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JET BUNDLE GEOMETRY OF SCALAR FIELD THEORIES

2308.00017 w/. M. Alminawi & I. Brivio

(1)

1.1 Basic ideas

- Jet bundles $\{J^r\}$ = a "nested" set of spaces, labelled by $r \in \mathbb{Z}_{\geq 0}$, that provide coord-free way to deal w/. $\leq r$-derivs of maps.

- Scalar fields in QFT = maps $\phi : \underset{\text{spacetime}}{\Sigma} \to \underset{\text{target}}{M}$

- EFTs = perturbative expansions in both # fields ϕ^N & # derivs ∂^D.

- Recap: geometry on M has been used to re-sum all op's w/. $D=2$ $(N \leq \infty)$:

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j$$

Ex: Higgs ϕ^i in SMEFT (linear $O(4)$ action)

$$g_{ij} = A\left(\phi^2/\Lambda^2\right) \delta_{ij} + B\left(\phi^2/\Lambda^2\right) \phi_i \phi_j$$

Amplitudes ~ tensors on M by coord. inv. (~ gauge redundancy)

$$\text{e.g.}\ \frac{\partial A_{ij \to kl}}{\partial s}\bigg|_{s=0} \sim R_{ijkl}$$

↑
b/c $D=2$.

1.2 EFT geometry beyond $D=2$

- Geometry (i.e. metrics $\{g^r\}$) on $\{J^r\}$ can systematically capture complete EFT expansion in N and D:

	N	D
g^0	∞	≤ 2
g^1	∞	≤ 4
g^2	∞	≤ 6
...		
g^r	∞	$\leq 2(r+1)$

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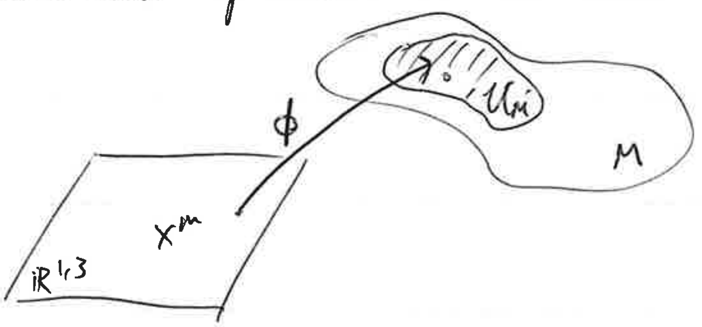
- this geometric organising structure reflects 'HEFT-like' power counting:

$$\mathcal{L}_{HEFT}\left(\frac{h}{v}, \frac{\partial}{\Lambda}\right) = \mathcal{L}^{(0)} + \frac{1}{\Lambda^2}\mathcal{L}^{(1)} + \frac{1}{\Lambda^4}\mathcal{L}^{(2)}$$

resum at each order in ∂/Λ	$N=\infty$ $D=2$ $\to$ 0-jet	$D=4$ $\to$ 1-jet (need $\mathcal{L}^{(0)}$ loops)	$D=6$ $\to$ 2-jet (needs $\mathcal{L}^{(\leq 1)}$ loops)
Dim$^{\underline{n}}$:	8	24	64

② Field Space Formalism: $D=2$

- s-t $(\Sigma, \eta) \sim$ Mink, global coords $\{x^\mu\}$ + Poinc. $x^\mu \to L^\mu{}_\nu x^\nu + a^\mu$
- sc. fields = maps $\phi: \Sigma \to M$ = smth. n-d $\mathbb{R}$-mfd., generally curved. Internal symm. $G \times M \to M$



- local coords u^i on $\mathcal{U}_m \subset M$
- $\phi^i(x) \equiv (u^i \circ \phi)(x)$ are "field values" at x.

EFT Lagrangian $D=2$:

1. Invariant metric $g = g_{ij}(u)\, du^i du^j$
2. pull back along ϕ $\quad g \to \phi^* g = g_{ij}(\phi(x))\, \partial_\mu \phi^i \partial_\nu \phi^j\, dx^\mu dx^\nu$
3. contract w/. inverse spacetime metric $\eta = \eta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma$

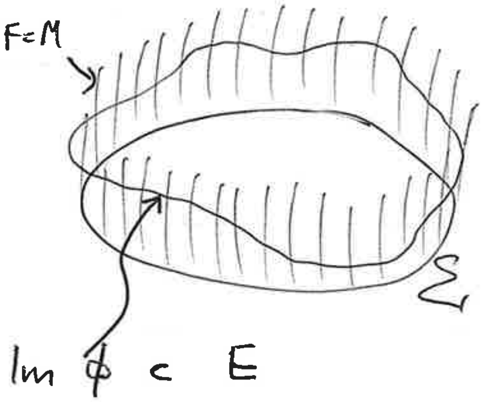
$$\mathcal{L} = \frac{1}{2}\eta^{-1} \circ \phi^* g = \frac{1}{2}\eta^{\mu\nu} g_{ij}(\phi)\, \partial_\mu \phi^i \partial_\nu \phi^j$$

* $D=0$ i.e. potential? Add by hand

* $D>2$? no obvious geo. interpretation.

