Monopole-Fermion Scattering and Generalised Symmetries

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EFT $\subseteq$ EFT, π day 2024

What's this about?

- What happens when a fermion bounces off a monopole?
- Spoiler: It scatters to the twisted sector of an ABJ defect.

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- What happens when a fermion bounces off a monopole?
- Spoiler: It scatters to the twisted sector of an ABJ defect.

Based on

- 2306.07318, *"Monopoles, Scattering, and Generalized Symmetries"*

with

- Marieke van Beest
- Diego Delmastro
- Zohar Komargodskii
- David Tong

and morally also with

- Gabriel Cuomo, Masataka Watanabe, Kantaro Ohmori

The Plan of the Talk

1. The Monopole-Fermion Problem
2. A Toy 2d Model
3. The Actual 2d Model
4. Lifting to 4d

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Our Setup

1. QED with N_f flavours:

$$U(1) \text{ gauge field} \quad + \quad N_f \text{ Dirac fermions}_{+1}$$

2. Place a charge-n monopole at the origin:

$$A = \frac{n}{2}(1 - \cos(\theta))d\phi$$

3. Attempt to scatter fermions off it.

Note

Unusually, we will write the fermions in terms of the variables

$$\psi_{i=1...N_f} \qquad \tilde{\psi}_{i=1...N_f}$$

where ψ_i and $\tilde{\psi}_i$ are both left-handed Weyl fermions, with opposite charges $+1$ and -1.

Partial waves: A scalar field...

A scalar field has a partial wave expansion

$$\phi(\mathbf{x}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \phi_{\ell m}(r, t) Y_{\ell m}(\theta, \phi)$$

We will denote this expansion by a table

$SU(2)_{\text{rot}}$
$\vdots$
$\mathbf{5}$
$\mathbf{3}$
$\mathbf{1}$

where $\mathbf{2}\ell + \mathbf{1}$ represents the multiplet of fields $\phi_{\ell m}$.

Partial waves: A scalar field + magnetic flux...

In the presence of magnetic flux n, this is shifted to

$SU(2)_{\text{rot}}$
$\vdots$
$\mathbf{\lvert n \rvert + 5}$
$\mathbf{\lvert n \rvert + 3}$
$\mathbf{\lvert n \rvert + 1}$

Partial waves: A Weyl fermion + magnetic flux...

Tensoring with the spin-$\frac{1}{2}$ representation,

	$SU(2)_{\text{rot}}$	
	$\vdots$	$\vdots$
$\lvert\mathbf{n}\rvert + \mathbf{5}$	$\lvert\mathbf{n}\rvert + \mathbf{4}$	$\times 2$
$\lvert\mathbf{n}\rvert + \mathbf{3}$	$\lvert\mathbf{n}\rvert + \mathbf{2}$	$\times 2$
$\lvert\mathbf{n}\rvert + \mathbf{1}$	$\lvert\mathbf{n}\rvert$	$\times 1$

we get the result for a Weyl fermion.

Partial waves: A Weyl fermion + magnetic flux...

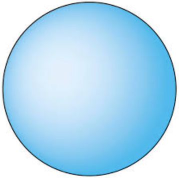
The equation of motion says each 2d field is chiral. Specifically, it moves either *in* or *out:*

direction	$SU(2)_{\text{rot}}$
$\vdots$	$\vdots$
in + out	$\lvert\mathbf{n}\rvert + \mathbf{4}$
in + out	$\lvert\mathbf{n}\rvert + \mathbf{2}$
in if $n > 0$, out if $n < 0$	$\lvert\mathbf{n}\rvert$

We observe only that the lowest partial wave is chiral.

