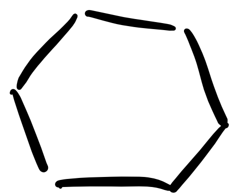
Towards Derived Reid's recipe for dimers.

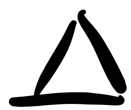
in progress, joint w/ Ali Craw

IPMU, Dec 2023.

Set-up: Consider a Gorenstein toric affine 3-fold $X = \operatorname{Spec} R$

Eg: $\mathbb{C}^3/\Gamma$

$\to$ represent this as  P

here P is .

Goal: Understand the image of a collection of distinguished objects through derived McKay corresp for X & a crepant res'n of X.

A fundamental step towards this is associating a quiver / dimer model living on a real 2-torus.

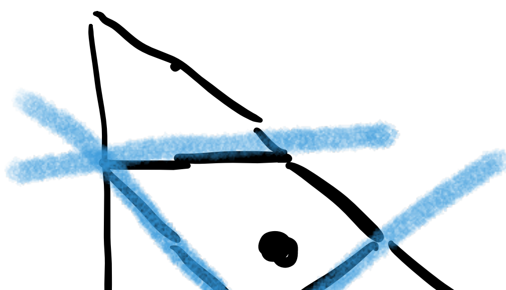
vertex of Quiver $\leftrightarrow$ tile in dimer

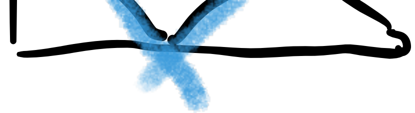
arrow $\longrightarrow$ $\leftrightarrow$ edge in dimer o—●

"elem" cycles in quiver $\leftrightarrow$ white / black nodes in dimer.

Know how to do this for $P = \Delta$ & Q = McKay quiver.

For a general P, ~~Ishii~~-Ueda '15 construct this dimer recursively





IU '08, '15
Broomhead '12

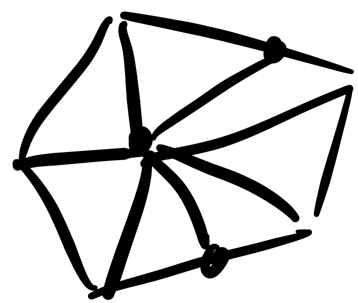
$\exists$ a quiver Q with relations
w/ jacobian algebra

$$A = {}^{\Bbbk Q}\!/\!{}_{\text{relations}}$$

such that $\forall$ choice of a generic stab. condition
θ we obtain a toric crepant resolution

$$\mathcal{M}_\theta(A, v) \longrightarrow X$$

$$v \mapsto (1, 1, \dots, 1)$$



$\exists$ T tautological bundle

line bundles.

$$T = \bigoplus_{i \in Q_0} L_i$$

$\Rightarrow$ the derived equivalence

$$\Psi : D^b(\mathrm{mod}\text{-}A) \longrightarrow D^b(\mathrm{Coh}(\mathcal{M}_\theta(A, v)))$$

$$\Psi = T^\vee \otimes_A -$$

FOR US: Restrict to case where we fix a vertex $0 \in Q_0$ such that

$$\theta = (-d, 1, \_\_ 1).$$

We distinguish the A-modules S_i $\forall\, i \in Q_0$ which are the A-mods associated to the trivial path at vertex i of Q.

"vertex simples".

Compute $\Phi(S_i)$, $i \neq 0$.

is called Derived Reid's Recipe

& is known Cautis - Craw - Logvinenko

for Δ.

In gen we know:

(Bocklandt – Craw – Quintero-Vélez '15).

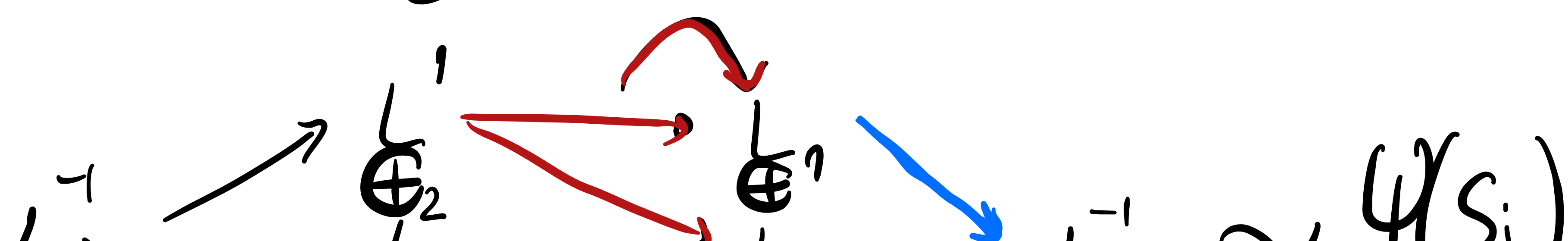
1) $i \neq 0$, $\exists!\, k \in \{0, -1\}$ st

