EFT of Cold Fermions near the Critical Point

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(Envisaging Future Trajectories in Effective Field Theory 3-13-24)

Many-body fermionic systems at quantum critical points have no quasi-particle description, ``Non-Fermi-Liquids”

- Usually described by quasi-particles coupled to ``critical bosons”

- No realistic expansion parameter (large N)

Consider Cold atomic gases, removes the need to treat the phonons

$$f_l(k) \equiv \frac{e^{2i\delta_l(k)-1}}{2ik} = \frac{1}{k\cot\delta_l(k) - ik}$$

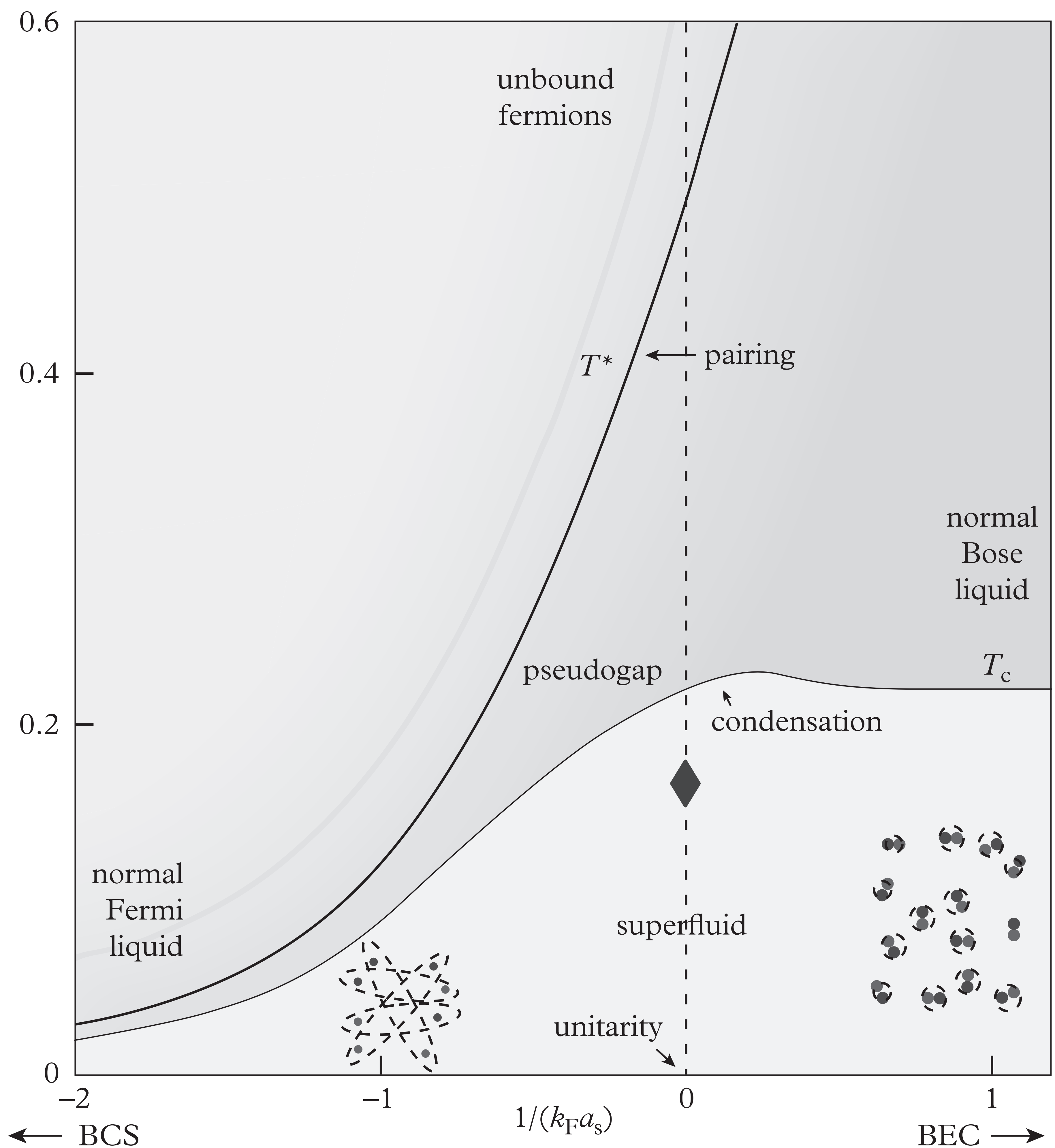
$$k\cot\delta_0(k) \equiv -\frac{1}{a} + \frac{1}{2}k^2 R_0 + ...$$

When a is large, there exists a bound state at $k = i/a$

e.g. a square well $$a = r_0\left(1 - \frac{\tan(\kappa_0 r_0)}{\kappa_0 r_0}\right)$$ $$\kappa_0 = \sqrt{mV_0}$$

Fine tuning leads to a large scattering bound state E=0 with infinite scattering length. $$\kappa_0 r_0 = \frac{(2n-1)\pi}{2}$$

- **Scattering amplitude becomes independent of range as well as other UV**
- **parameters when a gets large (NRCFT as a diverges many body problem)**


0.6
0.4
0.2
0
unbound
fermions
pairing
T^*
normal
Bose
liquid
pseudogap
T_c
condensation
normal
Fermi
liquid
superfluid
unitarity
−2
−1
0
1
$1/(k_F a_s)$
BCS
BEC

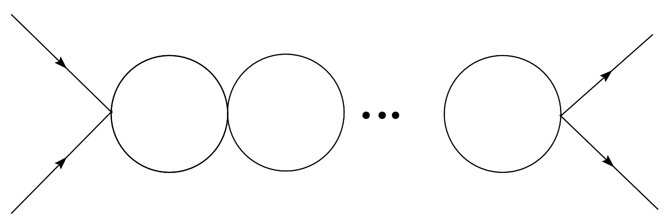
Consider dilute Cold Fermionic gases whose interaction is tunable through Feshbach resonance.