Via the Index Theorem

This structure arises because the partial waves are governed by the spectrum of $\not{D}$ on S^2 with n units of magnetic flux:



$$\text{index}(\not{D}) = \int_{S^2} \frac{F}{2\pi} = n$$

Thus,

$$n > 0 \implies n \text{ positive-chirality zero-modes}$$

$$n < 0 \implies n \text{ negative-chirality zero-modes}$$

Partial-wave Decompositions

- Decomposing ψ_i into partial waves,

in-mover or out-mover?	$SU(2)_{\text{rot}}$
$\vdots$	$\vdots$
in + out	$\lvert\mathbf{n}\rvert + \mathbf{4}$
in + out	$\lvert\mathbf{n}\rvert + \mathbf{2}$
in	$\lvert\mathbf{n}\rvert$

- Decomposing $\tilde{\psi}_i$ into partial waves,

in-mover or out-mover?	$SU(2)_{\text{rot}}$
$\vdots$	$\vdots$
in + out	$\lvert\mathbf{n}\rvert + \mathbf{4}$
in + out	$\lvert\mathbf{n}\rvert + \mathbf{2}$
out	$\lvert\mathbf{n}\rvert$

- We assume $n > 0$. Otherwise in and out are reversed.

Partial-wave Decompositions

- The higher partial waves experience a centrifugal barrier ($V(r) \propto 1/r$), and can be ignored.
- Only the lowest partial wave penetrates into the core of the monopole and experiences the UV physics.
- Therefore, the scattering reduces to the lowest partial wave:

fermion	in-mover or out-mover?	$SU(2)_{\text{rot}}$	$U(1)_Q$
ψ_i	in	$\lvert\mathbf{n}\rvert$	$+1$
$\tilde{\psi}_i$	out	$\lvert\mathbf{n}\rvert$	-1

- How do these scatter?
- Answer: $\psi_i \to \tilde{\psi}_i^*$. This preserves all the symmetries. Right?

What about flavour symmetry?

The theory has an additional $SU(N_f)$ flavour symmetry

fermion	in-mover or out-mover?	$SU(2)_{\text{rot}}$	$U(1)_Q$	$SU(N_f)$
ψ	in	$\lvert\mathbf{n}\rvert$	$+1$	$\mathbf{N_f}$
$\tilde{\psi}$	out	$\lvert\mathbf{n}\rvert$	-1	$\mathbf{N_f}$

that we can *ask* be preserved.

Note that for $n = 1$ and N_f even, the theory UV completes to

$$\begin{aligned} & SU(2) \text{ gauge field} \\ + \quad & N_f \text{ Weyl doublets } \begin{pmatrix} \psi_i \\ \tilde{\psi}_i \end{pmatrix} \\ + \quad & \text{'t Hooft-Polyakov Monopole} \end{aligned}$$

so it *should* be possible to preserve $SU(N_f)$.

Scattering

- Now how do these scatter?
  - If you said $\psi_i \to \tilde{\psi}_i^*$, this breaks $SU(N_f)$ since $\mathbf{N_f} \neq \bar{\mathbf{N}}_\mathbf{f}$.
  - If you said $\psi_i \to \tilde{\psi}_i$, this breaks $U(1)_Q$ since $1 \neq -1$.
- In fact there is no possible final state consistent with both $SU(N_f)$ and $U(1)_Q$.

Reminder of charges

fermion	$U(1)_Q$	$SU(N_f)$
ψ	$+1$	$\mathbf{N_f}$
$\tilde{\psi}$	-1	$\mathbf{N_f}$

Interpretations

- The scattering

$$\psi_{i,m} \to \tilde{\psi}_{i,m} - \frac{2}{nN_f} \sum_{i',m'} \tilde{\psi}_{i',m'}$$

  works. The question is how to interpret that fraction.
- Using fermionisation/bosonisation, we can view the fermions as kinks in 2d free scalar fields $\phi_{i,m}$ and $\tilde{\phi}_{i,m}$. Callan showed that an incoming kink scatters into fractional kinks with the above quantum numbers.
- Working in the full $SU(2)$ gauge theory and performing collective coordinate quantisation on the monopole confirms these quantum numbers.
- So, what's going on?