③ Field Space to Bundles : $D \leq 2$

- Map $\phi : \Sigma \to M$ can be traded for a section ϕ of fibre bundle $\pi : E \to \Sigma$ w/. base Σ & fibre M.



- fibred coords (x^μ, u^i)
- $\pi(x^\mu, u^i) = x^\mu$
- section $\phi : \Sigma \to E$ is an inverse of π, i.e. $\pi \circ \phi = id_\Sigma$

(Actually more general : E can be globally non-trivial $E \neq \Sigma \times M$).

- Metric on E allows more general $\mathcal{L}$: As before :

1. $g^{(0)} = g_{IJ}\, dx^I dx^J = (dx^\mu \; du^i)\begin{pmatrix} g_{\mu\nu}(u) & g_{\mu j}(u) \\ g_{\nu i}(u) & g_{ij}(u) \end{pmatrix}\begin{pmatrix} dx^\nu \\ du^j \end{pmatrix}$

$\uparrow$ $\partial_\mu g_{IJ} = 0$ by Poincaré (translations)

2. & 3. $\mathcal{L}^{(0)} = \frac{1}{2}\eta^{-1} \cdot \phi^* g^{(0)} = \frac{1}{2}\eta^{\rho\sigma}\Big[g_{\rho\sigma}(\phi) + \underbrace{2g_{\rho i}\partial_\sigma\phi^i}_{=0 \text{ by Poincaré.}} + g_{ij}(\phi)\partial_\rho\phi^i\partial_\sigma\phi^j\Big]$

$\Rightarrow \mathcal{L}^{(0)} = \frac{1}{2} g_{ij}(\phi)\, \partial_\mu\phi^i \partial^\mu\phi^j \; \underline{\underline{- V(\phi)}}$

choosing $g_{\rho\sigma} = \eta_{\rho\sigma}\left(-\frac{V(u)}{2}\right)$

We get the potential !

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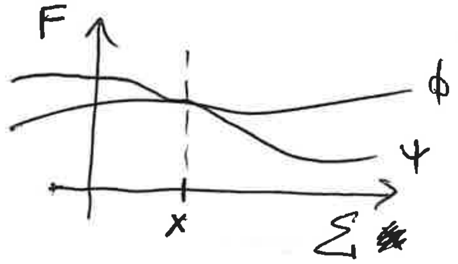
④. Bundles to Jet Bundles: $D \leq 2(r+1)$ [Saunders '1989]

- $J^0E \equiv E$, 0-jet bundle.
- Build up J^rE, $r > 0$, iteratively:
  - Two sections ϕ, ψ of (E, π, Σ) are "1-equivalent" at x iff

$$\phi^i(x) = \psi^i(x) \quad \text{and} \quad \partial_\mu \phi^i(x) = \partial_\mu \psi^i(x) \quad \forall \mu .$$



i.e. agree in value & 1st deriv. at point x.

  - "r-equivalence":

$$\phi^i|_x = \psi^i|_x \quad \& \quad \partial_\mu \phi^i|_x = \partial_\mu \psi^i|_x \quad \& \dots \& \quad \left.\frac{\partial \phi^i}{\partial x^{\mu_1} \dots \partial x^{\mu_r}}\right|_x = \left.\frac{\partial \psi^i}{\partial x^{\mu_1} \dots \partial x^{\mu_r}}\right|_x$$

  - These conditions are <u>coordinate independent</u>
  - Define 'r-jet of ϕ at x' = r-eq. class containing ϕ at x.
  - Then $J^rE := \{ j^r_x \phi \mid x \in \Sigma, \phi \in \Gamma_x(\pi) \}$

[Dimension Formula: $\dim \Sigma = d$, $\dim M = n \Rightarrow \mathrm{Dim}\, J^rE = d + n\dfrac{(d+r)!}{d!\,r!}$]

