Exc surfs in 3-folds & derived symms

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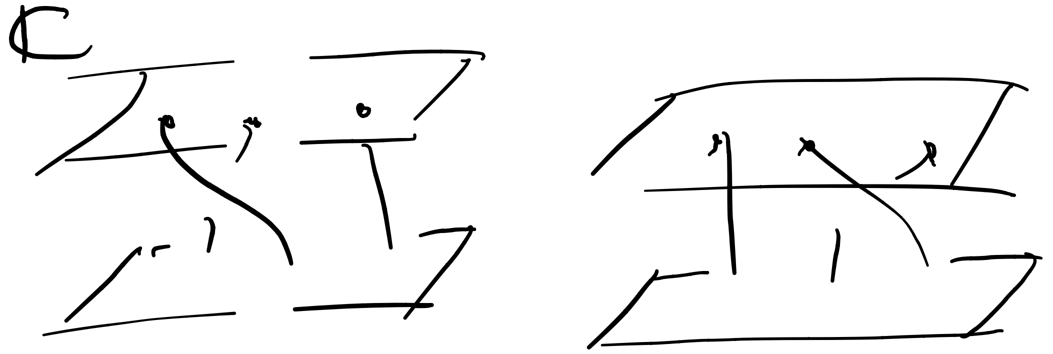
with Luyu Zheng

$$\mathrm{Br}_n = \pi_1\left(\mathbb{C}^n - [x_i = x_j] / S_n\right)$$

eg $n=3$

gens



σ_1 σ_2

rel: $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

X^2

$\downarrow$

$\mathbb{C}^2/\mu_n$

$$Br_{n+1} \xrightarrow{\text{f.ful}} D(X^2)$$

[Seidel-Thomas]

– Type D, E

[Bri, Brav-Thomas]

X^3

$$\pi_1(\mathbb{C}^n - \mathcal{H}) \xrightarrow{\text{f.ful}} D(X^3)$$

[D-Segal, D-Wemyss, Hirano-W]

flopping contraction $\downarrow$

sing

X^3

Q: which $G \to D(X^3)$?

Have X toric resol$^{\underline{n}}$ (crepant)

$\downarrow$

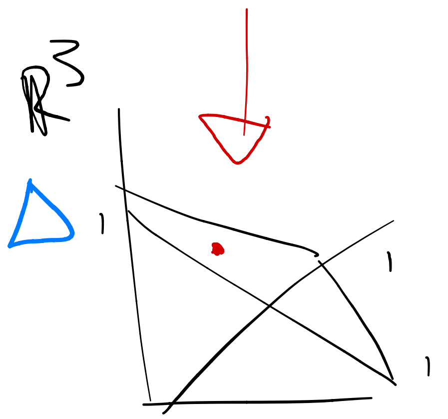
sing $\mathbb{C}^3/\mu_n$

$\mu_n = \{\varepsilon^n = 1\}$

[Nakamura, Reid, Craw, Ito-Nakajima]

ε action: $(\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2}, \varepsilon^{\alpha_3})$

<u>Write</u> $\frac{1}{n}(\alpha_1, \alpha_2, \alpha_3)$

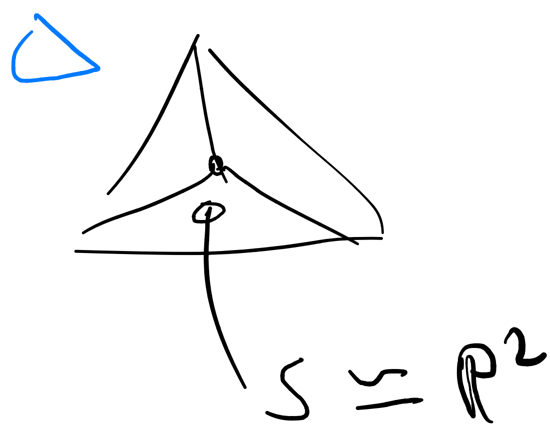


$$\mathrm{Fan}(X) = \mathrm{Cone}\left(\text{triangulat}^{\underline{n}} \text{ of } \Delta\right)$$

