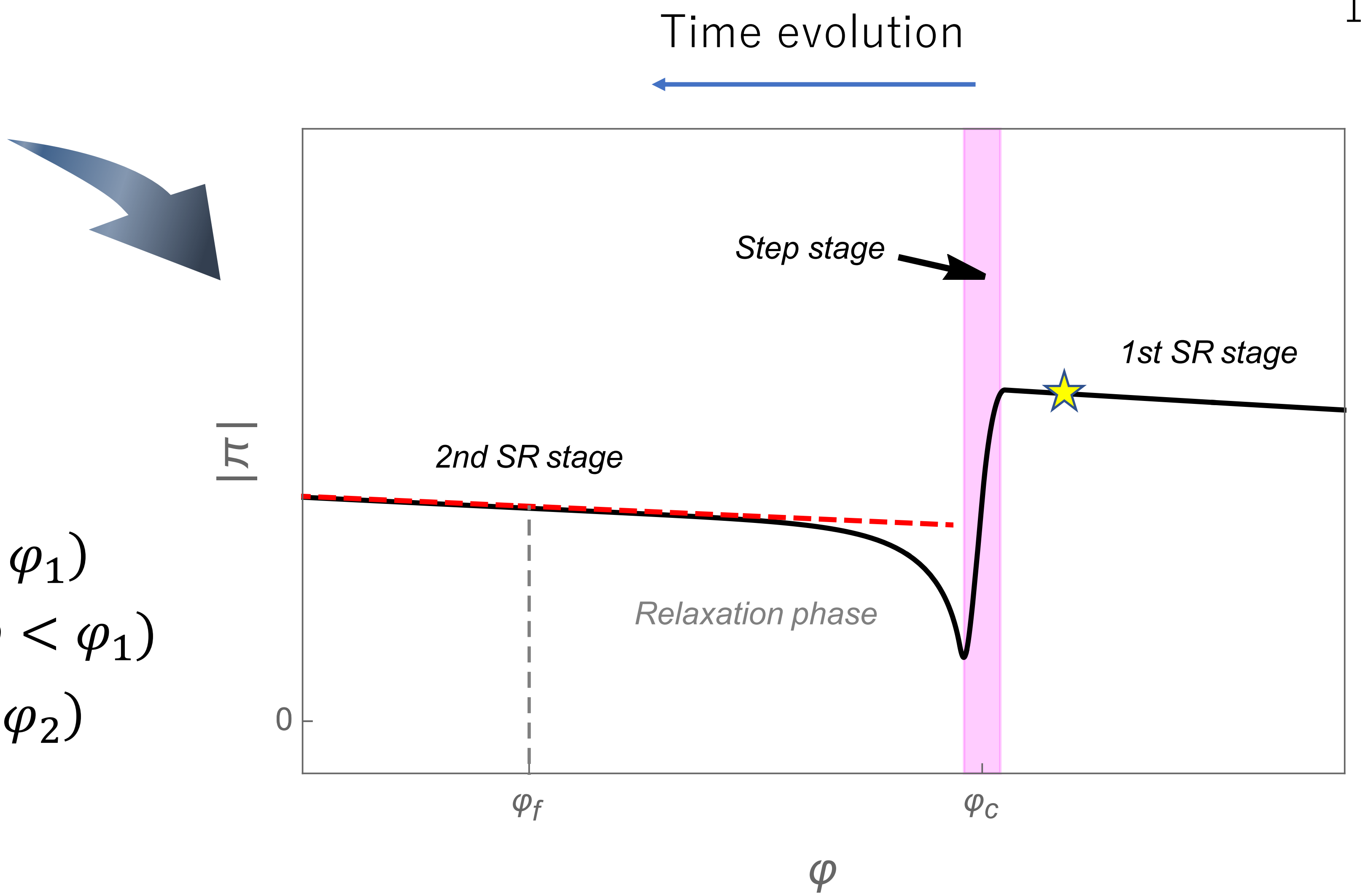


$$v(\varphi) = \begin{cases} v_{sr1}(\varphi) & (\varphi \geq \varphi_1) \\ f_{step}(\varphi) & (\varphi_2 < \varphi < \varphi_1) \\ v_{sr2}(\varphi) & (\varphi \leq \varphi_2) \end{cases}$$

$$f_{step}(\varphi) = \begin{cases} A_1 + B_1(\varphi - \varphi_{min})^2 & (\varphi_c \leq \varphi < \varphi_1) \\ A_2 + B_2(\varphi - \varphi_{max})^2 & (\varphi_2 < \varphi < \varphi_c) \end{cases}$$

- Requiring that the potential and its first φ-derivative are continuous at $\varphi = \varphi_c, \varphi_1$ and φ_2, the six constants are thereby determined.





(i) 1st SR stage $(\varphi \geq \varphi_1)$

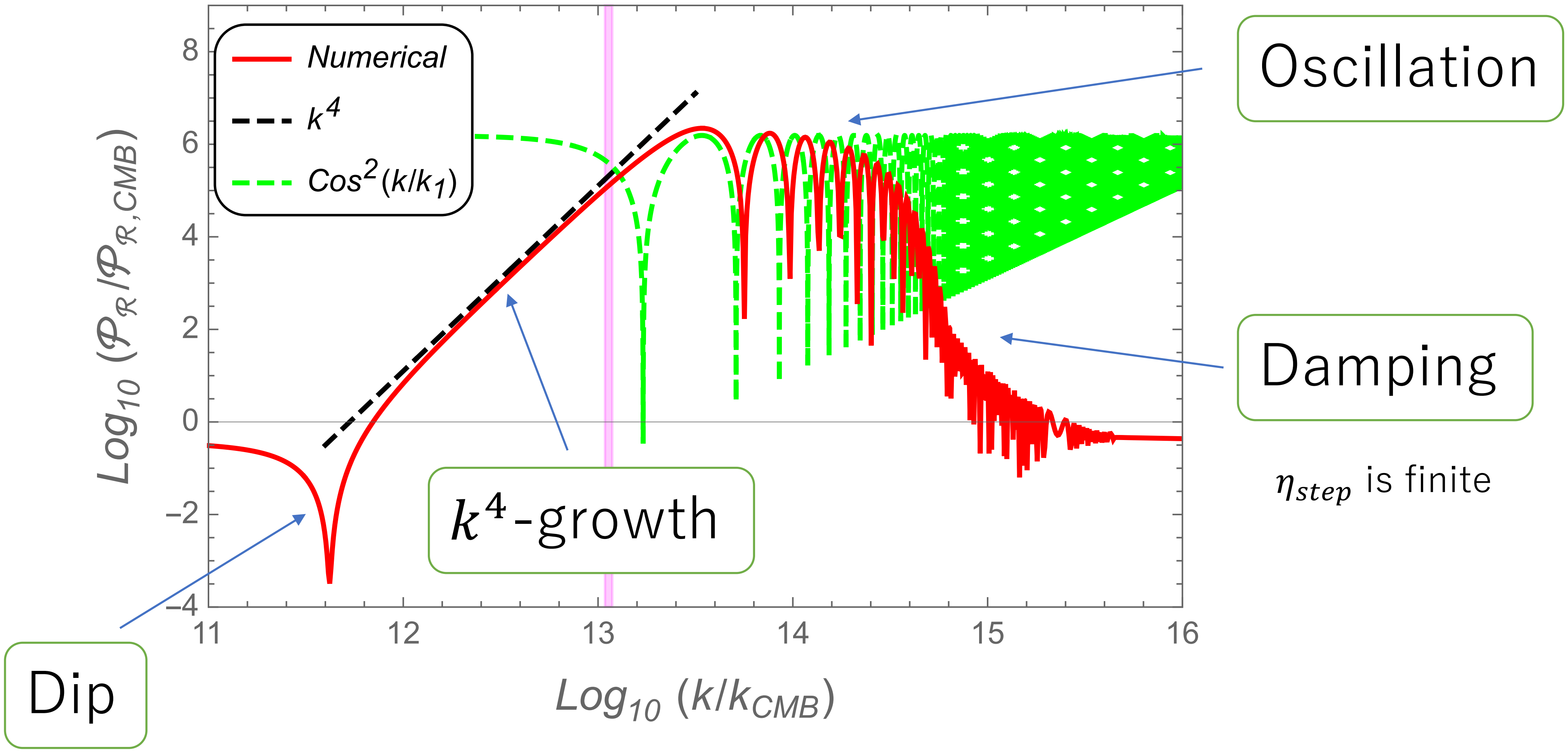
(ii) Step stage $(\varphi_2 < \varphi < \varphi_1)$

(iii) 2nd SR stage $(\varphi \leq \varphi_2)$

- Due to the presence of the upward step, there are three distinct stages of the background evolution.