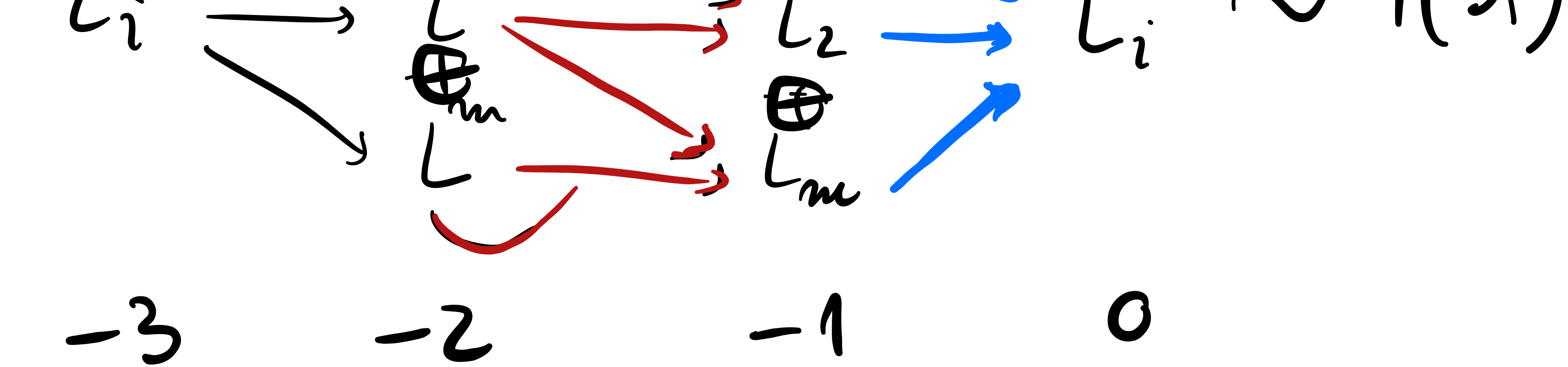
$$H^k(\Psi(S_i)) \neq 0.$$

$k = 0$, $\Psi(S_i) \simeq H^0(\Psi(S_i))$

$k = -1$ $\Psi(S_i) \simeq H^{-1}(\Psi(S_i))[1].$

2) $\Psi(S_i)$ can be computed from a wheel:

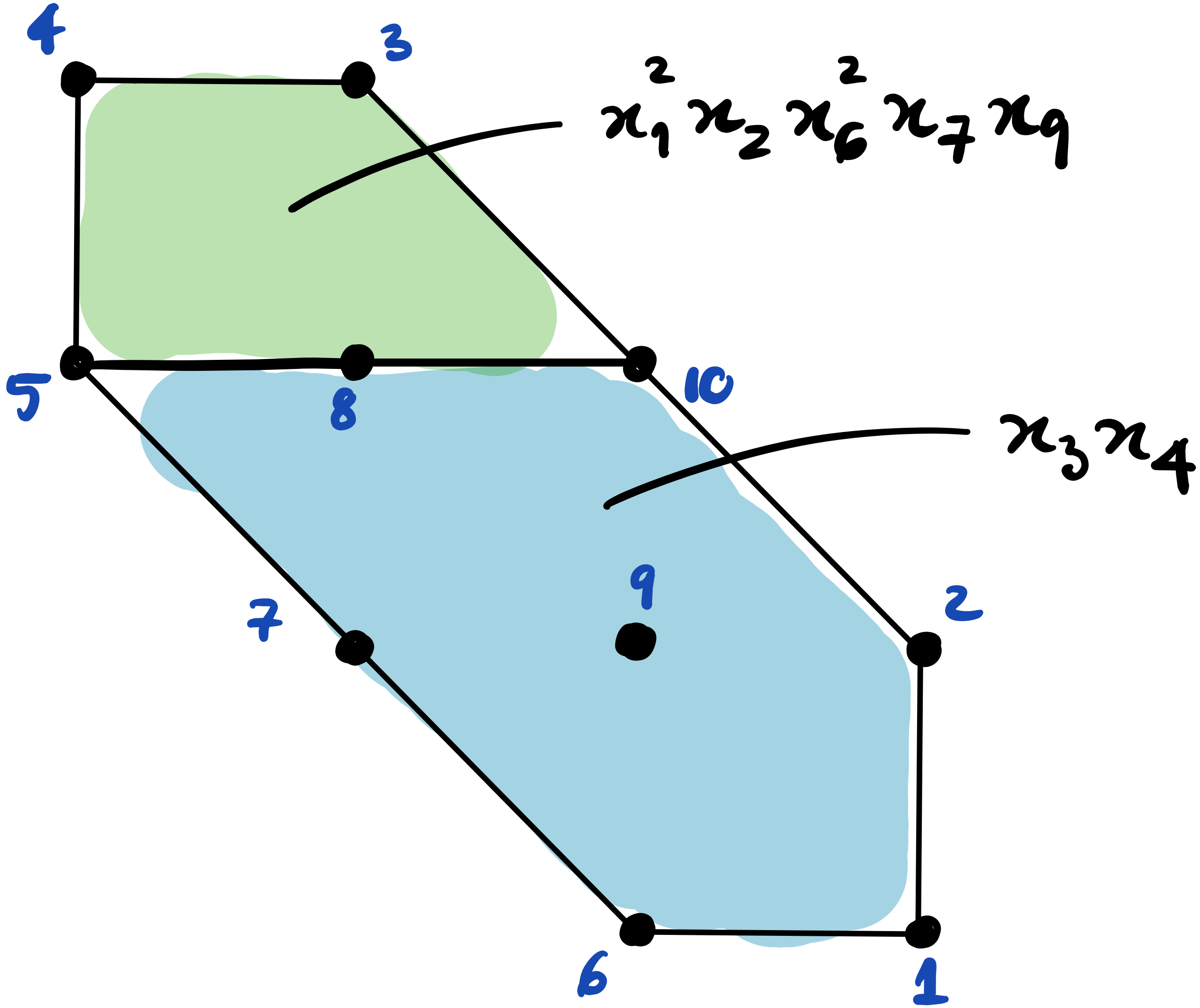




$\mathcal{M}_\theta(A, v)$ $\qquad T = \bigoplus_{i \in Q_0} L_i$

(choice of
$\rightarrow \theta$ ensures L_i are globally generated

Sections of L_6 $\qquad$ induces



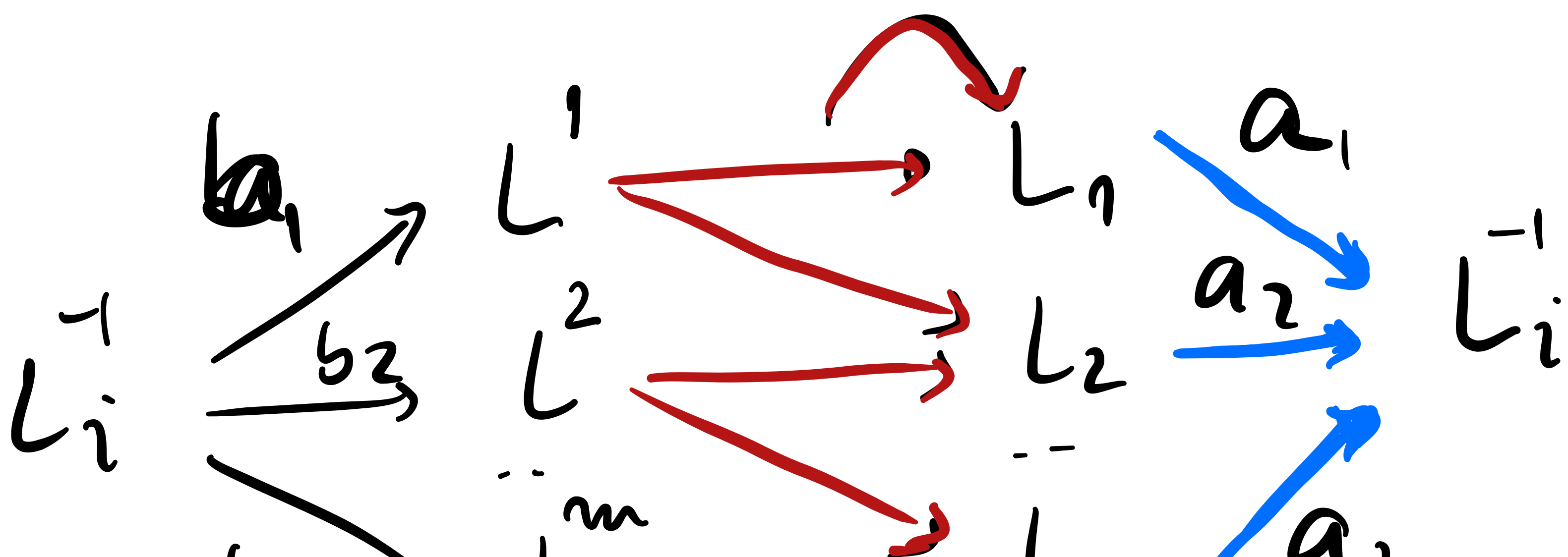
a partition of P

corresp to sections of L_i.

The paths in G can be understood as maps between the L_i