UV Theory

Quantum Mechanics in zero Range approximations, valid for large scattering lengths (strong coupling)

$$L_0 = i\psi^\dagger \partial_0 \psi + \frac{1}{2}\psi^\dagger \vec{\nabla}^2 \psi. \qquad L_I^{(1)} = g(\psi_\uparrow^\dagger \psi_\downarrow^\dagger)(\psi_\uparrow \psi_\downarrow)$$

$$g(\mu_0) = \left(\frac{4\pi}{-\mu_0 + 1/a}\right)$$



$$\hat{g}(\mu) = \frac{\mu}{4\pi} g(\mu),$$

This theory has a non-trivial UV fixed point

$$\mu \frac{d}{d\mu}\hat{g} = \hat{g}(1 + \hat{g})$$

$$g \sim a$$

Tuning to infinite scattering length leads to an NRCFT

IR Theory

Use this action in the context of a grand canonical ensemble, (μ, T)

At infinite scattering length we are sitting at a quantum critical point. We may like to think of the resulting Goldstones as being the ``critical bosons”, and we are back to where we started in terms of our inability to keep calculational control w/o resorting to any unphysical limit.

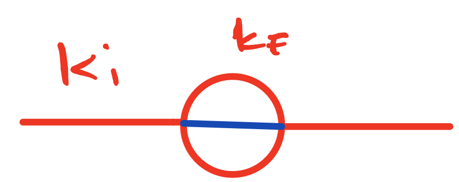
However, given that we can solve the UV theory exactly, you would think we could utilize the short distance information such that we can gain predictive power.

``Anomaly Matching”

Away From the critical point we expect FLT to correctly describe the physics above Tc.

Basic Assumptions of FLT

1) In the UV there exists quasi-particle excitations

Im 

$$\lim_{E \to 0} \Gamma \leq E^2$$

2) No long range forces

3) Rotational invariance: cold atoms as opposed to metals (here but but not in general)

Rapid Review for non-experts

Fermi Liquid Theory is the ``standard model” of CMT

At temperatures above Tc it is described by a weakly coupled IR fixed point with one a marginal coupling functions. For a spherically symmetric Fermi surface we can decompose these function into an infinite number of couplings, generically called ``Landau Parameters”

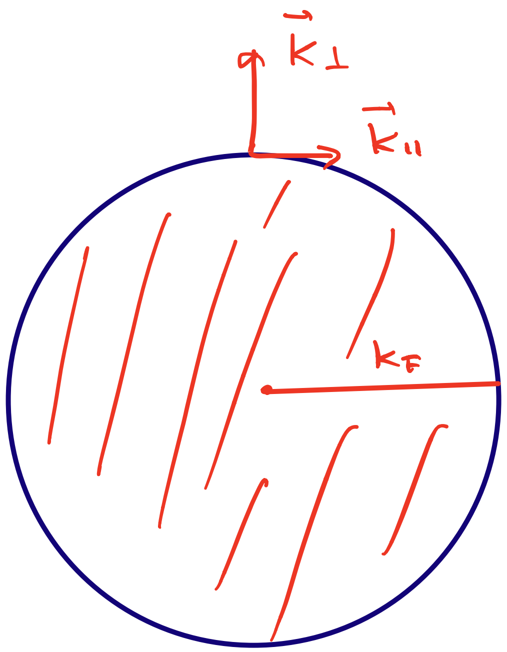
$$L_{int} = \sum_l g_l (\psi^\dagger \psi)^2$$

g_l **Determine all of the Low Energy properties of the system, e.g. compressibility, quasi-particle width.**

EFT of Fermi Surface (Shankar/Polchinski)

Power Counting Parameter: $\lambda \equiv E/E_F$

Momentum scaling: $k_\perp \sim \lambda \qquad k_{||} \sim 1$



$$[\psi(x), \psi^\dagger(0)] \sim \delta^d(x) \sim \lambda$$

Field Scalings:

$$\psi(x) \sim \lambda^{1/2}$$

Building Action via power counting

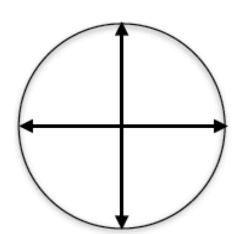
$$S = \int \underset{\lambda^{-1}}{dt}\, \overset{\lambda}{d^d p}\; \overbrace{\psi^\dagger_{\vec p}(t)\,(i\underset{\lambda}{\partial_t} - \underbrace{\varepsilon(\vec p) + \mu}_{\lambda})\,\psi_{\vec p}(t)}^{\lambda^{-1}}$$

$$\epsilon(\vec p) - \mu \approx \frac{\partial \epsilon}{\partial \vec k} \cdot \vec k_\perp = \vec v_F \cdot \vec k_\perp \sim \lambda$$

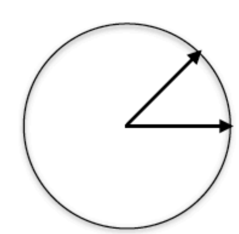
$$S_{int} = \frac{1}{2} \prod_i \underbrace{dt d^d p_i}_{\lambda^3} g(\vec p_i^{\,\|}) \underbrace{\psi^\dagger_{\vec p_1} \psi_{\vec p_2} \psi^\dagger_{\vec p_3} \psi_{\vec p_4}}_{\lambda^{-2}} \delta^{d-1}\Big(\sum \vec p_i^{\,\|}\Big) \underbrace{\delta\Big(\sum \vec p_i^{\,\perp}\Big)}_{\lambda^{-1}}$$

Only for special kinematic configurations does the delta function scale

BCS



$a)$



$b)$

Forward Scattering

Only possible marginal interaction

Any new marginal or relevant interaction expected to obstruct FL behavior

It is important to understand that this theory is ``solvable” in the sense that response function can be calculated in terms of bubble sums up to power corrections even at strong coupling. At least under the assumption that one partial wave dominates.

When can we use FLT to study Cold Atomic System Near Unitarity

— At Unitarity Non-linearly realized Space-Time Symmetries Prohibit FL description

IZR, P. Shrivistava 1712.07797

— Can we do perturbation theory away from this conformal fixed (where FLT is expected to arise) and retain the predictive power that comes with the enhanced symmetry?