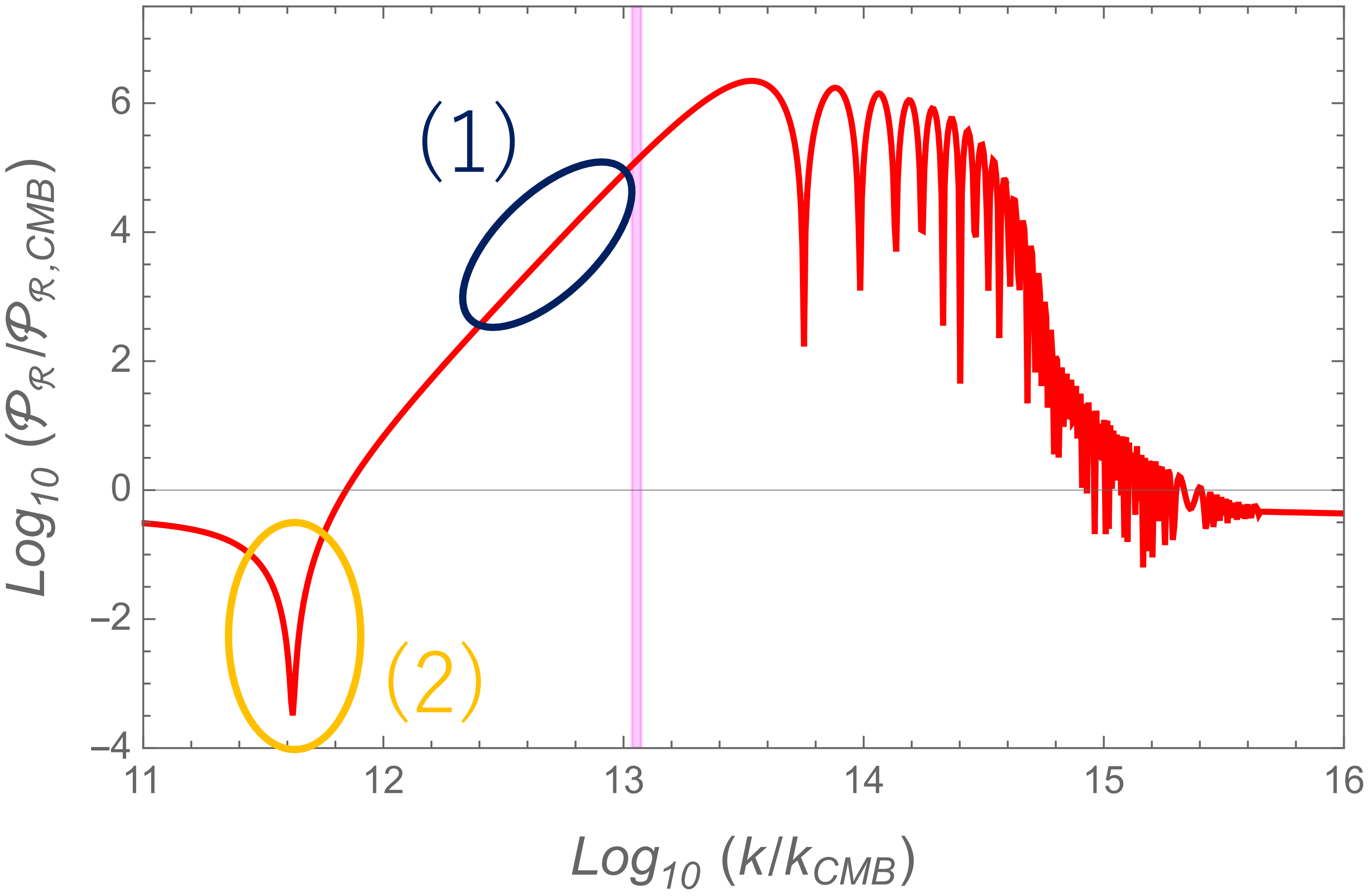
K. Inomata et al. (2022)
Y.-F. Cai et al. (2022)

Power spectrum in Step model



η_{step} is finite

- In our study, we calculate PDF at two different scales,
  (1) Scale exiting the Hubble horizon just before the step stage
  (2) Dip scale



3. PDF and CCDF

Result of δN calculation

- Summing up the contributions to δN from the three stages,

$$\mathcal{R} = \underbrace{\beta\delta\varphi}_{\delta N^{(1)}} + \underbrace{\frac{\kappa g}{3}\left(1-\sqrt{1+\frac{2\gamma}{g^2}\delta\varphi+\frac{\gamma^2}{g^2}\delta\varphi^2}\right)}_{\delta N^{(2)}} \underbrace{- \frac{1}{2\omega_{s2}}\log\left(1+\frac{2\gamma}{g^2}\delta\varphi+\frac{\gamma^2}{g^2}\delta\varphi^2\right)}_{\delta N^{(step)}}$$

where

$$\beta = -\frac{1}{\pi} > 0\,, \qquad \gamma = \frac{\eta_1\beta}{2}\left(\frac{\pi}{\pi_1}\right)^{\frac{6}{\eta_1}} \simeq \frac{\eta_1\beta}{2}\left(\frac{k}{k_1}\right)^3$$

$$g \equiv \frac{\pi_2}{\pi_1}$$

$$\omega_{s2} \equiv \sqrt{\frac{-6B_2}{A_2}} \simeq \frac{\sqrt{2}\,|\pi_1|}{\Delta\varphi}$$

$$\kappa \equiv \sqrt{\frac{\epsilon_{V1}}{\epsilon_{V2}}}$$

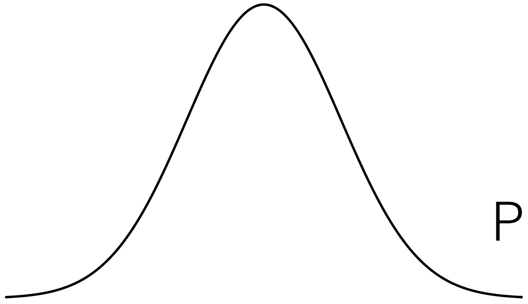
$$\mathcal{R} = \beta\delta\varphi + \frac{\kappa g}{3}\left(1 - \sqrt{1 + \frac{2\gamma}{g^2}\delta\varphi + \frac{\gamma^2}{g^2}\delta\varphi^2}\right) - \frac{1}{2\omega_{s2}}\log\left(1 + \frac{2\gamma}{g^2}\delta\varphi + \frac{\gamma^2}{g^2}\delta\varphi^2\right)$$

$$P[\mathcal{R}] \propto \begin{cases} \exp\left[-\frac{\mathcal{R}^2}{2\left(\beta - \frac{\kappa\gamma}{3g} - \frac{\gamma}{\omega_{s2} g^2}\right)^2 \sigma_{\delta\varphi}^2}\right] & : \text{(Gaussian)} \\ \left(1 - \frac{3\mathcal{R}}{g\kappa}\right)\exp\left[-\frac{1}{2\sigma_{\delta\varphi}^2}\frac{9g^2\mathcal{R}^2}{\gamma^2\kappa^2}\left(1 - \frac{3}{2}\frac{\mathcal{R}}{g\kappa}\right)^2\right] & : \text{(Cutoff)} \\ \exp(-2\omega_{s2}\mathcal{R})\exp\left[-\frac{1}{2\sigma_{\delta\varphi}^2}\frac{g^2}{4\gamma^2}\left(\exp(-2\omega_{s2}\mathcal{R}) - 1\right)^2\right] & : \text{(Exponential tail)} \end{cases}$$

$$\mathcal{R}_{cutoff} = \frac{g\kappa}{3}$$

Y.-F. Cai et al. (2023)

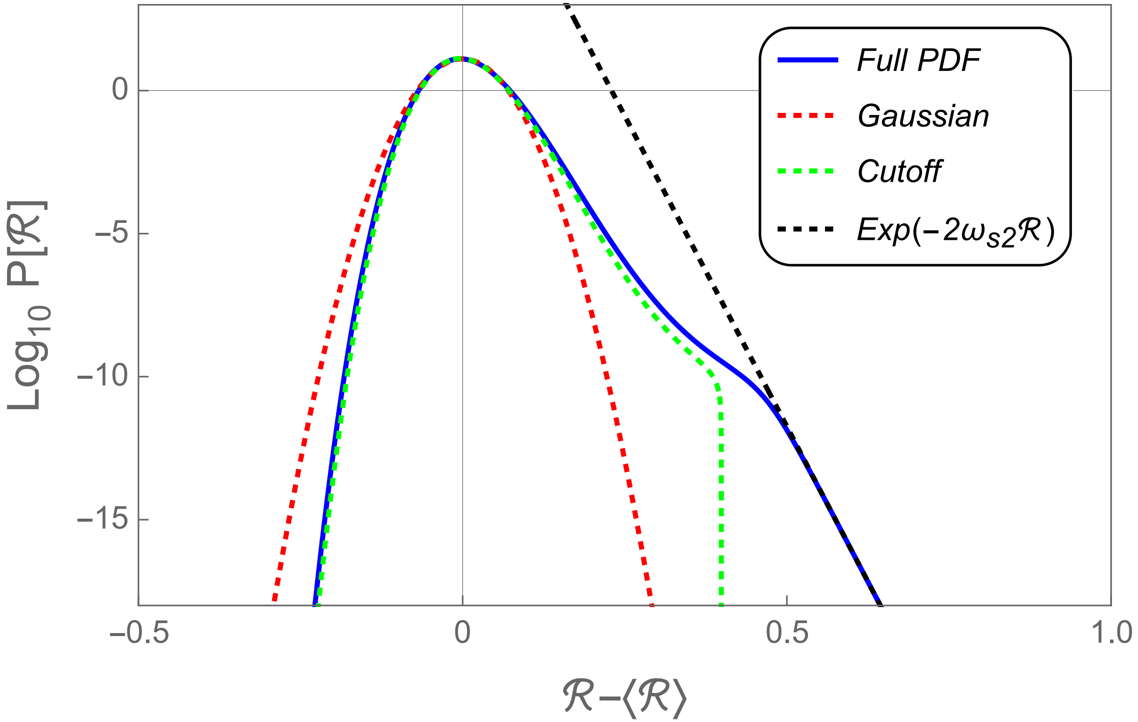
$P[\delta\varphi]$ (Gaussian)