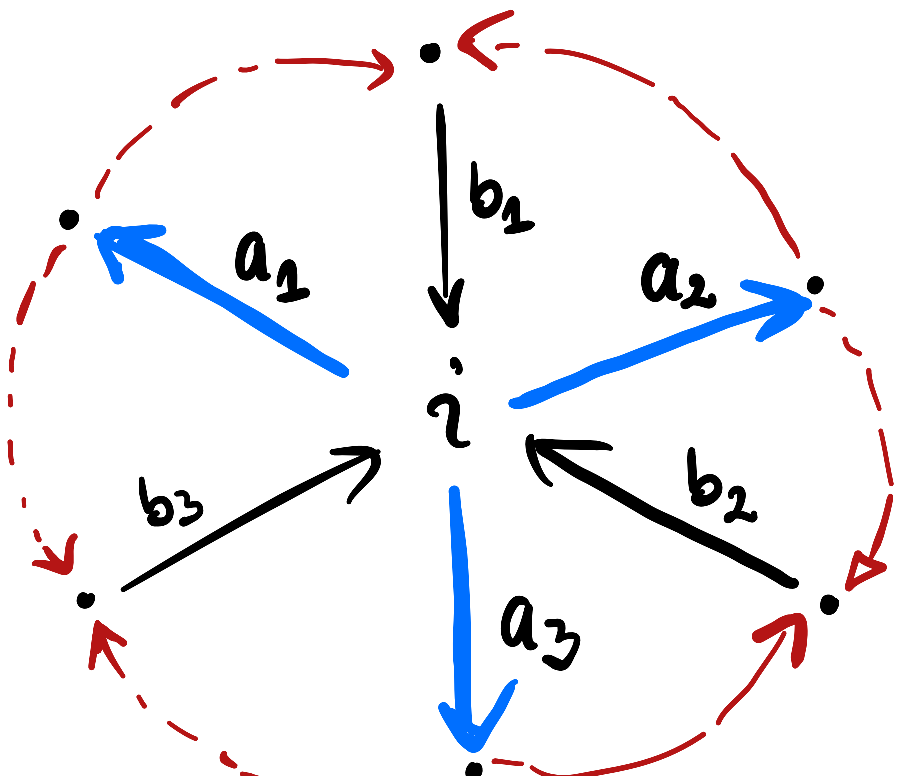
IU'15 $\exists$ an iso

$$A \xrightarrow{\sim} \mathrm{End}(T)_{\mathrm{op}} = \bigoplus_{i,j} \mathrm{Hom}(L_i, L_j)$$



arrows in Q w/ head at L_i

arrows in Q with tail at L_i



The wheel inside the quiver

$\Psi(S_i)$ in terms of L_i

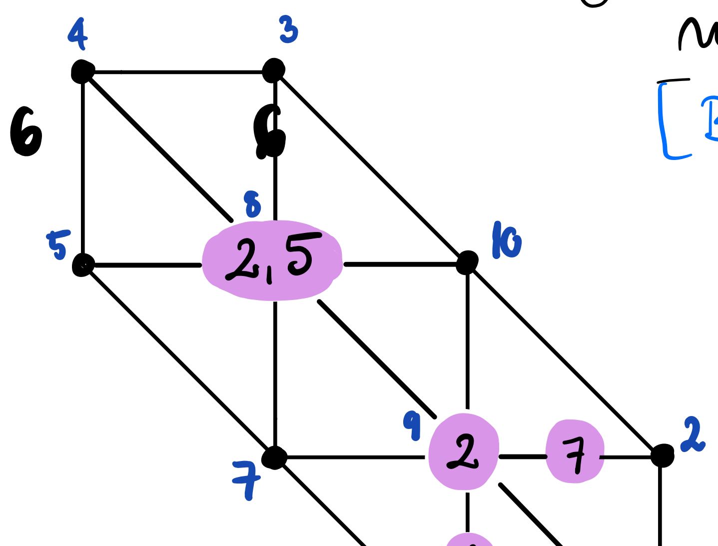
restrictions & twists of Craw-H-Tapia Amador '21