— Can we make first principles predictions by being near enough from the fixed point that we have approximate conformal symmetry but far enough away that we can still calculate using FLT?

To make progress we will consider FLT in the language of space-time symmetry breaking, as we will utilize the relevant Ward identities to gain predictive power.

All our predictive power will be based upon the Space-Time Symmetries of FLT

Galilean Group: $H, \vec{P}, \vec{L}, M, \vec{K}$

Spontaneously Broken: $\vec{K}$ **("Framid")** **(X: Broken Generators)**

Where are the boost Goldstones?

Order Parameter

Goldstones Thm when S.T. symmetries are broken: $\langle \Omega \mid [X, \phi] \mid \Omega \rangle \neq 0$

$$\sum_n \delta^{(d-1)}(\vec{p}_n) \left[\langle \Omega \mid j_0^X(0) \mid n \rangle \langle n \mid \phi(0) \mid \Omega \rangle e^{iE_n t} - \langle \Omega \mid \phi(0) \mid n \rangle \langle n \mid j_0^X(0) \mid \Omega \rangle e^{-iE_n t} \right] \neq 0$$

Zero momentum/energy state produced by current must exist in the spectrum

Broken current can interpolate for a multi-particle state that saturates the Ward identity.

There is a general methodology which allows us to determine when Goldstone bosons need not appear in the spectrum via the ``inverse Higgs mechanism” (Volkov (73) Ogievetsky (74))

$[\bar{P}, X] \propto X'$ $\qquad$ $X, X' \in$ **Broken Generators**

$\bar{P}$ $\qquad$ **unbroken translation**

X, X' $\qquad$ **Then the associated Goldstones are redundant**

In the case of Framids there is no inverse Higgs constraint since the only broken generators are boosts. Where are the boost Goldstones?

This is no puzzle, we've seen this before:

HQET is a Fermi liquid with N=1

Coset construction of HQET

$$\mathcal{L}_\phi = \frac{i}{2}\left(\phi^\dagger(t,\vec{x})(\partial_t + \vec{\eta}\cdot\vec{\partial} + \frac{i}{2}m\vec{\eta}^2)\phi(t,\vec{x}) - [(\partial_t + \vec{\eta}\cdot\vec{\partial} - \frac{i}{2}m\vec{\eta}^2)\phi^\dagger(t,\vec{x})]\phi(t,\vec{x})\right)$$
$$+ \frac{c_1}{2m}((\vec{\partial} + im\vec{\eta})\phi^\dagger(t,\vec{x}))(\vec{\partial} - im\vec{\eta})\phi(t,\vec{x}).$$

$$\vec{\eta} \to \vec{\eta} + \vec{v} \qquad \phi \to e^{\frac{i}{2}mv^2 t + i\vec{v}\cdot\vec{x}}\phi$$

But in this trivial vacuum there are no collective excitations. Treat boost Goldstone as Lagrange multiplier that imposes constraint leading to boost invariance.

Nothing more than RPI constraints $\quad c_1 = 1$

What happens in the case of FLT where Multi-particle states can saturate Ward identity (as opposed to HQET)

Space-time Coset construction:

$$U = e^{i\bar{P}\cdot x} e^{i\pi\cdot X}$$

Parameterizes G/H

$$U^{-1}\partial_\mu U = E_\mu^A(\bar{P}_A + \nabla_A \pi^a X^a + A_A^b T^b)$$

Extract: $E, A, \nabla\pi$

Capital Roman: H irreps

Greek: G irreps

Covariant building blocks ensue

$$\nabla_A \psi \equiv (E_A^\mu \partial_\mu + iT^q A_A^q)\psi$$

FLT

$$U = e^{iP\cdot x} e^{-i\vec{K}\cdot\vec{\eta}(x)}$$

$$E_0^0 = 1 \quad E_i^j = \delta_i^j, \quad E_0^i = \eta^i, E_i^0 = 0$$

$$A_i = -\eta_i, A_0 = -\frac{1}{2}\vec{\eta}^2$$

Only break boosts

$$S_0 = \int d^d x dt \; \psi^\dagger \left[i(\partial_0 + \eta^i \partial_i) + \frac{1}{2} m \vec{\eta}^2 + \varepsilon(i\partial_i + m\eta_i) \right] \psi$$

$$S_{int} = \prod_{i,a} \int d^d k_i dt \;\, g(\vec{k}_i + m\vec{\eta})\psi^\dagger_{k_1}(t)\psi_{k_2}(t)\psi^\dagger_{k_3}(t)\psi_{k_4}(t)\delta^d(\sum_i k_i)$$

Note: At this point we have not assumed anything about what scattering kinematics are allowed or not.

This is where things become interesting, as we now have a coupling function.

As we did in HQET, treat the framid as a L.M

$$= \sum_{\vec{\mathbf{k}}(\theta)} \int d^d x dt \;\, \psi^\dagger_{\vec{\mathbf{k}}(\theta)}(x) \left[i\partial_0 - \vec{\eta}(x)\cdot\vec{v}_F(\theta)(m - m^\star) + i\vec{v}_F(\theta)\cdot\vec{\partial} \right] \psi_{\vec{\mathbf{k}}(\theta)}(x) + \ldots$$

$$\frac{\partial \varepsilon}{\partial \vec{k}} = \vec{v}_F = \frac{\vec{k}_F}{m^\star}$$

$$S_{int} = \prod_i \int d^d k_i dt \sum_j \frac{m}{2}\vec{\eta}\cdot\frac{\partial g(k^j)}{\partial \vec{k}^j}\psi^\dagger_{k_4}(t)\psi^\dagger_{k_3}(t)\psi_{k_2}(t)\psi_{k_1}(t)\delta^d(\sum_i k_i)$$

Constraint from treating framid as L.M.