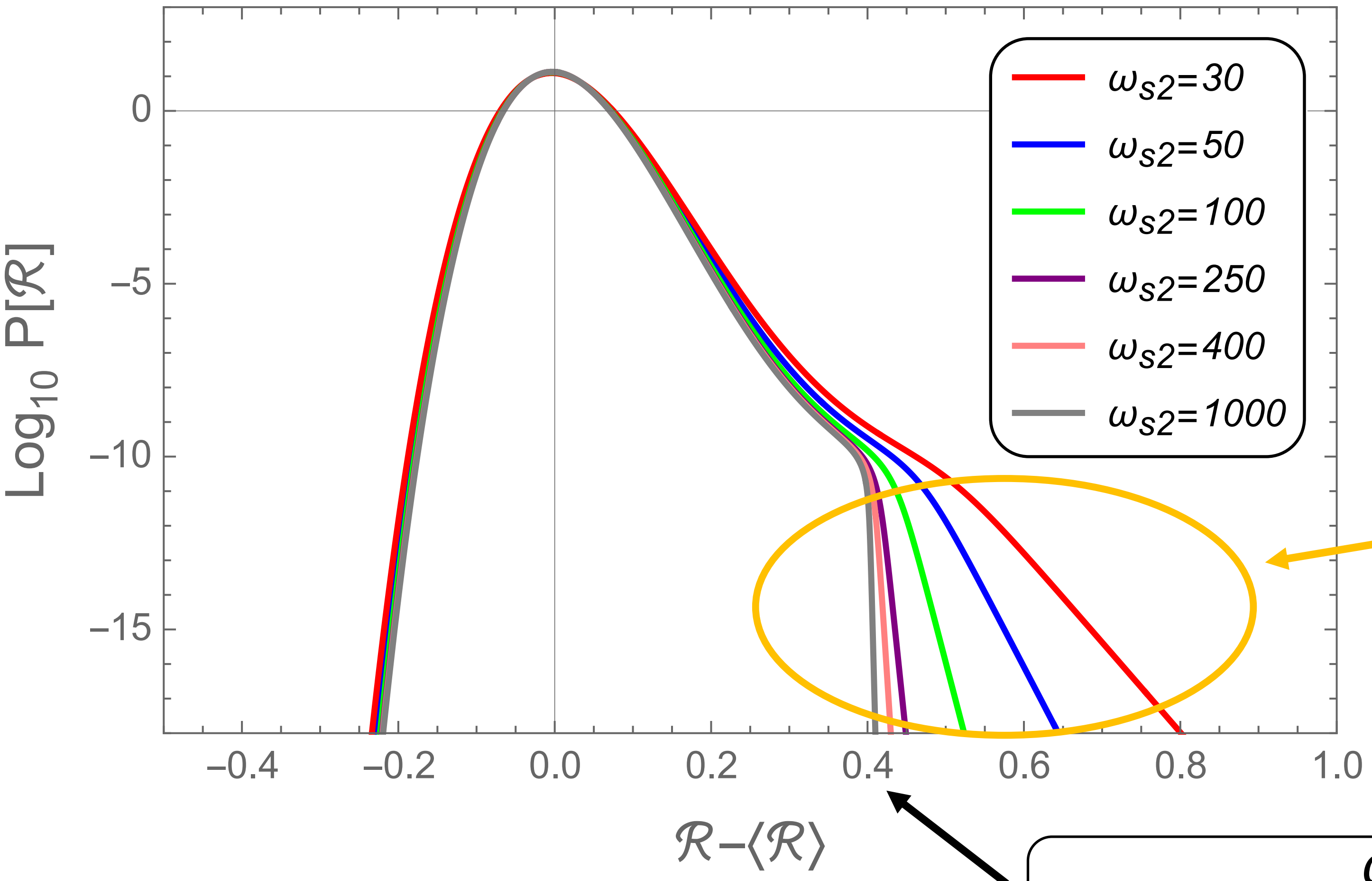


δN formalism

Probability conservation

$$P[\mathcal{R}] = P[\delta\varphi]\left|\frac{d\delta\varphi}{d\mathcal{R}}\right|$$





$$\omega_{s2} \propto \frac{1}{\Delta\varphi}$$

Tail behavior

$$P[\mathcal{R}] \propto \exp(-2\omega_{s2}\mathcal{R})$$

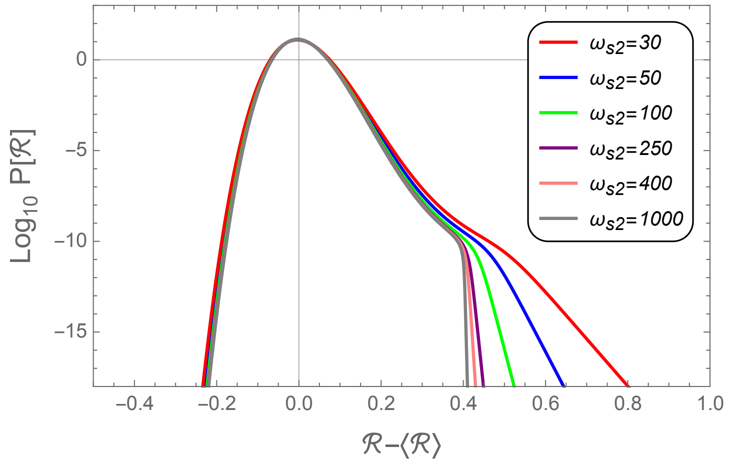
$$\mathcal{R}_{cutoff} = \frac{g\kappa}{3}$$

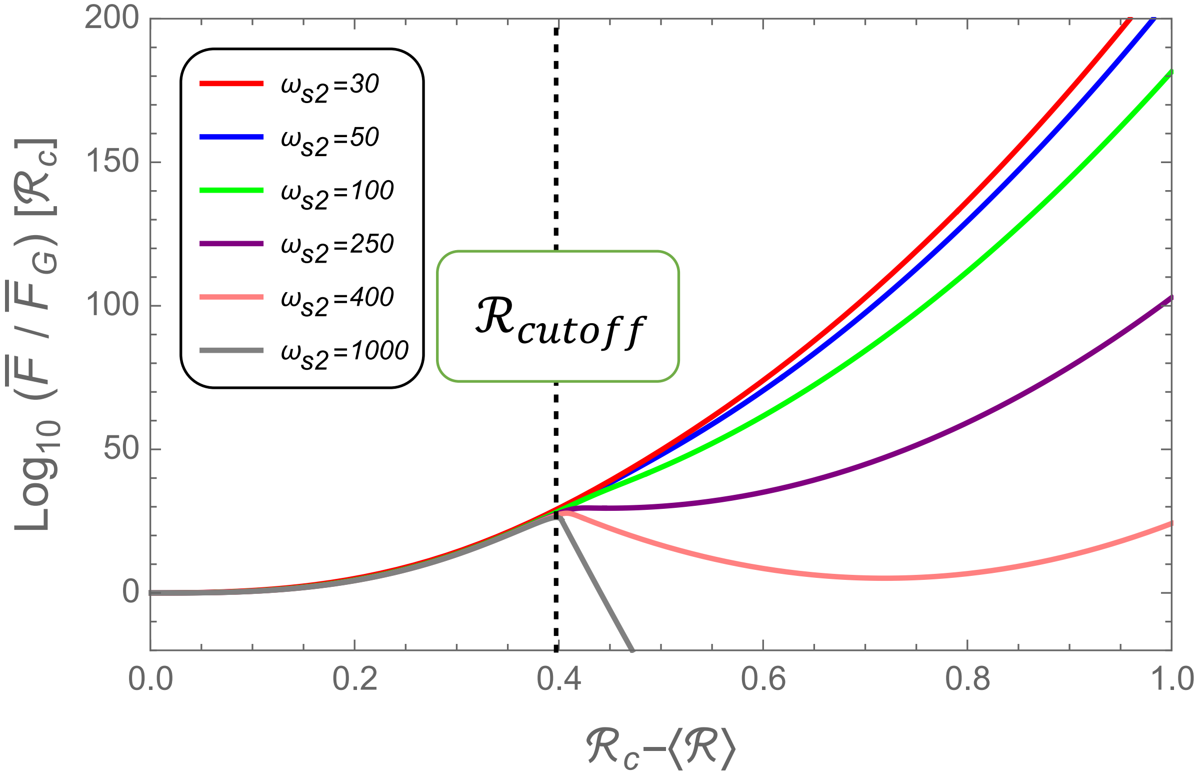
- The slope of the tail becomes steeper as $\Delta\varphi$ decreases (i.e. the step is steeper).