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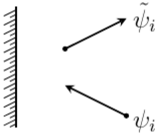
The 3450 Model

Consider a 2d system of two free left-moving complex fermions ψ_i, two free right-moving complex fermions $\tilde{\psi}_i$, and a boundary:



We give the fermions charges

	ψ_1	ψ_2	$\tilde{\psi}_1$	$\tilde{\psi}_2$
$U(1)$	3	4	5	0

chosen so that $3^2 + 4^2 = 5^2$, so the $U(1)$ is non-anomalous.

The 3450 Model

- Because the $U(1)$ is non-anomalous, there exists a symmetry-preserving boundary condition.
- But how do the ψ_i scatter off this boundary?

$$\psi_1 \to ?\,\tilde{\psi}_1 + ?\,\tilde{\psi}_2$$

- Again, there is no possible final state consistent with the $U(1)$ symmetry.
- The 3450 model is therefore a close analogue of the fermion-monopole system.

Solving a Problem by Making it Harder

Let us introduce a second $U(1)'$ global symmetry that we *also* demand be preserved by the boundary:

	ψ_1	ψ_2	$\tilde{\psi}_1$	$\tilde{\psi}_2$
$U(1)$	3	4	5	0
$U(1)'$	4	-3	0	5

It is chosen so that $U(1)'$ is non-anomalous and has no mixed anomaly with $U(1)$.
Such a boundary condition exists, and can be explicitly constructed.

The Scattering

Because we now have as many symmetries are there are fermions, we can now understand the scattering problem more explicitly:

$$\psi_1 \to \frac{3}{5}\tilde{\psi}_1 + \frac{4}{5}\tilde{\psi}_2$$
$$\psi_2 \to \frac{4}{5}\tilde{\psi}_1 - \frac{3}{5}\tilde{\psi}_2$$

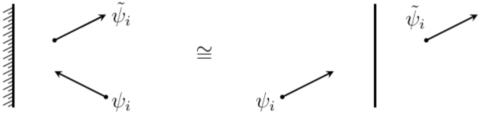
where this is derived using the table

	ψ_1	ψ_2	$\tilde{\psi}_1$	$\tilde{\psi}_2$
$U(1)$	3	4	5	0
$U(1)'$	4	-3	0	5

But how to interpret those fractions?

From Boundaries to Interfaces

- The conformal boundary can be unfolded into a conformal interface:



- Since the unfolded theory is purely right-moving, the interface is topological:

Conformal vs Topological

This is because a conformal interface obeys

$$(L_n - \bar{L}_{-n})\mathcal{I} = \mathcal{I}(L_n - \bar{L}_{-n})$$

where L_n are the Virasoro generators. A topological interface obeys the stronger condition

$$L_n\mathcal{I} = \mathcal{I}L_n$$
$$\bar{L}_n\mathcal{I} = \mathcal{I}\bar{L}_n$$

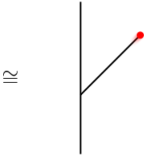
But in a purely right-moving theory, $\bar{L}_n = 0$, so these conditions are equivalent.

Scattering through Topological Interfaces

- As a particle scatters through the interface, we are free to deform the interface around the particle like a lasso:

It's as if the scattering never happened!

- The two lines in the rope of the lasso can be fused together:



This leaves the final state as a twist operator at the end of another topological line.

Computing the Fusion

Let us denote the topological interface by $\mathcal{I}$. Then we can compute the fusion

$$\mathcal{I} \times \mathcal{I}^\vee = \sum_{i=1}^{5} T^i$$

where T is the generator of a $\mathbb{Z}_5$ symmetry that acts as

$$T: \quad \tilde{\psi}_1 \to e^{2\pi i \frac{3}{5}} \tilde{\psi}_1 \qquad \tilde{\psi}_2 \to e^{2\pi i \frac{4}{5}} \tilde{\psi}_2$$

Thus the line attached to the twist operator must be some T^i.