("DIHC": Dynamical Higgs Constraint)

Boost invariance constraint, EOM for framid field

Impose: $O_B = 0$

$$O_i^B = \int \frac{d^d p}{(2\pi)^d}\, \psi_p^\dagger(t) \left(p_i - m\frac{\partial \varepsilon_p}{\partial p_i}\right) \psi_p(t) - \frac{m}{2} \int \prod_{a=1}^{4} \frac{d^d p_a}{(2\pi)^d} \delta^{(d)}\Big(\sum_i p_i\Big)\Big(\sum_i \frac{\partial g(p_a)}{\partial p_{i,a}}\Big)\psi_{p_4}^\dagger(t)\psi_{p_3}^\dagger(t)\psi_{p_2}(t)\psi_{p_1}(t).$$

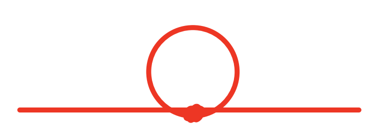
We have not imposed any restrictions on kinematics at this point

Conserved charges (time independent)

non-conserved

$$\langle \vec{k} \mid O_B \mid \vec{k}\rangle = 0.$$

$$k_i = m\frac{\partial \varepsilon(k)}{\partial k_i} + \frac{2m}{(2\pi)^d}\int d^d p\Big(\frac{\partial g(k,p,p,k)}{\partial p_i} + \frac{\partial g(k,p,p,k)}{\partial k_i}\Big)\theta(p_F - p)$$



$$\frac{k_F}{m} = v_F + \frac{2p_F}{(2\pi)^2}\int d\theta \ \cos\theta \sum_l g_l P_l(\cos\theta)$$

$$\frac{m^\star}{m} = 1 + \frac{1}{3}\frac{2m^\star}{(2\pi)^2} g_1$$

Alternative derivation of Landau relation. But we have only worked to 1-loop?

Constraint must be RG invariant: This implies that the beta function must vanish!

For generic kinematics this will not be true, but it IS true for forward scattering!!

We see that boost invariance ONLY allows for forward scattering *. This is an alternative derivation of the famous statement that FLT has an infinite number of conserved quantities 1712.07795 (IZR/Srivistava).

This also shows that constraint hold to all order in PT.

*** BCS is not ruled out because it does not contribute to the two point function (kinematically disallowed).**

Now Suppose we tune to the fixed point in the UV theory

Schrodinger Group $$(H, \vec{P}, \vec{L}, \vec{K}, M, C, D)$$

If we tune the atomic interaction via Feschbach resonance to fixed point (diverging scattering length) then the Fermi sea will break

$$\vec{K}, C, D$$

Inverse Higgs constraint: $$[H, C] = iD$$

So in generating an action we only need to add the dilaton mode

We will be interested in what happens as we perturb away from unitarity

As we move away from unitary the Dilaton gets gapped (pseudo-Goldstone).

First Build action via coset construction:

$$U = e^{iHt} e^{-i\vec{P}.\vec{x}} e^{-i\vec{K}.\vec{\eta}} e^{-iD\phi} e^{-iC\xi}$$

-Impose DIHC on framid (Landau Condition)

-Eliminate special conformal GB via IHC

$$S_\psi = \int dt d^3x e^{-\frac{5\phi}{\Lambda}} [\tilde{\psi}^\dagger (i e^{\frac{2\phi}{\Lambda}} \partial_t \tilde{\psi} - (\epsilon(e^{\frac{\phi}{\Lambda}} i\vec{\partial}) - \mu_F)\tilde{\psi})$$

$$\tilde{\psi} = e^{\frac{3\phi}{2\Lambda}} \psi$$

$$S_{int} = \int d^d x dt e^{-5\phi/\Lambda} f_0 (\tilde{\psi}^\dagger \tilde{\psi})^2$$

Only kept l=0 coupling for reasons which will become clear

Expanding out to generate the Dilaton coupling

$$S_\psi = \int d^3x\,dt\ i\psi^\dagger \partial_t \psi + \psi^\dagger \vec{v}_F \cdot \vec{\partial}\psi - \frac{2\phi\mu_F}{\Lambda})\psi^\dagger\psi + ...$$

We have no calculational control over this theory due to the dilaton coupling (no quasi-particles). How let us deform the UV theory away from criticality

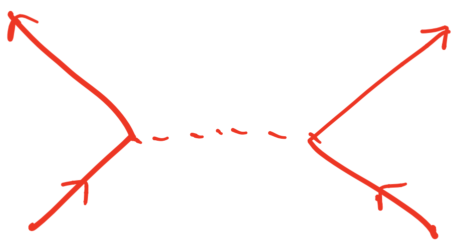
In the IR this will gap the Dilaton

$$\delta L = \frac{1}{2} m_\phi^2 \phi^2$$

For large enough scattering length we will have a light dilaton

$$m_\phi \ll E_F$$

Generate local interactions when $E \ll m_\phi$



$$\sim \frac{\mu_F^2}{\Lambda^2} \frac{1}{m_\phi^2}$$

Dominates scattering $m_\phi^2 / E_F^2 \ll 1$

We have two unknown parameters: m_ϕ, Λ

Dilaton mass can be fixed by "anomaly matching"

$$\partial_\mu s^\mu = m_\phi^2 \Lambda \, \phi$$ **(EFT)**

$$\partial_\mu s^\mu = (g(\mu) + \beta(g))(\chi^\dagger \chi)^2$$ **(UV Theory)**

$$D^0(0) = \Lambda \int d^3x \; \pi(\vec{x}, 0) \qquad \int_x [D^0(0), \partial_\mu s^\mu(\vec{x}, 0)] = \int d^3x \; m_\phi^2 \Lambda^2$$ **(EFT)**

$$D^0(0) = \int d^3x (\frac{3}{2}\chi^\dagger(\vec{x}, 0)\chi(\vec{x}, 0) + \chi^\dagger(\vec{x}, 0)\vec{x} \cdot \vec{\partial}\chi(\vec{x}, 0))$$ **(UV Theory)**

$$\int_x [D^0(0), \partial_\mu s^\mu(\vec{x}, 0)] = 3 \int d^3x \; (g(\mu) + \beta(g)) \; (\chi^\dagger \chi)^2$$