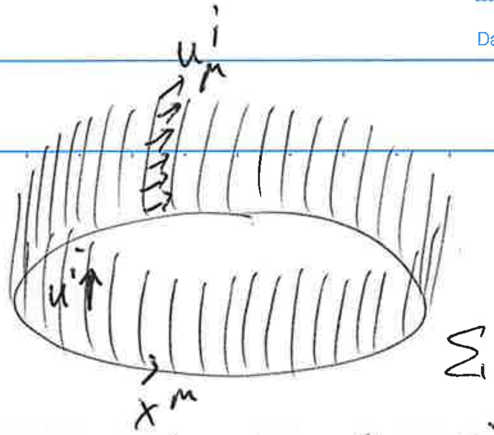
Each is a bundle over the last:

$$\begin{tikzcd} & J^1E \arrow[dl, "\pi_{1,0}"'] \arrow[dr, "\pi_1"] & \\ E \arrow[rr, "\pi"'] & & \Sigma \end{tikzcd}$$

$\pi_{1,0} : j^1_x \phi \to \phi(x)$

$\pi_{1,0}^{-1}(x, m \in M_x) \simeq T^*_x(\Sigma) \otimes T_m(M)$

~ spacetime derivatives



⑤. 1-jet: $D \leq 4$

Example: QM on line, $\Sigma = \mathbb{R}$, $M = \mathbb{R}$

- 0-jet: $E = \mathbb{R}_t \times \mathbb{R}_u \xrightarrow{\pi} \mathbb{R}_t$, $\phi : t \mapsto (t, \phi(t))$ a section
- 1-jet: $J^1E \cong T^*\mathbb{R}_t \times \mathbb{R}_u$, coords (t, u, u_t), where $u_t \circ j^1\phi = \dot{\phi}(t)$.

General: local coords

$$(x^\mu, u^i, u^i_\mu) . \quad \underset{\substack{\uparrow \\ \text{a point} \\ \text{in } J^1E}}{j^1_x\phi} = (x^\mu, \underbrace{\phi^i(x), \partial_\mu \phi^i(x)}_{\substack{\text{field \& 1st derivs} \\ \text{encoded in a coord-free} \\ \text{way via } (J^1E, \phi)}})$$

▷ Fields?

- Physical field is always $\phi \in \Gamma(\pi)$.
- 'Prolong' ϕ to section $j^1\phi \in \Gamma(\pi_1)$

$= $ section of $\pi_1 : J^1E \to \Sigma$ passing through $j^1_x\phi$

▷ Symmetries?

- 'Internal' means $f_g : (x^\mu, u^i) \mapsto (x^\mu, f_g^i(x,u))$ on E, s.t. $f_{g_1} \circ f_{g_2} = f_{g_1 g_2}$ $\forall g_{1,2} \in G$

'Prolong': $j^1 f_g : (x^\mu, u^i, u^i_\mu) \mapsto (x^\mu, f_g^i, \partial_\mu f_g^i + u^j_\mu \partial_j f^i)$ $\forall g$.

(CHAIN RULE!)

Ex: SMEFT? $f_O^i = O^i{}_j u^j$ $\quad O \in O(4)$, global $(\partial_\mu f^i = 0)$

$\Rightarrow j^1 f_O : u_\mu^i \to O^i{}_j u_\mu^j$ Simple!

Just contract all i indices w/ $O(4)$-invariant tensors (δ_{ij}).

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$$\left[\begin{array}{c} \begin{array}{ccc} J^1E & \xrightarrow{j^1f} & J^1E \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow[f']{} & \Sigma \end{array} \quad f' \neq \mathrm{id}_\Sigma \end{array}\right]$$

Can also prolong Poincaré:

$$j^1 f_{a,L} : (x^\mu, u^i, u^i_\mu) \mapsto (L^\mu{}_\nu x^\nu + a^\mu, u^i, (L^{-1})_\mu{}^\nu u^i_\nu)$$

Simple! contract any μ index using $\eta^{\mu\nu}$!

▷ Geometry? Most general metric consistent w/. $G \times$ Poinc. symmetry:

$$g^{(1)} = g_{IJ}\, dx^I dx^J = g_{\mu\nu}\, dx^\mu dx^\nu + 2g_{\mu i}\, dx^\mu du^i + g_{ij}\, du^i du^j$$

$$+ \underbrace{2g_{\mu j}^{\ \nu}\, dx^\mu du^j_\nu} + \underbrace{2g_{ij}^{\ \nu}\, du^i du^j_\nu} + \underbrace{g_{ij}^{\mu\nu}\, du^i_\mu du^j_\nu}$$

fⁿs of u^i & u^i_μ.