eg $\frac{1}{3}(1,1,1)$

$X = K_{\mathbb{P}^2}$

$= \mathcal{O}_{\mathbb{P}^2}(-3)$



Thm [ST] ~ 00 S rat surf $\subset X$ dim 3, K_X triv

Φ

special case

$$\Sigma = \mathcal{O}_S \in \mathrm{Coh}(X)$$

$$\Rightarrow \quad \exists\, T : D(X) \xrightarrow{\sim} D(X)$$
$$\Sigma \mapsto \Sigma[-2]$$

Note $T \cong \mathrm{id}$ on $\Sigma^{\perp}$

Rem Σ "3-spherical"

eg $\Sigma = \mathcal{O}_{\mathbb{P}^2}$ $\quad X = K_{\mathbb{P}^2}$

Thm (cont)

For $\Sigma_1, \underline{\qquad\qquad}, \Sigma_n$ as above with

$$\mathrm{Ext}^1(\Sigma_i, \Sigma_j) = \begin{cases} k & j = i+1 \\ 0 & \text{o'wise} \end{cases}$$

$$\Longrightarrow \quad \begin{aligned} Br_{n+1} &\xrightarrow{\text{f'ful}} D(X) \\ \sigma_i &\longmapsto T(\Sigma_i) \end{aligned}$$

Ext^1-quiver

$$\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \cdots \longrightarrow \overset{n}{\circ}$$

Say A_n-chain

Intersection surfs:

$$\begin{tikzcd}
 & X & \\
S_1 \arrow[ur, hook] & & S_2 \arrow[ul, hook'] \\
 & C \arrow[ul, hook, "\mathcal{O}^k"] \arrow[ur, hook', "\ell"'] &
\end{tikzcd}$$

$\deg N_C \in \mathbb{Z}$

$C \cong \mathbb{P}^1$

$$\mathcal{E}xt^1_X(\mathcal{O}_{S_1}, \mathcal{O}_{S_2}) = \mathcal{O}_{\mathbb{P}^1}(\ell)$$

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^1}(\ell) \\
\mathcal{O}_{S_2}(C) \arrow[u, mapsto, "\mathrm{res}"']
\end{tikzcd}$$

Note ω_X triv $\Rightarrow k + \ell = -2$

$\deg K_{\mathbb{P}^1}$

Prop [ST] surfs $S_i \subset X$

$$\begin{tikzcd}
S_1 & S_2 & S_3 & \cdots & S_n \\
& C_{12} \arrow[ul, "-2"] \arrow[u, "0"] & C_{23} \arrow[ul, "-2"] \arrow[u, "0"] & \cdots &
\end{tikzcd}$$

$$\Longrightarrow \quad Br_{n+1} \xrightarrow{\text{f'ful}} D(X)$$
$$\sigma_i \longmapsto T(\mathcal{O}_{S_i})$$

Pf $\operatorname{Ext}^1(\mathcal{O}_{S_1}, \mathcal{O}_{S_2}) = \Gamma(\mathcal{O}_{C_{12}})$
$= k$

eg S_i S_{i+1}

C

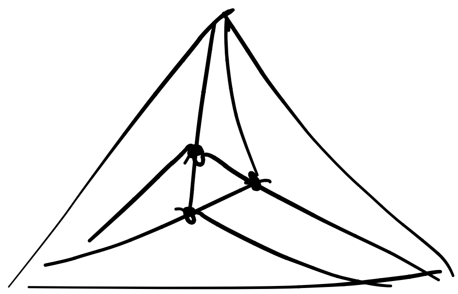
$\rightsquigarrow$ $\{\mathcal{O}_{S_i}\}$ A_n-chain