(CCL'12) this is formulated in terms of Combinatorial Reid's

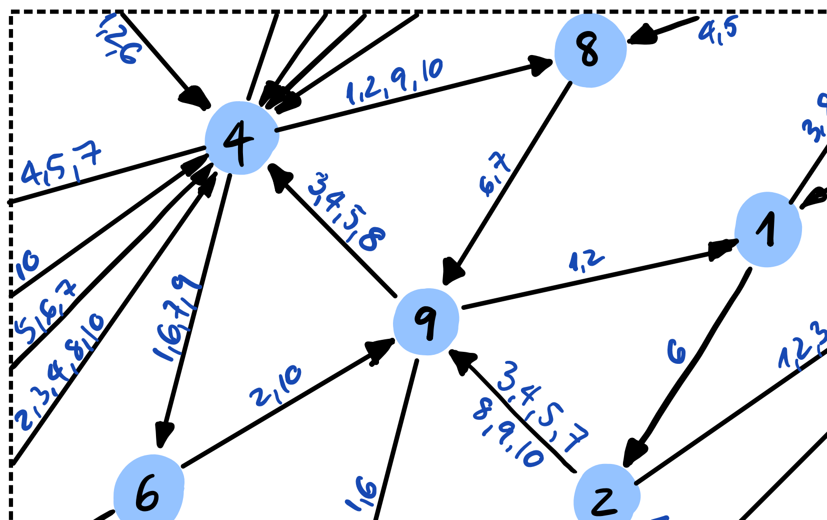
<u>recipe</u>: for each $i \in Q_0$ associates a configuration of connected compact torus-inv. curves / surfaces in the fan of $\mathcal{M}_\theta(A,v)$

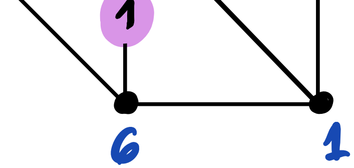
$$\Psi(S_6) = L_6^{-1}(-D_5 - D_{10})|_{D_8}$$

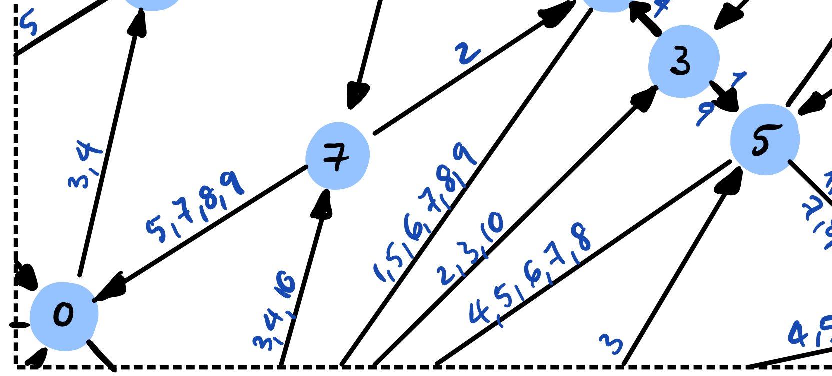
A toric variety with GRR markings [BCQV15]



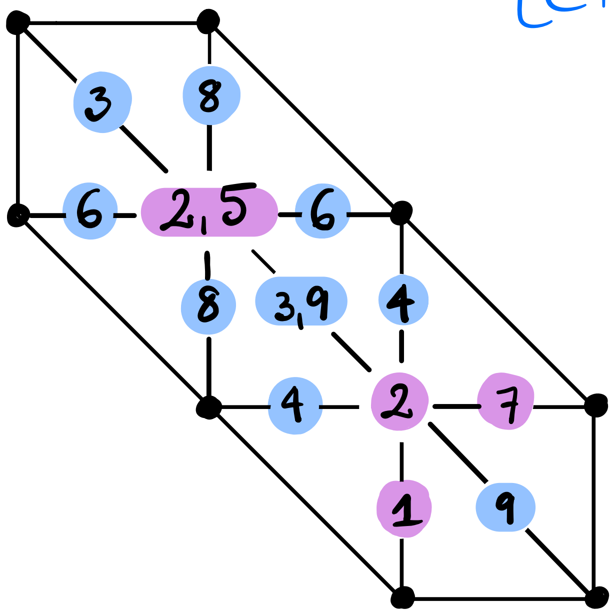
Its associated quiver







The combinatorial markings [CHTA 21]



(BCQV'15) $i \neq 0$

$H^0(\Psi(S_i)) \neq 0$

$$\Psi(S_i) = \begin{cases} L_i^{-1}|_{Z_i} \\ L_i^{-1}|_{C_i} \end{cases}$$

a union of divisors labeled by i in CRR

irred. curve labelled by i

in the Quiver.