Now we equate the vacuum matrix elements of the commutators in full and EFT

$$\boxed{m_\phi^2 \Lambda^2 = \frac{3}{4\pi a}\langle g^2 \ \chi_\uparrow^\dagger \chi_\uparrow \chi_\downarrow^\dagger \chi_\downarrow \rangle \equiv \frac{3}{4\pi a}\,\mathcal{C}}$$

$\mathcal{C}$ **Contact (Tan) parameter (controls thermodynamics of system)**

Integrating out dilaton leads to $$L_{int} = (f_0 + \frac{8\pi a \mu_F^2}{3\mathcal{C}})(\psi^\dagger\psi)^2$$

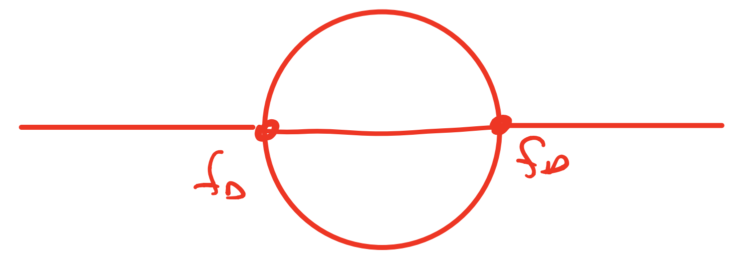
UV modes down by Ef, independent of a.

$$f_D \equiv \frac{8\pi a \mu_F^2 m}{3\hbar^4 k_F^4 \tilde{C}}$$

Thus we have a prediction for the l=0 Landau parameter when : $\left(\frac{E_F}{E}\right)^2 \gg ak_f \gg 1.$

corrections: $(E^2/m_\phi^2, m_\phi^2/E_F^2)$

Predictions for Observables

quasi-particle width $\Gamma(E) \sim Im$ 

$$\Gamma(E,T) = \frac{m}{9\pi\tilde{\mathcal{C}}^2}\left(\sqrt{\frac{m}{m^\star}}\frac{a\mu_F^2}{\hbar E_F^2}\right)^2 (E^2 + (\pi k T)^2)$$

$$\frac{\Delta\Gamma_T}{\Gamma} \sim O\left(\frac{1}{k_F a}\right) + O\left(k_F a\left(\frac{E^2}{E_F^2}\right)\right)$$

Compressibility:

$$\kappa = \frac{V}{N}\left(\frac{\partial N}{\partial \mu}\right)_{T,V} = \lim_{q\to 0,\omega\to 0}\langle\rho(\omega,q)\rho(-\omega,-q)\rangle.$$

$$\langle \rho(\omega)\rho(-\omega)\rangle_{\omega\to 0} = \frac{\bigcirc}{1 - f_D \bigcirc}$$

$$\frac{\kappa}{\kappa_0} = \frac{1}{1 - \hat{f} + \frac{4}{3\pi}\frac{k_F a}{\tilde{C}}\left(\left(\frac{m}{m^\star} - 1\right)^2\right)}$$

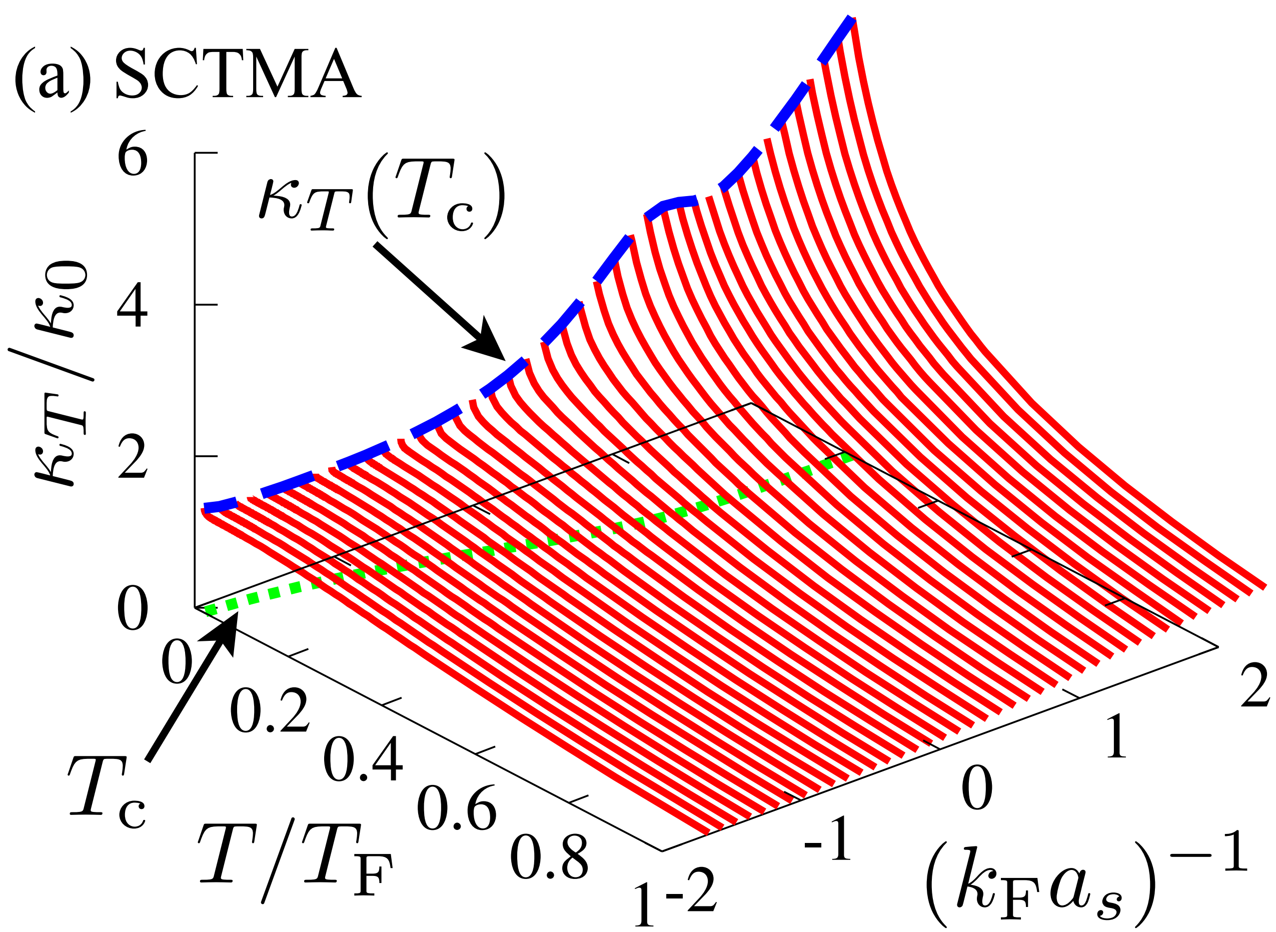
Short distance contribution (unknown) which is power suppressed