Wrapped-up chain

eg $\quad \frac{1}{7}(1,2,4)$

$$\frac{1}{7}(1,2,4) \xrightarrow{\times 2} \frac{1}{7}(2,4,1) \xrightarrow{\times 2} \frac{1}{7}(4,1,2)$$

$$\text{mod } 1 \left\{ \begin{array}{l} \frac{1}{7}(2,4,1) \\ \\ \frac{1}{7}(4,1,2) \end{array} \right.$$

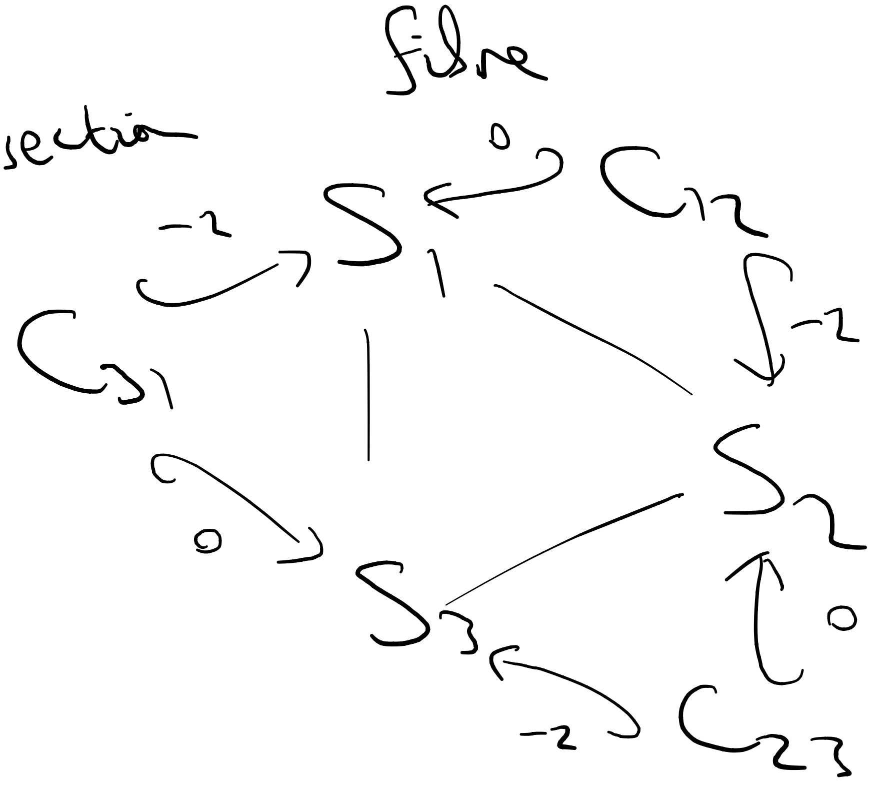


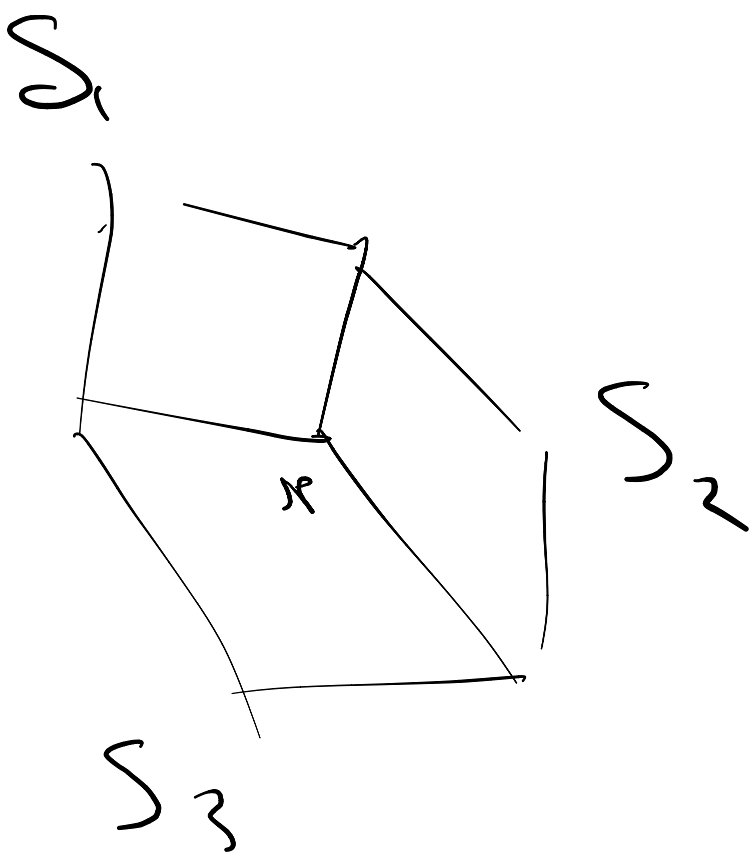
$S_i \simeq \Sigma_2$

__Def__ $\quad \Sigma_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$

$$\downarrow$$

$$\mathbb{P}^1$$





<u>Note</u> $v = \bigcap S_i = \bigcap C_{ij}$

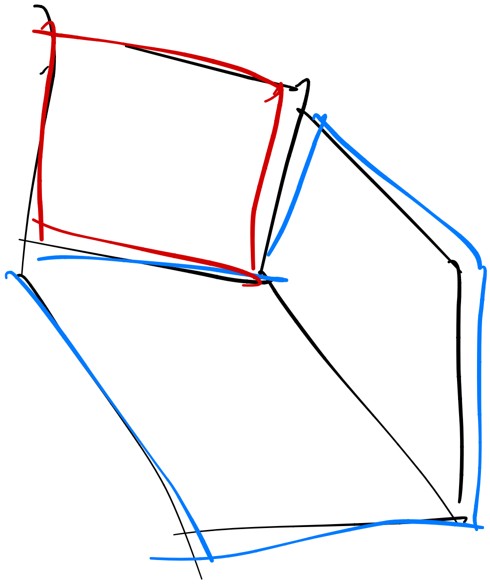
<u>Write</u> $T_i = T(G_{S_i})$

Get $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

But also ...

Claim $T_1^{-1}T_3^{-1}\mathcal{O}_{S_2} = T_2\mathcal{O}_{S_3}$

$$\underbrace{T_1^{-1}T_3^{-1}\mathcal{O}_{S_2}}_{\varphi} = \underbrace{T_2\mathcal{O}_{S_3}}_{\varphi} \simeq F$$



$= \text{unique ext}^n$ of $\mathcal{O}_{S_2}$ by $\mathcal{O}_{S_3}$

$$\Rightarrow \quad \mathcal{O}_{S_2} = \underbrace{T_3T_1T_2}_{\Phi}\mathcal{O}_{S_3}$$

$$\Rightarrow \quad T_2 = \Phi T_3 \Phi^{-1}$$

$$\Rightarrow \quad T_2T_3T_1T_2 = T_3T_1T_2T_3$$

Think rel^n associated to triple intersec^n x

$$G = \left\{ \rho_1, \rho_2, \rho_3 \;\middle|\; \text{braid rel}^{\text{ns}}, \; \rho_1\rho_2\rho_3\rho_1 = \text{cycles} \right\}$$

Thm (mod Claim) [DZ]

$$G \hookrightarrow D(X)$$
$$\rho_i \longmapsto T_i$$

Question f'ful?