CCDF

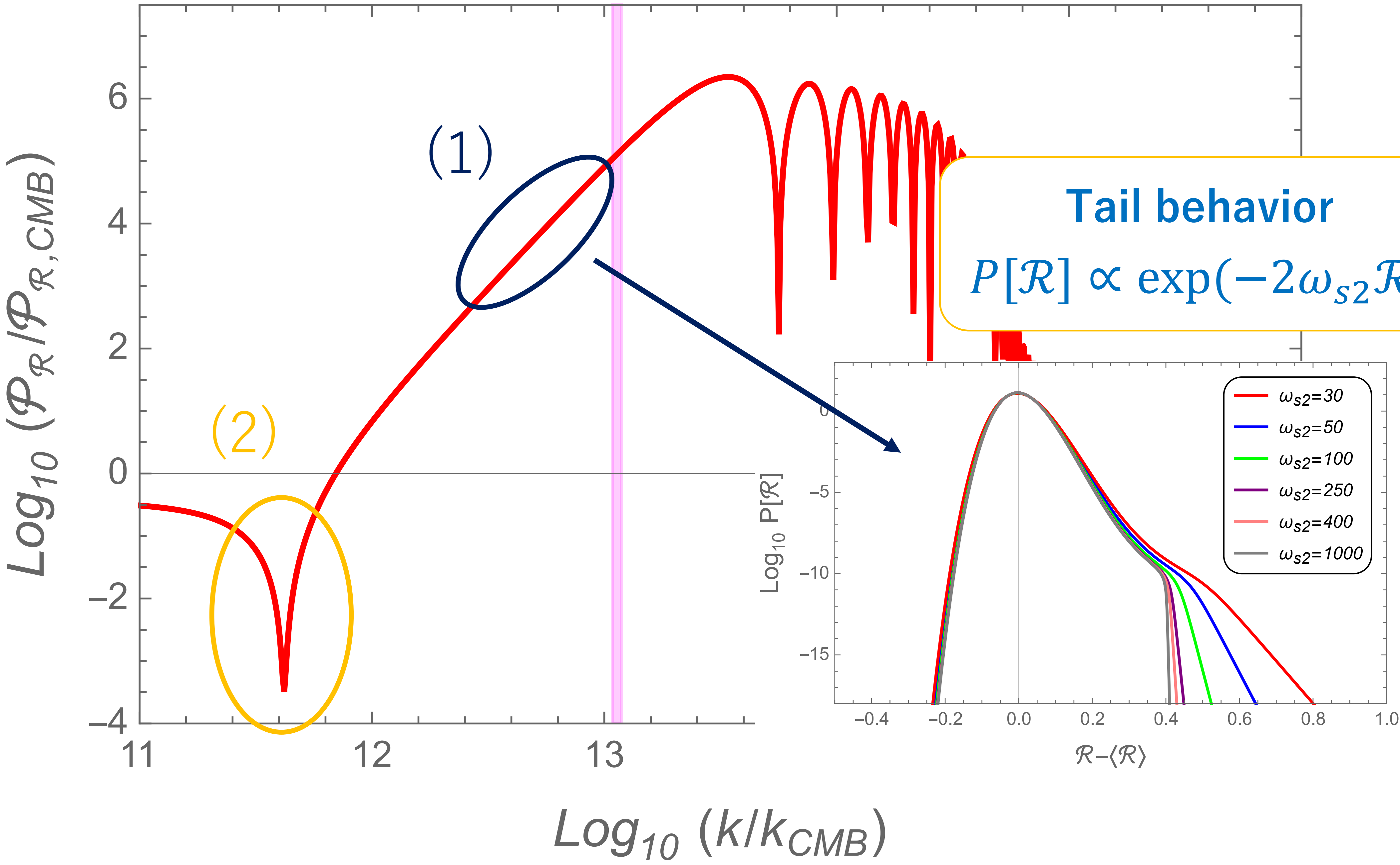
$$\bar{F}[\mathcal{R}_c] = \int_{\mathcal{R}_c}^{\infty} P[\mathcal{R}] d\mathcal{R}$$

- With the evaluation of the PBH abundance in mind, we calculate the CCDF of curvature perturbation.





- **The CCDF dramatically changes depending on the value of ω_{s2}, if $\mathcal{R}_c - \langle\mathcal{R}\rangle$ is larger than the cutoff value.**

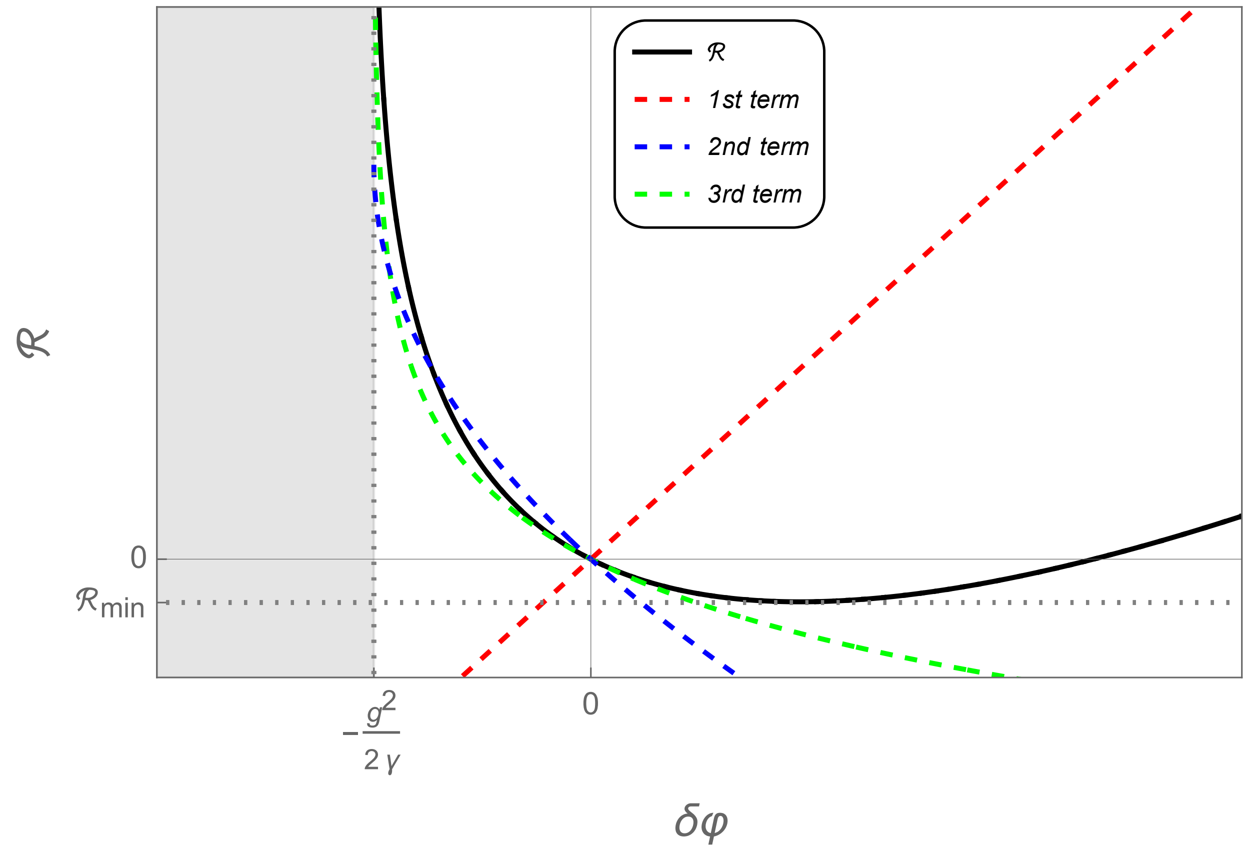


Tail behavior

$P[\mathcal{R}] \propto \exp(-2\omega_{s2}\mathcal{R})$

4. Highly asymmetric PDF

$$\mathcal{R} = \beta\delta\varphi + \frac{\kappa g}{3}\left(1 - \sqrt{1 + \frac{2\gamma}{g^2}\delta\varphi + \frac{\gamma^2}{g^2}\delta\varphi^2}\right) - \frac{1}{2\omega_{s2}}\log\left(1 + \frac{2\gamma}{g^2}\delta\varphi + \frac{\gamma^2}{g^2}\delta\varphi^2\right)$$



$$\beta \equiv -\frac{1}{\pi} \qquad \gamma \equiv \frac{\eta_1\beta}{2}\left(\frac{k}{k_1}\right)^3 \qquad \kappa \equiv \sqrt{\frac{\epsilon_{V1}}{\epsilon_{V2}}}$$

For $\beta > \kappa\gamma/3$ and $\gamma > 0$

$\mathcal{R} \to +\infty$ as $\delta\varphi \to -\frac{g^2}{2\gamma}$

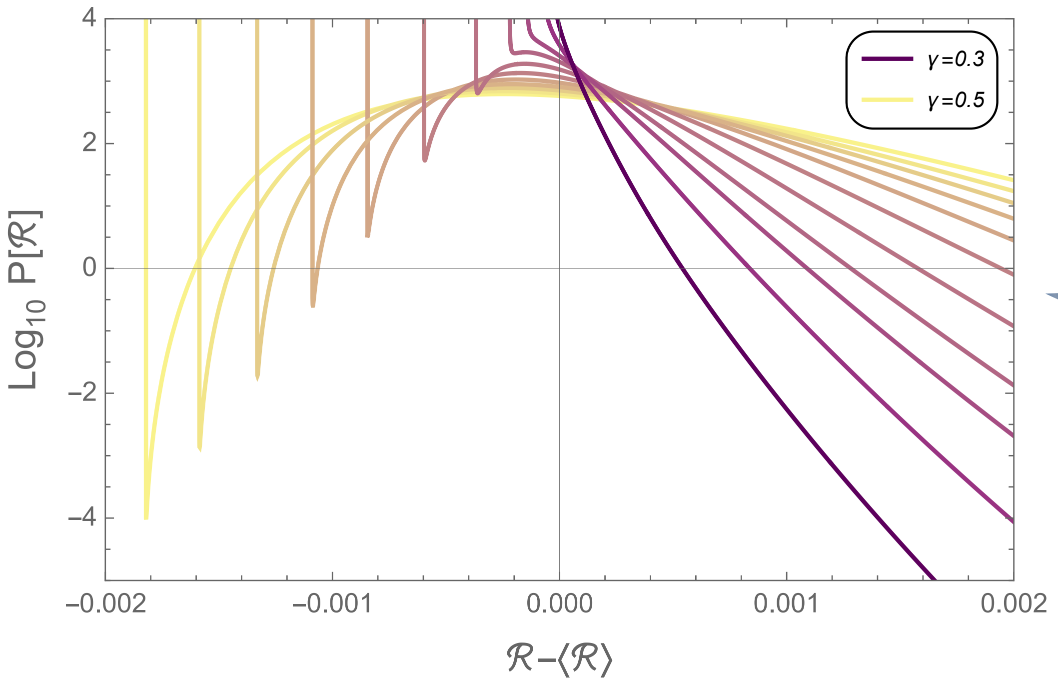
$\mathcal{R} = 0$ at $\delta\varphi = 0$