$(\tilde{C}(a), m^\star)$ **Measure quantities.**

prediction holds in the range -1<kf a<-10.

Compare to numerical simulations

Kagamihara et al. ArXive: 2206.02406



$$\frac{\kappa}{\kappa_0} = \frac{1}{1 - \hat{f} + \frac{4}{3\pi}\frac{k_F a}{\tilde{C}}\left(\frac{m}{m^\star} - 1\right)^2)}$$

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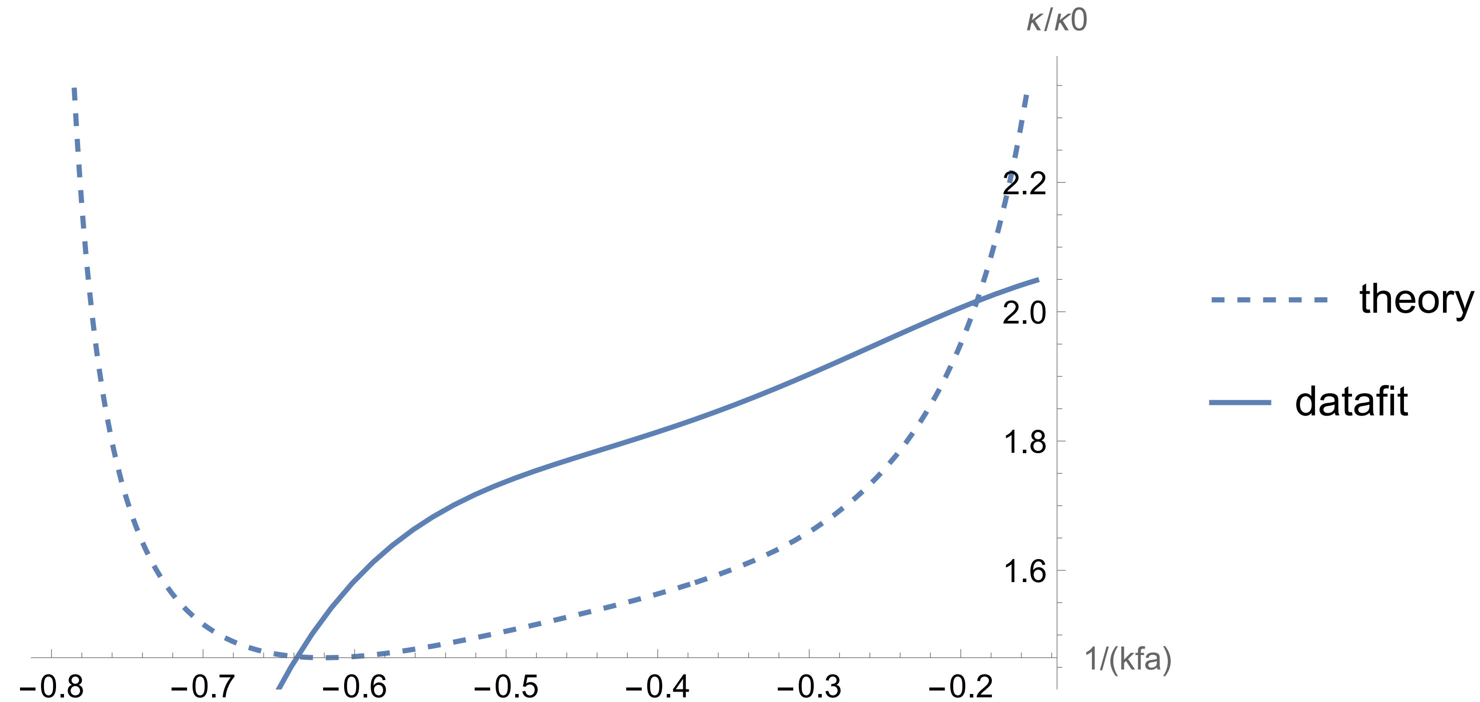

If we account for fact that f0 is independent of a, we can us the data at one point and fix f0, shifting the coupling such that