Scattering to Twisted States

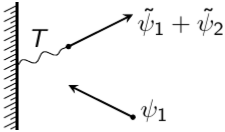
- It turns out that there is always a unique twist vacuum for each T^i. (A twist operator of lowest dimension.)
- Let us *also* denote it by T^i (notation abuse).
- Then the scattering is

$$\psi_1 \to T + \tilde{\psi}_1 + \tilde{\psi}_2$$
$$\psi_2 \to T^3 + \tilde{\psi}_1 + \tilde{\psi}_2^*$$

- The fractional particle number is entirely due to T. Indeed, T adds $\tilde{\psi}_1$-number $\frac{3}{5}$ and $\tilde{\psi}_2$-number $\frac{4}{5}$, mod 1.

Back to Boundaries

- Finally, we can pass back to the boundary picture by folding back up again:



- T is now the generator of the $\mathbb{Z}_5$ symmetry

$$\psi_1 \to \psi_1 \qquad \psi_2 \to \psi_2 \qquad \tilde{\psi}_1 \to e^{2\pi i \frac{3}{5}} \tilde{\psi}_1 \qquad \tilde{\psi}_2 \to e^{2\pi i \frac{4}{5}} \tilde{\psi}_2$$

that acts *only* on the right-movers.

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Fermion-Monopole Scattering in 2d

- Recall that the problem of fermions scattering off a monopole is an effectively 2d problem of fermions scattering off a boundary.
- The fermions have the following quantum numbers that must be preserved:

	$U(1)_Q$	$SU(N_f)$	$SU(2)_{\text{rot}}$
ψ	$+1$	$\mathbf{N_f}$	$\lvert\mathbf{n}\rvert$
$\tilde{\psi}$	-1	$\mathbf{N_f}$	$\lvert\mathbf{n}\rvert$

- We can run very similar arguments as for the 3450 model to understand the scattering in terms of twist operators.

From Fusion

Just like for the 3450 model, we can

- Explicitly determine the conformal boundary condition, which was constructed by Affleck and is known as the *dyon state.*
- Unfold it to a topological interface $\mathcal{I}$.
- Compute the self-fusion $\mathcal{I} \times \mathcal{I}^\vee$.

The result is

$$\mathcal{I} \times \mathcal{I}^\vee = \sum_{i=1}^{nN_f / \gcd(nN_f,2)} T^i$$

where T acts by

$$T: \quad \psi_{i,m} \to \psi_{i,m} \qquad \tilde{\psi}_{i,m} \to e^{2\pi i \frac{-2}{nN_f}} \tilde{\psi}_{i,m}$$

and generates a $\mathbb{Z}_{nN_f / \gcd(nN_f,2)}$ symmetry.

From Fusion

- Then the scattering is

$$\psi_{i,m} \to T + \tilde{\psi}_{i,m}$$

- The line T does not touch the non-abelian transformation properties, but adds $+2$ units of charge.
- This is consistent with our earlier puzzling formula

$$\psi_{i,m} \to \tilde{\psi}_{i,m} - \frac{2}{nN_f} \sum_{i',m'} \tilde{\psi}_{i',m'}$$

The fractional sum is interpretated as the twist operator T.

From an Ansatz

On the other hand, suppose we do *not* know the explicit boundary condition. Can we make progress?

- Assume the line acts on all right-movers by a phase:

$$T: \quad \psi_{i,m} \to \psi_{i,m} \qquad \tilde{\psi}_{i,m} \to e^{i\theta}\tilde{\psi}_{i,m}$$

- T commutes with $SU(N_f)$ and $SU(2)_{\mathrm{rot}}$, so cannot change the non-abelian transformation properties.
- T has a mixed anomaly with $U(1)$, so adds charge

$$\underbrace{nN_f}_{\text{number of right-movers}} \times \underbrace{-1}_{\text{charge of each right-mover}} \times \frac{\theta}{2\pi}$$

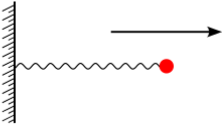
- Choose $\theta = 2\pi\frac{-2}{nN_f}$ so that T adds $+2$ units of charge.

From The Perspective of an Experimenter

Twist operators can always be created even in the absence of a boundary. They are not surprising.

But their endpoints always come in pairs of opposite charge.

The novelty of the monopole boundary condition is that the twist line can end on it *without* the junction carrying charge.