$\mathcal{R} \to +\infty$ as $\delta\varphi \to +\infty$

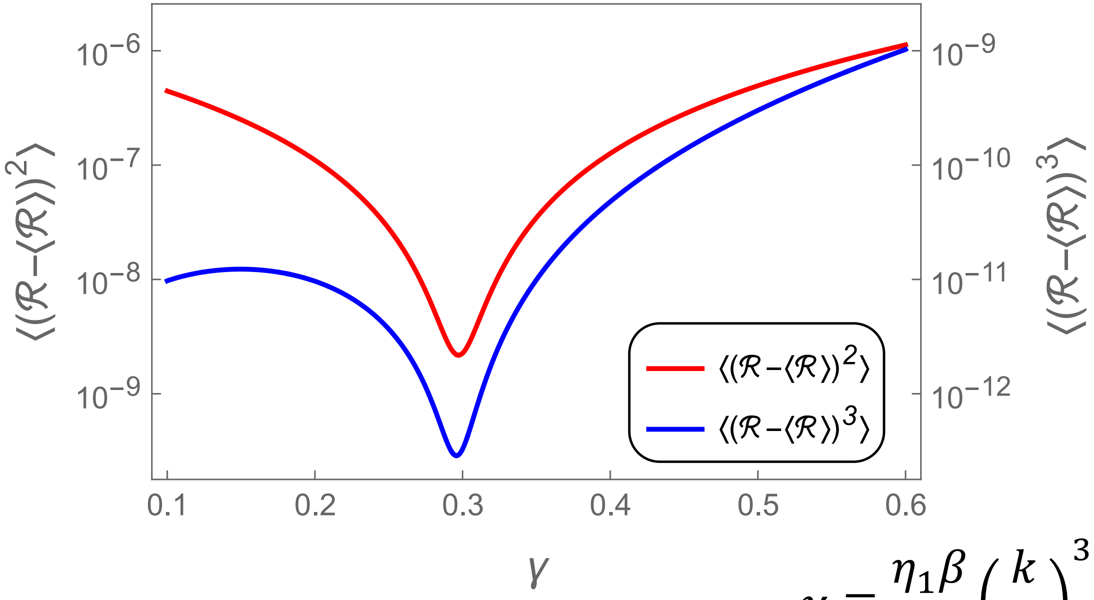
Local minimum of $\mathcal{R}$ ($\mathcal{R}_{min}$) exists !!

Extreme case ($\gamma = 0.3,\ \mathcal{R}_{\min} = 0$)

$$\left.\frac{d\mathcal{R}}{d\delta\varphi}\right|_{\delta\varphi=0} = 0 \Rightarrow \mathcal{R} = \left.\frac{d\mathcal{R}}{d\delta\varphi}\right|_0 \delta\varphi + \frac{1}{2}\left.\frac{d^2\mathcal{R}}{d\delta\varphi^2}\right|_0 \delta\varphi^2 + \cdots$$



Highly asymmetric PDF



$$\gamma \equiv \frac{\eta_1 \beta}{2}\left(\frac{k}{k_1}\right)^3$$

At the dip scale, the highly asymmetric PDF is realized

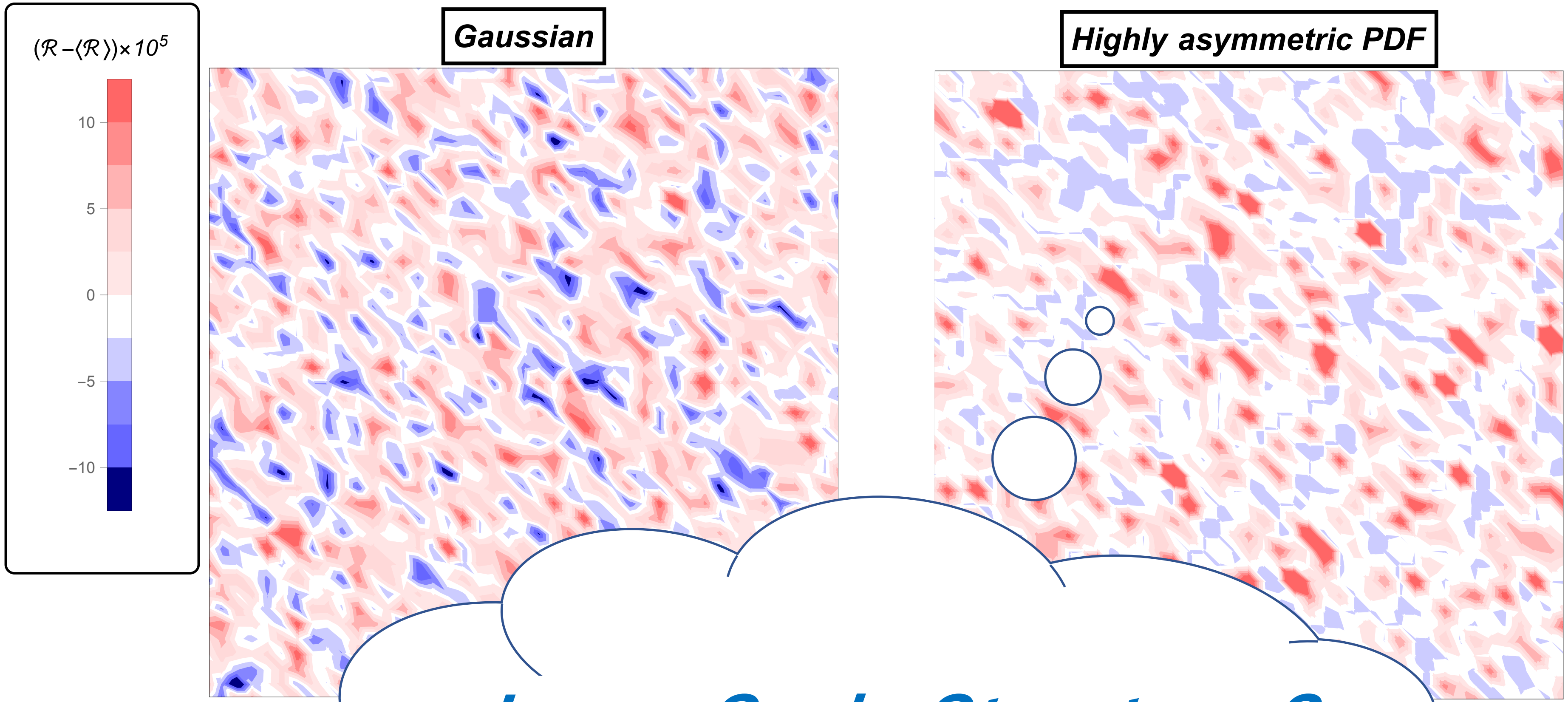


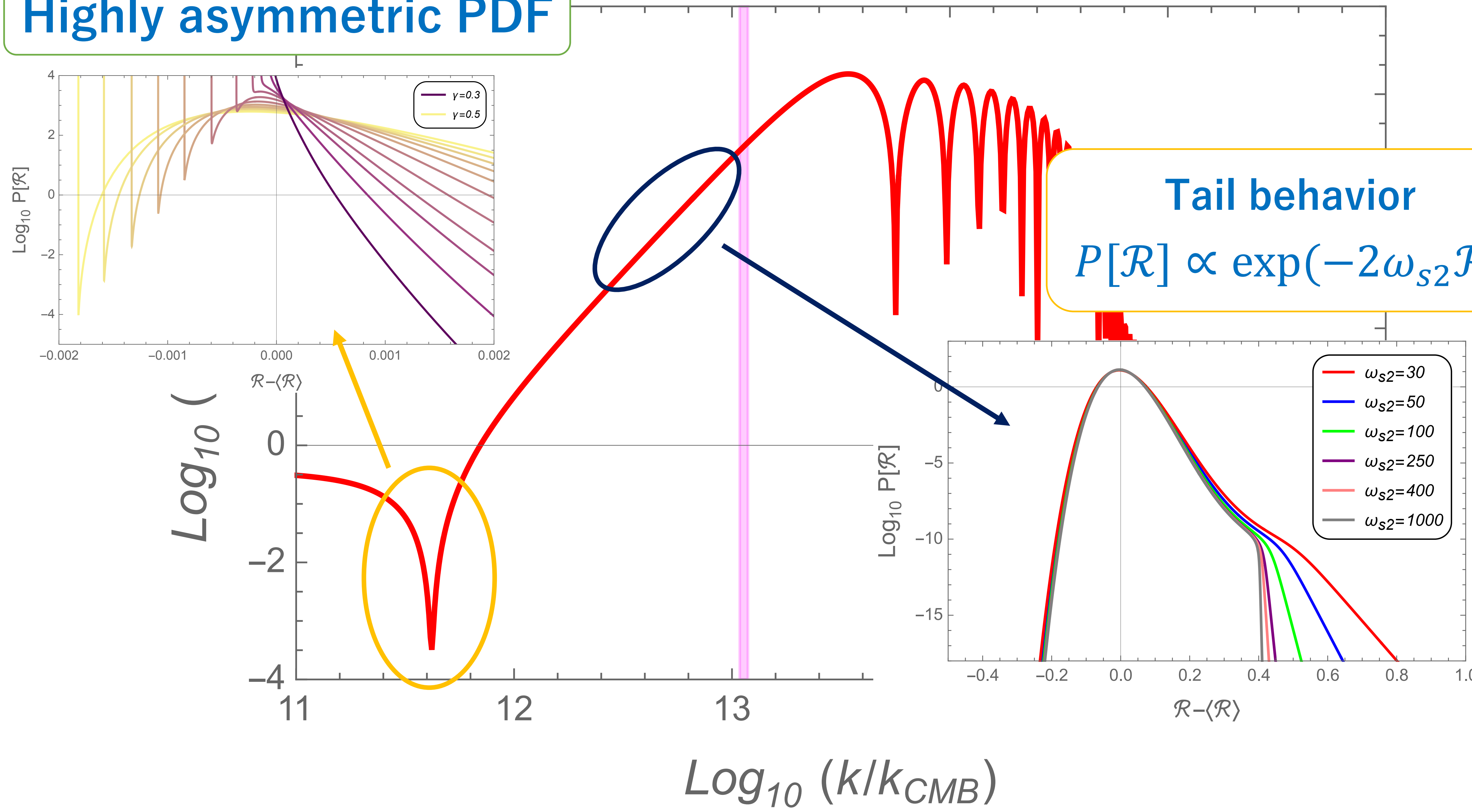
Illustration of the di… on different PDFs.
$\mathcal{R}$ is randomly assign… …ice points
according to the Gaussia… …etric PDF (right)