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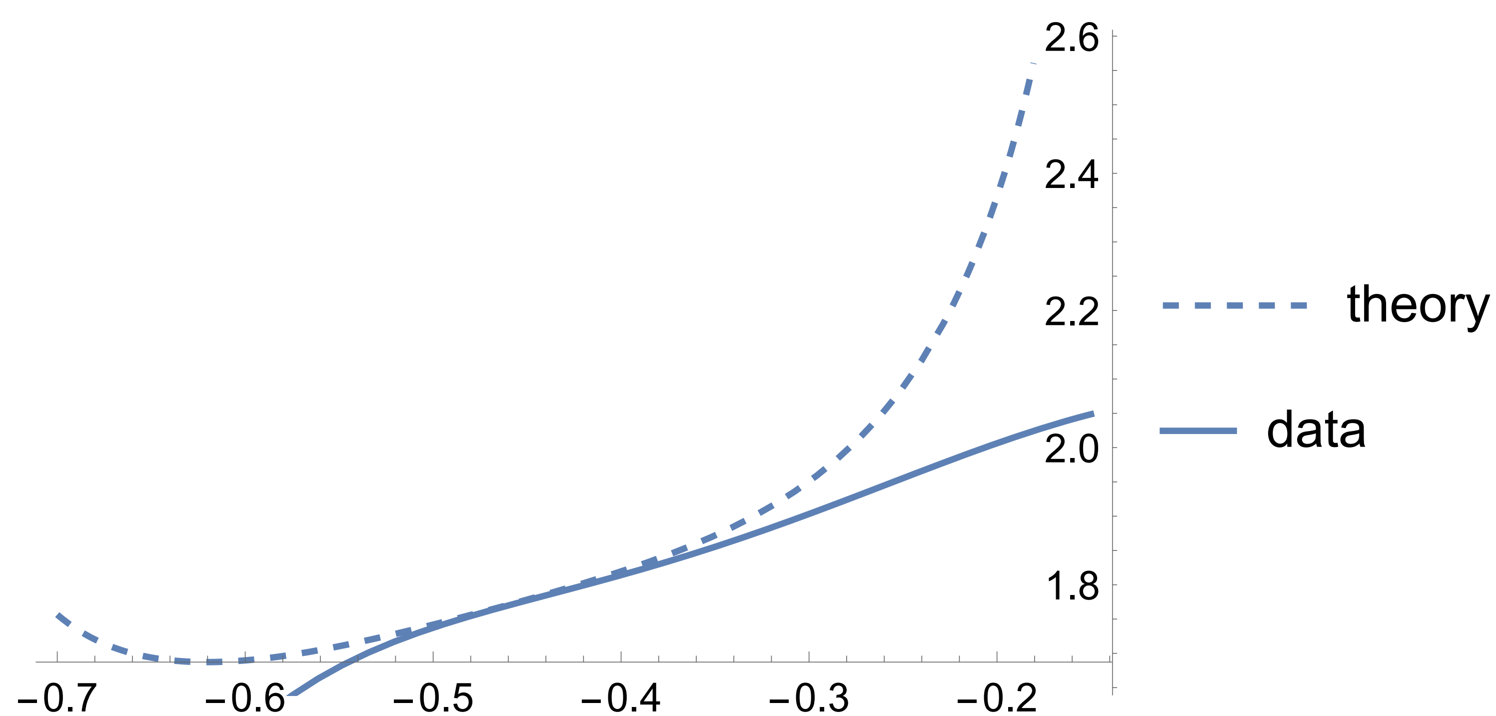

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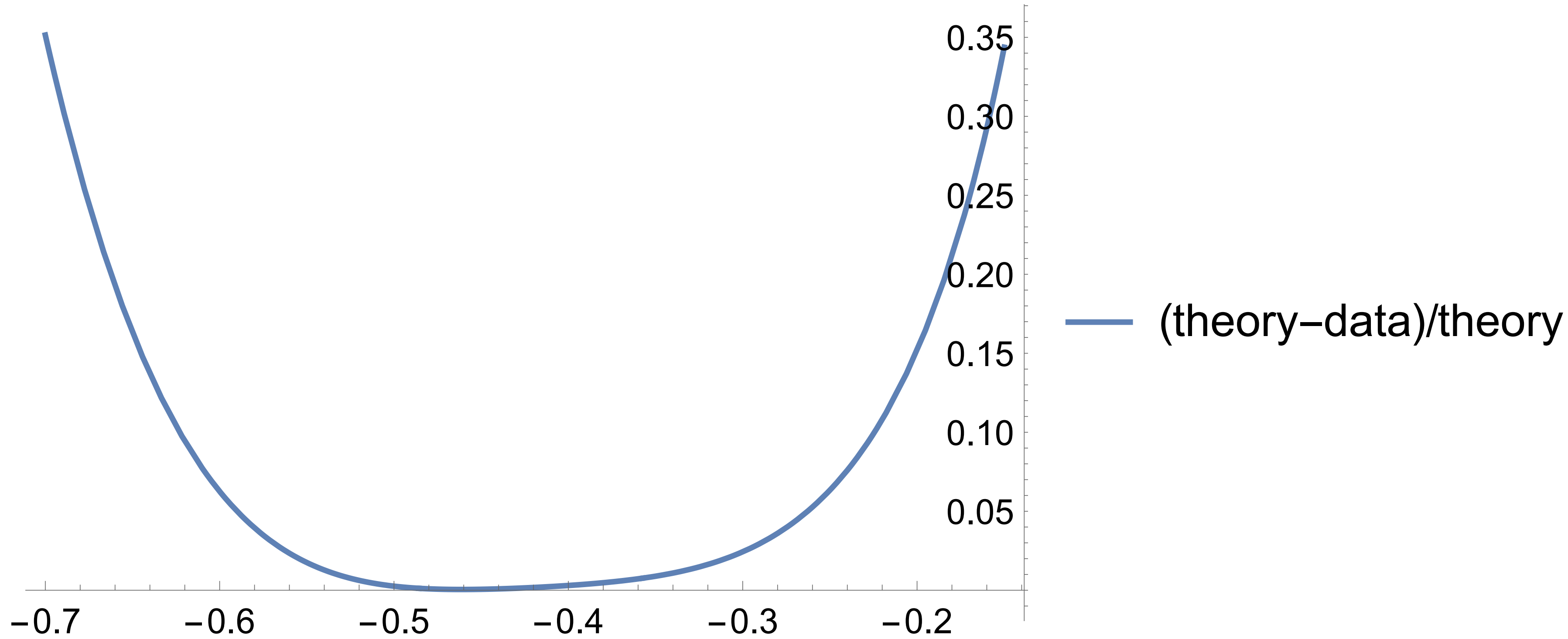