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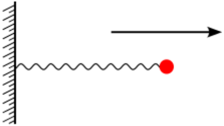
Summary So Far

We have seen that in the 2d reduction of monopole scattering, an incoming ψ scatters to $\tilde{\psi}$ plus a twist line T:



But what does this mean in 4d?

Lifting the Twist Line

The twist line T is characterised by its effect on the chiral lowest partial-wave:

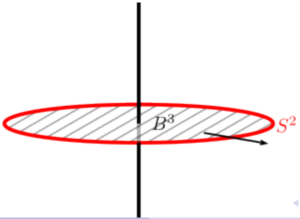
$$\tilde{\psi}_{i,m}(t,r) \to e^{2\pi i \frac{-2}{nN_f}} \tilde{\psi}_{i,m}(t,r) \quad \text{for } r \leq r_0$$

But this does *not* determine its effect on the higher partial waves, and therefore on the 4d fermion.

A natural guess for the effect on the 4d fermion would be

$$\tilde{\psi}_{i}(t,\mathbf{x}) \to e^{2\pi i \frac{-2}{nN_f}} \tilde{\psi}_{i}(t,\mathbf{x}) \quad \text{for } |\mathbf{x}| \leq r_0$$

The analogue of T in 4d is a 3-dimensional surface operator.



Axial Transformations

But does such a surface operator exist?
Recall that gauge transformations act as

$$\psi_i \to e^{i\theta}\psi_i \qquad \tilde{\psi}_i \to e^{-i\theta}\tilde{\psi}_i$$

while axial transformations act as

$$\psi_i \to e^{i\alpha/2}\psi_i \qquad \tilde{\psi}_i \to e^{i\alpha/2}\tilde{\psi}_i$$

Up to a gauge transformation, that is the same as $\tilde{\psi}_i \to e^{i\alpha}\tilde{\psi}_i$.
The $U(1)_A$ axial symmetry has an ABJ anomaly, which breaks it to discrete angles

$$\alpha \in \frac{2\pi}{N_f}\mathbb{Z}$$

Axial Transformations

Allowed:

$$\alpha \in \frac{2\pi}{N_f}\mathbb{Z}$$

Wanted:

$$\alpha = -\frac{2\pi}{N_f}\frac{2}{n}$$

Except for $n = 1, 2$, the symmetry we want is not a symmetry!

Non-invertible ABJ Defects

- In 2205.05086 and 2205.06243, a construction of $U(1)_A$ defects for any rational angle was given.
- The catch is that the new defects are non-invertible.

$$U_A\left(\alpha = \frac{2\pi}{N_f}\frac{p}{q}\right) = \exp\left(i\alpha\int J_A\right)\mathcal{A}^{q,p}[dA/q]$$

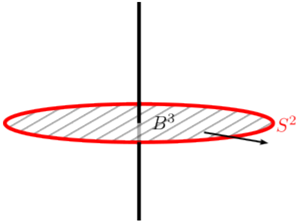
where

- the rational number $\frac{p}{q}$ is in lowest terms
- $\mathcal{A}^{q,p}[B]$ is the minimal TQFT with $\mathbb{Z}_q^{[1]}$ symmetry with anomaly p mod q.

Summary of the Scattering

So, we have argued that the scattering is

$$\psi_i \to \tilde{\psi}_i +$$



where the operator supported on the B^3 is an ABJ defect

$$U_A\left(\alpha = -\frac{2\pi}{N_f}\frac{2}{n}\right)$$

The surface carries charge 2, so charge is conserved: $1 = -1 + 2$.