Large Scale Structure ?

Void region ?

In the highly asymmetric PDF, the distribution is significantly biased

Highly asymmetric PDF



Tail behavior

$P[\mathcal{R}] \propto \exp(-2\omega_{s2}\mathcal{R})$

Summary

- We studied an inflationary model in which the inflaton potential includes a finite-width upward step between two SR stages.

- Using the δN formalism, we calculated the PDF of the curvature perturbation. For $\mathcal{R} < \mathcal{R}_{cutoff}$, the PDF follows the Cutoff PDF, while for $\mathcal{R} > \mathcal{R}_{cutoff}$ the **exponential tail $\boldsymbol{P[\mathcal{R}] \propto exp(-2\omega_{s2}\mathcal{R})}$** is dominant.
- The CCDF was also calculated, and we find the significant impact on the PBH abundance of the exponential tail.

- We also show that the **PDF becomes highly asymmetric on a particular scale** exiting the horizon before the step, at which the curvature power spectrum has a **dip**. This asymmetric PDF may leave an interesting signature in the **large scale structure such as voids**.

Back up slides

δN formalism

A.A. Starobinsky (1985)

M.Sasaki and E.D.Stewart (1996)

- According to the δN formalism,

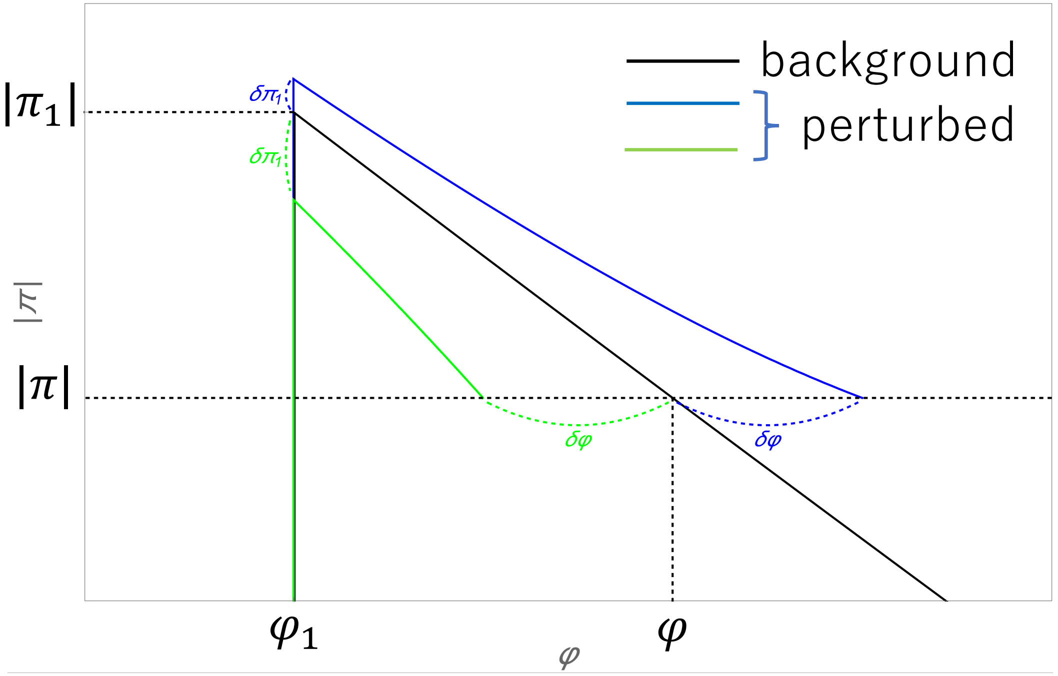
$$\mathcal{R} = \delta N = \underbrace{N\left(\varphi + \delta\varphi, \pi + \delta\pi\,; \varphi_f, \pi_f\right)}_{\text{Perturbed trajectory}} - \underbrace{N\left(\varphi, \pi\,; \varphi_f, \pi_f\right)}_{\text{Background trajectory}} \qquad \pi = \frac{d\varphi}{dn}$$

where $\delta\varphi$ and $\delta\pi$ are the initial scalar field perturbation and velocity perturbation, respectively.

- Using the δN formalism, we can write the curvature perturbation as a function of $\delta\varphi$.

1st SR stage

- In this stage, the background trajectory is on the SR attractor. However, the perturbed trajectories are not on the SR attractor.
- The scalar field perturbation $\delta\varphi$ gives rise not only to $\delta N^{(1)}$ but also to $\delta\pi_1$.



$$\delta N^{(1)} \simeq -\frac{\delta\varphi}{\pi}$$

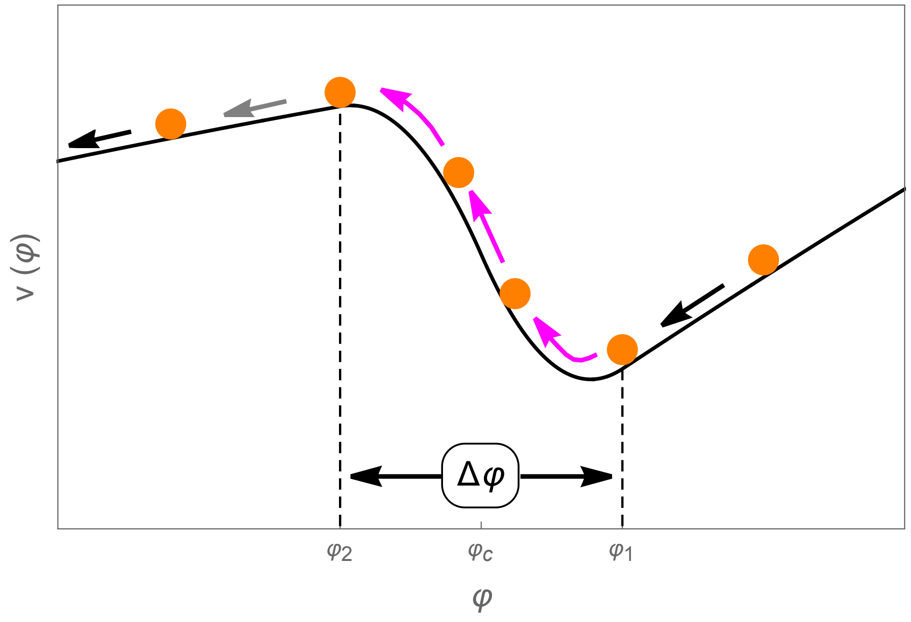
$$\delta\pi_1 \simeq -\frac{\eta_1}{2}\left(\frac{\pi}{\pi_1}\right)^{\frac{6}{\eta_1}}\delta\varphi$$