↳ Today: $0 \neq i$, i is

$$\begin{tikzcd} {} & & {} \arrow[dl] \\ & i \arrow[ul] \arrow[dr] & \\ {} \arrow[ur] & & {} \end{tikzcd}$$

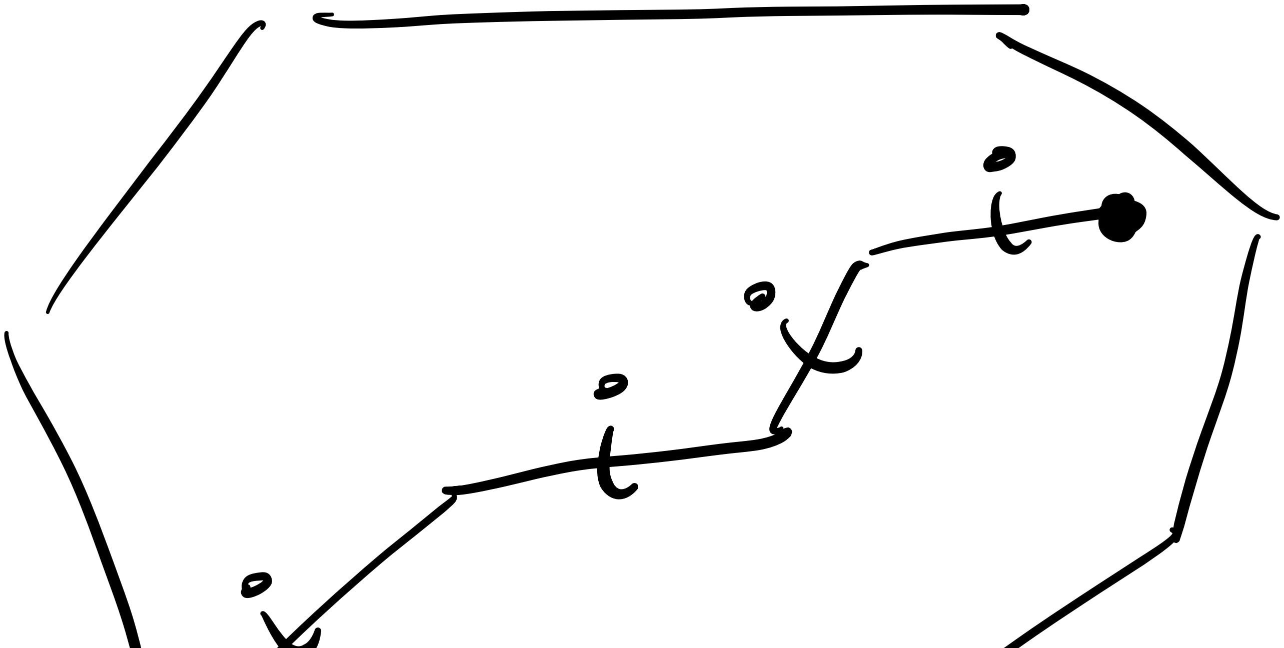
and $H^{-1}(\Psi(S_i)) \neq 0$ (this is the case in BCQV where the support of $\Psi(S_i)$ is a <u>chain</u> of irreducible divisors)

$$\begin{tikzcd} & L^1 \arrow[r] \arrow[r, bend left] \arrow[dr] & L_1 \arrow[dr] & \\ L_i^{-1} \arrow[ur] \arrow[dr] & & & L_i^{-1} \\ & L^2 \arrow[r] & L_2 \arrow[ur] & \end{tikzcd}$$

Thm: $\Psi(S_i) = L_i^{-1}(-F_0 - F_{m+1})\big|_{Z_i}$ where

$Z_i = F_1 \cup \dots \cup F_m$ and

$F_0, F_1, \dots, F_m$ are the divisors in the chain labelled by i.



The new markings in the - arrow case