We can also calculate the temperature dependence of the result

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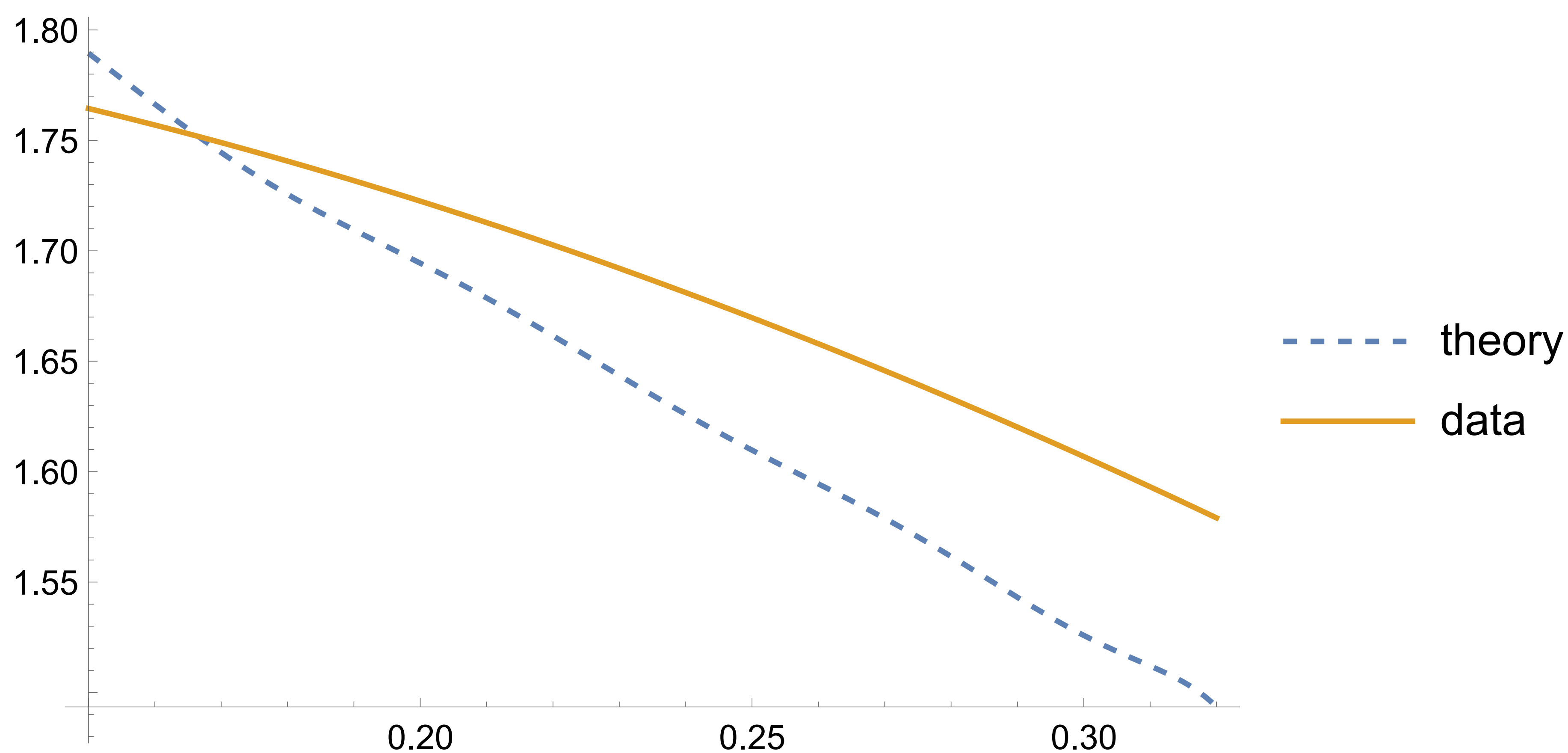

- Applications to other nearly critical systems

$\mathscr{A}(\varphi \to \pi^a(p_1)\,\pi^b(p_2))$ **(Voloshin+Zakharov): (Georgi, Manohar, Grinstein)**

$$\mathscr{L}_{\text{eff}} = -\frac{n_h}{3b}\frac{\varphi}{v}\left(\frac{\beta(g)}{g}G^2\right)\Big|_{\mu = 1\ \text{GeV}} \qquad \partial_\mu s^\mu = \frac{1}{2}\frac{\beta(g)}{g}G^2$$

$$\partial_\mu s^\mu = -\frac{f^2}{2}\text{Tr}\,\partial_\mu \Sigma^\dagger\,\partial^\mu \Sigma - 2f^2 \Lambda(\text{Tr}\,M\Sigma + \text{h.c.})$$

$$\mathscr{L}_{\text{eff}} = -\frac{2n_h}{3b}\frac{\varphi}{v}\,\partial_\mu s^\mu - \sum_i m_i \bar{\psi}_i \psi_i \frac{\varphi}{v}\left(1 - \frac{2n_h}{3b}\right)$$

$$\mathscr{L}_{\text{eff}} = \frac{n_h}{3b}\frac{\varphi}{v} f^2\,\text{Tr}\,\partial_\mu \Sigma^\dagger\,\partial^\mu \Sigma + \frac{f^2}{2}\Lambda(\text{Tr}\,M\Sigma + \text{h.c.})\left(1 + \frac{2n_h}{b}\right)\frac{\varphi}{v}$$

$$\mathscr{A}(\varphi \to \pi^a \pi^b) = -\left[\frac{2}{3v}\frac{n_h}{b}m_\varphi^2 + \frac{1}{v}m_a^2\left(1 + \frac{2n_h}{3b}\right)\right]\delta_{ab}$$

Relevant Side Note:

S.T. Goldstone bosons can lead to long range forces, i.e none-derivatively coupled.

Sufficient criteria: $[X_i, \vec{P}] \neq 0$ **(Watanabe and Vishnawath 2014)**

$[\bar{P}^{\mu}, X] \neq 0$ **(IZR and Srivistava 2017)**

Notice that the framid (Boost GB) will be None-derivatively coupled which threatens the Fermi liquid picture.

We need to saturate the Boost Ward identities w/o any GB and no I.H.C

How can we stabilize Dilaton mass? C.C. Term is allowed

$$S_{int} = \int d^d x dt e^{-5\phi/\Lambda} \Lambda$$

Fine tuning needed keep dilaton light, but that is exactly what is provided by fine tuning the magnetic field