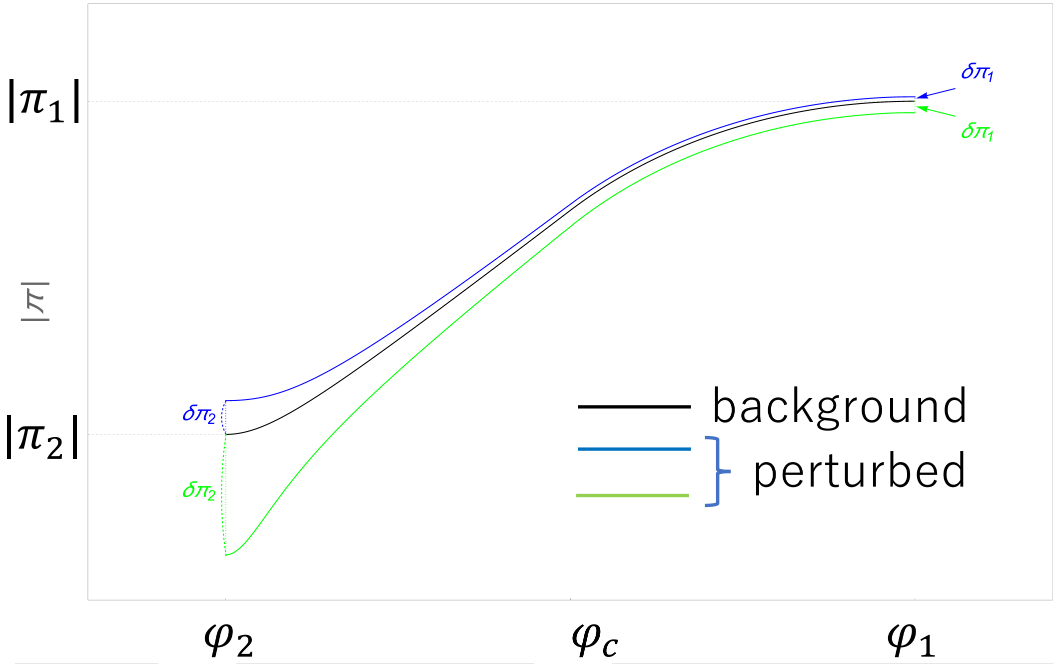
$\delta\pi_1$ induces $\delta N^{(step)}$ and $\delta N^{(2)}$

Step stage

From the energy conservation law,

$$\pi_2 = -\sqrt{\pi_1^2 + 6\log\left(\frac{v(\varphi_1)}{v(\varphi_2)}\right)}$$



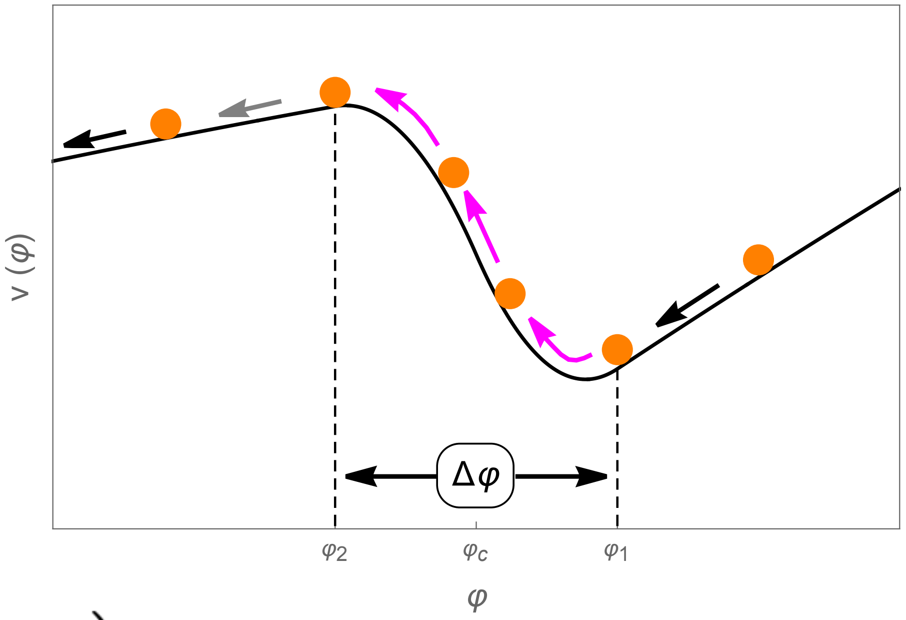


$$\delta\pi_2 = \pi_2\left(\sqrt{1 + \frac{2}{g^2}\frac{\delta\pi_1}{\pi_1} + \frac{1}{g^2}\left(\frac{\delta\pi_1}{\pi_1}\right)^2} - 1\right)$$

where $g \equiv \frac{\pi_2}{\pi_1} < 1$

Step stage

$$\delta\pi_2 = \pi_2 \left(\sqrt{1 + \frac{2}{g^2}\frac{\delta\pi_1}{\pi_1} + \frac{1}{g^2}\left(\frac{\delta\pi_1}{\pi_1}\right)^2} - 1 \right)$$



$$N^{(step)} \simeq \frac{1}{\omega_{s2}} \sinh^{-1}\left(\frac{\Delta\varphi}{2|\pi_2|}\omega_{s2}\right) \simeq \frac{1}{\omega_{s2}} \log\left(\frac{\Delta\varphi}{|\pi_2|}\omega_{s2}\right)$$

$$\omega_{s2} \equiv \sqrt{\frac{-6B_2}{A_2}} \simeq \frac{\sqrt{2}\,|\pi_1|}{\Delta\varphi}$$

$$\delta N^{(step)} \simeq -\frac{1}{\omega_{s2}} \log\left(1 + \frac{\delta\pi_2}{\pi_2}\right)$$

$$g \equiv \frac{\pi_2}{\pi_1}$$

- It is important to note that when $\delta\pi_2$ is comparable to $-\pi_2$, $\delta N^{(step)}$ may diverge to infinity.

2nd SR stage

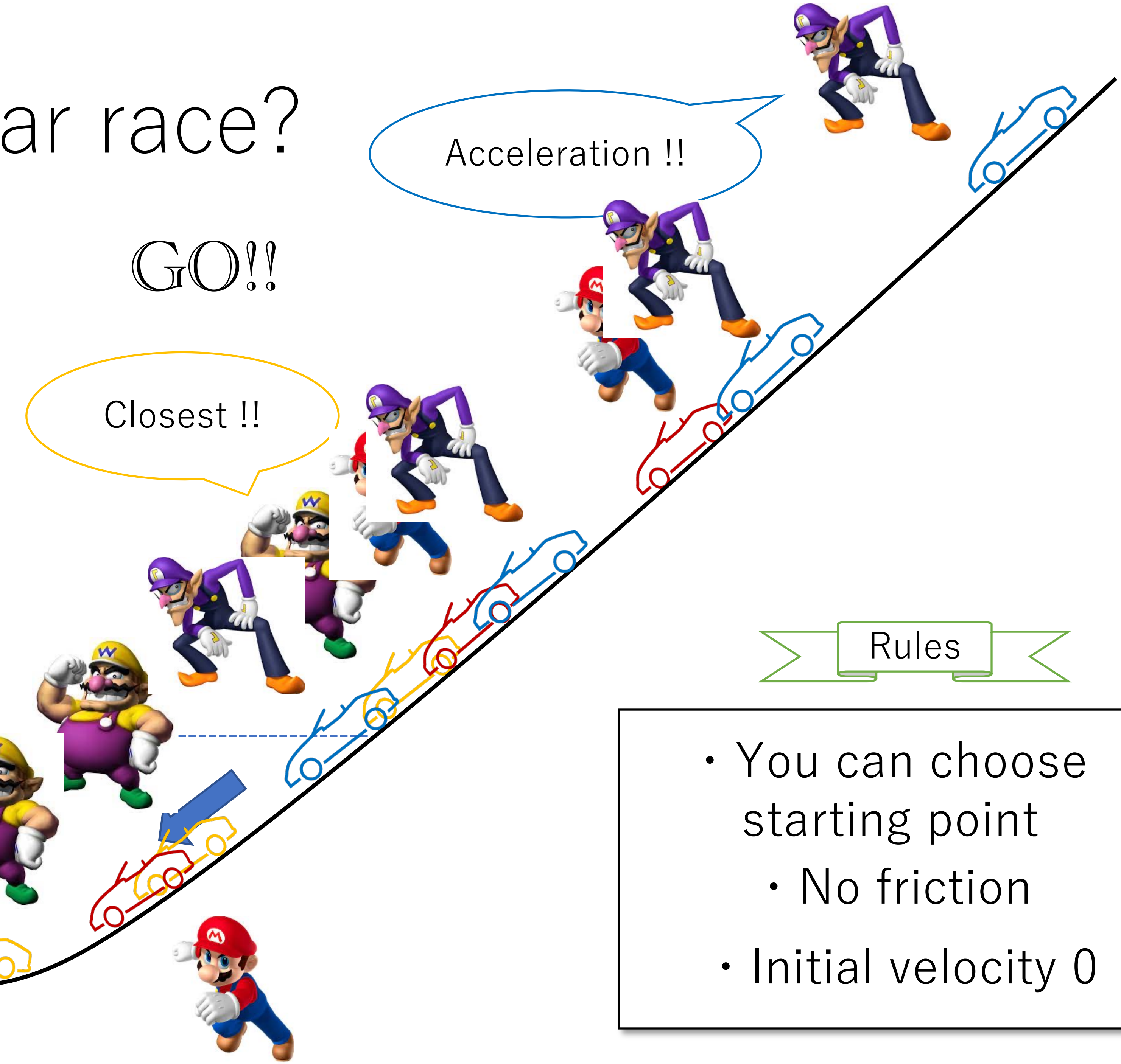
$$\delta\pi_2 = \pi_2 \left(\sqrt{1 + \frac{2}{g^2}\frac{\delta\pi_1}{\pi_1} + \frac{1}{g^2}\left(\frac{\delta\pi_1}{\pi_1}\right)^2} - 1 \right)$$

- The e-folding number $\delta N^{(2)}$ induced by $\delta\pi_2$ is given by

$$\delta N^{(2)} \simeq -\frac{\kappa g}{3}\frac{\delta\pi_2}{\pi_2} \simeq \frac{\kappa g}{3}\left(1 - \sqrt{1 + \frac{2}{g^2}\frac{\delta\pi_1}{\pi_1} + \frac{1}{g^2}\left(\frac{\delta\pi_1}{\pi_1}\right)^2}\right)$$

where $\kappa \equiv \sqrt{\frac{\epsilon_{V1}}{\epsilon_{V2}}}$ is the ratio of the slope of the potential before and after the step.