NEW STRUCTURES!!

▷ Lagrangian? Pull back along $j^1\phi$:

$u^i \to \phi^i(x)$, $u^i_\mu \to \partial_\mu \phi^i(x)$, $du^i \to \partial_\rho \phi^i dx^\rho$, $du^i_\mu \to \partial_\rho \partial_\mu \phi^i dx^\rho$

$$\mathcal{L} = \tfrac{1}{2}\eta^{-1} \cdot (j^1\phi)^* g^{(1)} = \tfrac{1}{2}\eta^{\mu\nu} g_{\mu\nu}(\phi, \partial\phi) + \ldots + \tfrac{1}{2} g_{ij}(\phi, \partial\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j$$

$$+ \ldots + g_{ij}^{\ \nu}(\phi, \partial\phi)\, \partial_\rho \phi^i \partial^\rho \partial_\nu \phi^j$$

$$+ \tfrac{1}{2} g_{ij}^{\mu\nu}(\phi, \partial\phi)\, \partial_\rho \partial_\mu \phi^i \partial^\rho \partial_\nu \phi^j$$

n.b.!

Caution! the map $g \to \mathcal{L}$ is highly redundant!

e.g.
$$\begin{aligned} g_{ij} &= \delta_{ij} &\to\quad & \partial_\mu \phi^i \partial^\mu \phi^i \\ g_{\mu i} &= \delta_{ik} u^k_\mu &\to\quad & \qquad '' \\ g_{\mu\nu} &= \delta_{ij} u^i_\mu u^j_\nu &\to\quad & \qquad '' \end{aligned}$$

* Nonetheless, this (redundant) description gives complete basis w/. $D \leq 4$
* 1-jet → r-jet ⇒ complete basis w/. $D \leq 2(r+1)$
[Essence: IBPs ⟶ no more than $r+1$ ∂'s on a single field ϕ]

* Ex: SMEFT: build inv. metrics from δ_{ij} [$\epsilon_{ijkl} \to$ top. terms only]

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(6) WIP

1/. Connection between amplitudes and tensors (Riemann) built from $g^{(r)}$.
(difficult technically due to e.g. redundancies, "heavy" formalism.)

▸ 0-jet amplitudes: $g_{IJ} = \begin{pmatrix} -\frac{1}{2}V(u)\eta_{\mu\nu} & 0 \\ 0 & g_{ij}(u) \end{pmatrix}$

no redundancies!

$\Gamma^I{}_{JK}$: $\Gamma^i{}_{jk} = \frac{1}{2} g^{il}\left(g_{jl,k} + g_{lk,j} - g_{kj,l}\right)$ (*)

$\Gamma^i{}_{\mu\nu} = -\frac{1}{2}\eta_{\mu\nu} g^{ij}\partial_j V$, $\Gamma^\mu{}_{\nu j} = \frac{1}{2V}\delta^\mu_\nu \partial_j V$, else $= 0$

$R^I{}_{JKL}$: $R^i{}_{jkl} = \ldots$ (as in field space picture) (*)

* suggests we can go to "normal fibre coordinates" ⤳ formula for A?

$R^i{}_{\mu k\nu} = -\frac{1}{2}\eta_{\mu\nu}\left[\partial_k\left(g^{ij}\partial_j V\right) + \Gamma^i{}_{kj}\, g^{jl}\partial_l V\right]$

$R^\mu{}_{\nu\rho\sigma} = -\frac{1}{4V}\left(\delta_\rho^\mu \eta_{\nu\sigma} - \delta_\sigma^\mu \eta_{\rho\nu}\right) g^{ij}\partial_i\partial_j V$

Consider $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \lambda\phi^4 + c\,\phi^2(\partial\phi)^2$ (w/. indices)

$$A_{ijkl} \sim \underset{\sim\partial^4_{ijkl}V}{\lambda_{ijkl}} + \underset{\sim R_{ijkl}}{C_{ijkl}\, s}$$

$$\Rightarrow A_{ijkl} \sim \eta^{\mu\nu}\left[\partial_i\partial_j R_{k\mu l\nu} + p_\mu q_\nu R_{ijkl}\right] \sim \nabla\nabla R$$??

2/. Derivative FRs via r-jet diffeos (requires careful truncation)
3/. HEFT v. SMEFT? $D \lessgtr 4$ criteria for "loryons"?
4/. Gauging + fermions
5/. Positivity bounds $\longleftrightarrow$ 'causal structure' on jet bundles?