Relation to $U(1)_m^{[1]}$

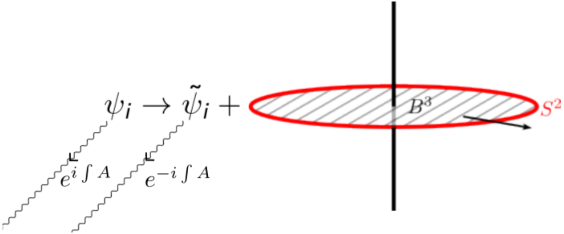
- Suppose our UV theory has dynamical monopoles of charge n, one of which is the monopole we are scattering off.
- Then although $U(1)_m^{[1]}$ is broken, a subgroup $\mathbb{Z}_n^{[1]} \subseteq U(1)_m^{[1]}$ survives.
- This implies the presence of symmetry operators $U_A\left(\alpha = \frac{2\pi}{N_f}\frac{k}{n}\right)$ with topological junctions with the monopole.
- Thus the surface $U_A\left(\alpha = -\frac{2\pi}{N_f}\frac{2}{n}\right)$ which implements monopole-scattering is indeed present in the EFT.

The Charge on the S^2

- How can a *surface* carry gauge charge?
  It has the wrong dimension!
  The gauge symmetry is a 0-form symmetry, so the charged objects are points.
- How, then, can we understand the fact the surface adds gauge charge $+2$?

Dressing with Wilson lines

Recall that when we discuss scattering in gauge theories, all charged particles must be dressed with Wilson lines to make them gauge invariant:



So, the question is: where does a charge $+2$ Wilson line attach to the ABJ defect?

The TQFT on the ABJ defect

In more detail, the TQFT on the ABJ defect is an abelian Chern-Simons theory,

$$S = \int_{B^3} \frac{1}{4\pi} a^T K da + \frac{1}{2\pi} a^T v dA$$

with a certain K matrix and v vector, satisfying $v^T K^{-1} v = -2/n$.

Example 1: When n is even, we have

$$K = -\left(n/2\right) \qquad v = (1)$$

Example 2: When n is odd, we have

$$K = -\begin{pmatrix} \frac{n+1}{2} & 1 \\ 1 & 2 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Monopole Operators

The Chern-Simons action

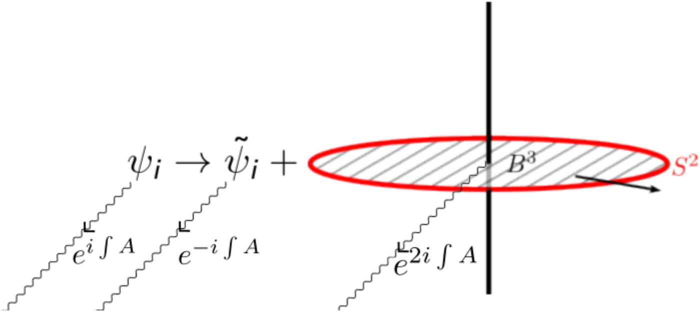
$$S = \int_{B^3} \frac{1}{4\pi} a^T K da + \frac{1}{2\pi} a^T v dA$$

causes monopoles to acquire electric charge.

- We have a background monopole in A of magnetic charge n.
- Its junction with B^3 acquires electric charge nv under a.
- We must add a **monopole operator for** a of magnetic charge $-K^{-1}nv$.
- This cancels the electric charge under a, but adds electric charge $-v^T K^{-1} nv = 2$ under A.

Dressing with Wilson lines (fixed)

Thus, the junction carries electric charge $+2$:



Despite this, we should *still* think of the charge density as being localised at the surface S^2.

Conclusion and Open Questions

Summary: A fermion scatters off a monopole into the twisted sector of an ABJ defect with angle $-\frac{2\pi}{N_f}\frac{2}{n}$.

- Yes, but *what does that mean*?
  What does an experimenter see in their particle detector?
  Is it a local excitation, or a surface?
- What about monopoles in more general chiral gauge theories?
  E.g. $U(1)$ + Weyl fermions of charges $1, 5, -7, -8, 9$.
  Does a suitable 2d boundary even exist?
  If it does, what symmetry could it possibly lift to in 4d?
- In particular, it seems there are sensible UV completions (the $SU(5)$ GUT) where certain monopoles still admit no obvious 2d boundary, and there is apparently no possible 4d symmetry defect. For these cases, monpole-fermion scattering is still mysterious.

The End