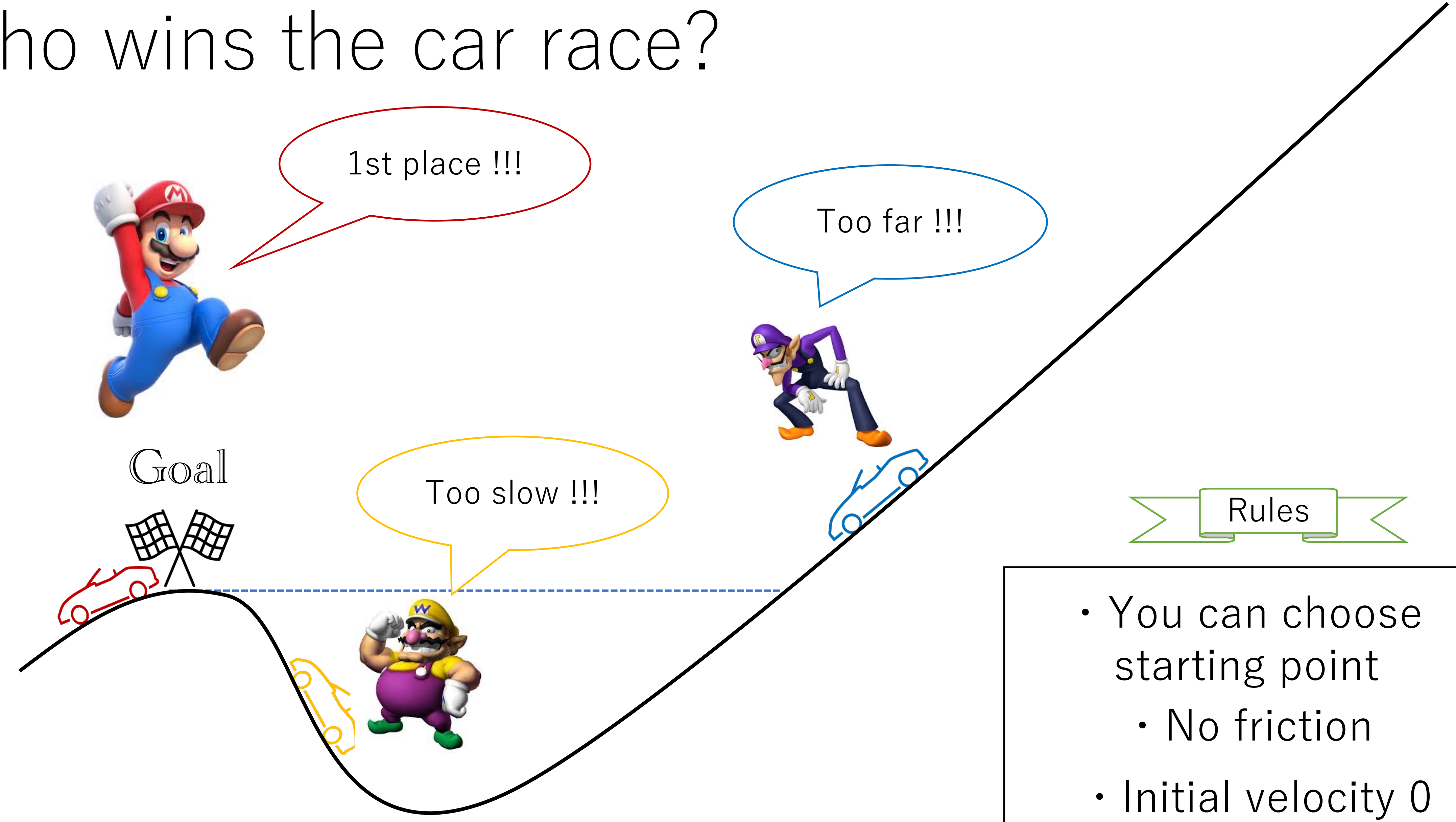
Who wins the car race?



Rules

- You can choose starting point
- No friction
- Initial velocity 0

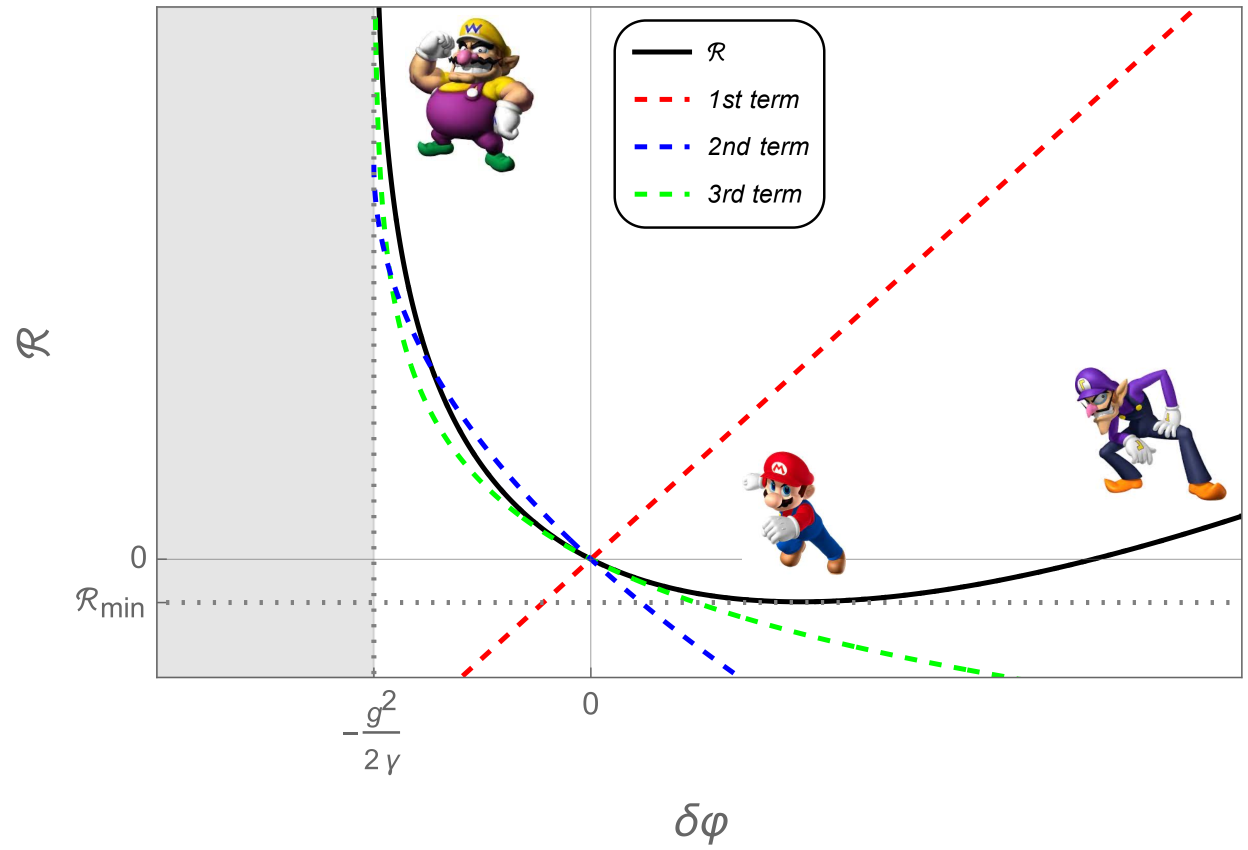
Who wins the car race?



Rules

- You can choose starting point
- No friction
- Initial velocity 0

$$\mathcal{R} = \beta\delta\varphi + \frac{\kappa g}{3}\left(1 - \sqrt{1 + \frac{2\gamma}{g^2}\delta\varphi + \frac{\gamma^2}{g^2}\delta\varphi^2}\right) - \frac{1}{2\omega_{s2}}\log\left(1 + \frac{2\gamma}{g^2}\delta\varphi + \frac{\gamma^2}{g^2}\delta\varphi^2\right)$$



$$\beta \equiv -\frac{1}{\pi} \qquad \gamma \equiv \frac{\eta_1\beta}{2}\left(\frac{k}{k_1}\right)^3 \qquad \kappa \equiv \sqrt{\frac{\epsilon_{V1}}{\epsilon_{V2}}}$$

For $\beta > \kappa\gamma/3$ and $\gamma > 0$

$\mathcal{R} \to +\infty$	as	$\delta\varphi \to -\frac{g^2}{2\gamma}$
$\mathcal{R} = 0$	at	$\delta\varphi = 0$
$\mathcal{R} \to +\infty$	as	$\delta\varphi \to +\infty$

A local minimum of $\mathcal{R}$ ($\mathcal{R}_{min}$) exists !!

$$\mathcal{R} = A\delta\varphi + B\delta\varphi^2 + \mathcal{O}(\delta\varphi^3)$$

$$f_{\rm NL}^{\rm local} \equiv \frac{5}{3}\frac{B}{A^2} = \frac{5}{2}\frac{\kappa g(1-g^2) + \frac{3}{\omega_{\rm s2}}(2-g^2)}{\left(\frac{3\beta g^2}{\gamma} - \kappa g - \frac{3}{\omega_{\rm s2}}\right)^2}$$