(Q, W) quiver w/ potential

$\mathrm{Br}(Q, W)$ braid group

[Grant-Marsh, Qiu]

eg

$$\begin{tikzcd}
 & \bullet \arrow[dr] & \\
\bullet \arrow[ur] & & \bullet \arrow[ll]
\end{tikzcd}$$

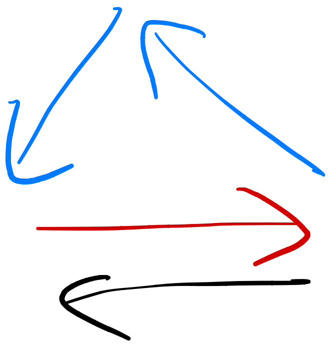
$W = a_1 a_2 a_3 +$ cycles

$\mathrm{Br}(Q, W) = G$

Mutation at vertex

Fomin–Zelev.

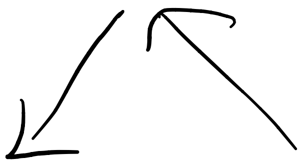
Step 1



reverse

compose

Step 2



remove 2-cycles

$$G \cong Br_4$$

check signs here

$$\left.\begin{aligned} \rho_1 &= \sigma_1 \\ \rho_2 &= \sigma_2 \\ \rho_3^{-1}\rho_2\rho_3 &= \sigma_3 \end{aligned}\right\} Br_4 \text{ gens}$$

So, after breaking symm:

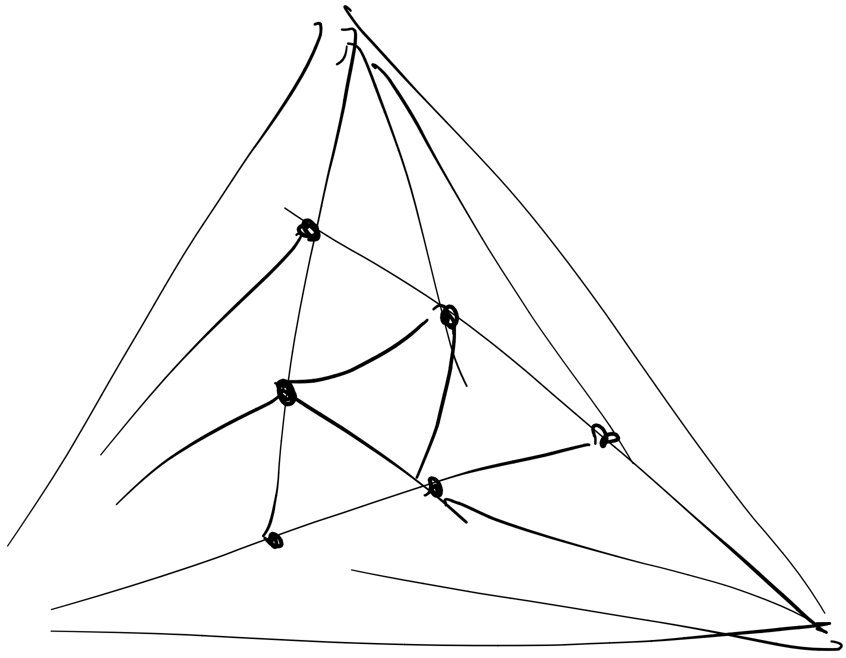
<u>Cor</u> $Br_4 \rightsquigarrow D(X)$

$\sigma_3 \longmapsto T(F)$!

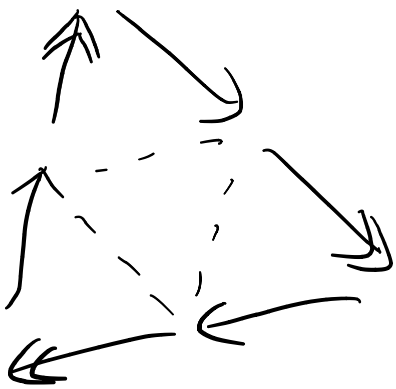
<u>Question</u> f'ful?

Further

$\frac{1}{13}(1,3,9)$



Ext^1-quiver Q



Some $G \curvearrowright D(X)$?

Q: general case?

Free subgps [Keating]