Non-commutative crepant resolutions of a special family of stable set rings

Koji Matsushita (Osaka University)

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Non-commutative (crepant) resolutions

- R : Cohen-Macaulay normal domain.
- For a reflexive R-module $M \neq 0$, let $E = \mathrm{End}_R(M)$ and we denote the global dimension of E by gldim E.

Definition

- We call E a non-commutative resolution (NCR) of R if gldim $E < \infty$.

In addition, suppose that R is Gorenstein.

- We call E a non-commutative crepant resolution (NCCR) of R if E is an NCR of R and E is a maximal Cohen-Macauly (MCM) R-module.

- The existence of NCCRs (of toric rings) is the one of the most well-studied problems.

Setting

Let

- $\Bbbk$ be an algebraically closed field of characteristic 0.
- $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{Z}_{>0}$.
- $C \subset \mathbb{R}^d$ be a d-dimensional strongly convex rational polyhedral cone.

Definition

We define the toric ring of C over $\Bbbk$ by setting

$$R = \Bbbk[C \cap \mathbb{Z}^d] = \Bbbk[t_1^{\alpha_1} \cdots t_d^{\alpha_d} : (\alpha_1, \cdots, \alpha_d) \in C \cap \mathbb{Z}^d].$$

- R is a d-dimensional Cohen-Macaulay normal domain.

Conic divisorial ideals

Conic divisorial ideals of R are the direct summands of $R^{1/k}$, where $R^{1/k} = \Bbbk[C \cap (1/k\mathbb{Z})^d]$ is regarded as an R-module.
(Smith-Van den Bergh (1997), Bruns-Gubeladze (2003))

Remark

- Since $R^{1/k}$ is an MCM R-module, conic divisorial ideals of R are also MCM R-modules.
- For $k \gg 0$, $\operatorname{gldim} \operatorname{End}_R(R^{1/k}) = d$, and hence $\operatorname{End}_R(R^{1/k})$ is an NCR of R.
(Špenko-Van den Bergh (2017), Faber-Muller-Smith (2019)).
- The endomorphism ring of the direct sum of some conic divisorial ideals of R may be an NCCR.
$\rightarrow$ NCCRs constructed in this way are called toric NCCRs.

Example

It is known that the following toric rings have toric NCCRs:

- Gorenstein toric rings whose divisor class groups are $\mathbb{Z}$.
  (Van den Bergh (2004))
- 3-dimensional Gorenstein toric rings.
  (Broomhead (2012), Ishii-Ueda (2015))
- Segre products of polynomial rings which have the same variables.
  (Higashitani-Nakajima (2019))
- Gorenstein Hibi rings whose divisor class groups are $\mathbb{Z}^2$.
  (Nakajima (2020))
- Gorenstein edge rings of complete multipartite graphs.
  (Higashitani-M. (2022)).

However, there is an example of a Gorenstein toric ring which have no toric NCCRs while toric rings always have NCRs.

Example (Špenko-Van den Bergh (2020))

The toric ring

$$R = \Bbbk[x_1x_3x_5, x_2x_4x_6, x_1x_4^3, x_3x_6^3, x_2^3x_5] \cong \Bbbk[a, b, c, d, e]/(ab^3 - cde)$$

has an NCCR, but has no toric NCCRs.

Therefore, it is interesting to ask when a Gorenstein toric ring has a toric NCCR.

Q. Do stable set rings have toric NCCRs?

If $\mathrm{Cl}(R) \cong \mathbb{Z}^r$, then we can rewrite R as the ring of invariants under the action of $G = \mathrm{Hom}(\mathrm{Cl}(R), \Bbbk^\times) \cong (\Bbbk^\times)^r$ on a polynomial ring as follows:

- $X(G)$; the character group of G (note that $X(G) \cong \mathrm{Cl}(R) \cong \mathbb{Z}^r$).
- $\mathfrak{p}_1, \dots, \mathfrak{p}_n$; the height 1 prime ideals of R.
- $\beta_i \in X(G)$; the weight corresponding to the prime divisor $\mathrm{div}(\mathfrak{p}_i)$.

Theorem

- We consider the action of G on $S = \Bbbk[x_1, \dots, x_n]$, which is the action induced by $g \cdot x_i = \beta_i(g)x_i$ for $g \in G$. Then we have $R \cong S^G$. (Bruns-Gubeladze (2003))
- Each element of $\{a_1\beta_1 + \cdots + a_n\beta_n \in \mathbb{R}^r : a_i \in [0,1)\} \cap \mathbb{Z}^r$ one-to-one corresponds to a conic divisorial ideal of R. (Špenko-Van den Bergh (2017))

Quasi-symmetric toric rings

Definition (Špenko-Van den Bergh (2017))

A toric ring R is quasi-symmetric if for every line $l \subset X(G)_{\mathbb{R}} := X(G) \otimes_{\mathbb{Z}} \mathbb{R}$ passing through the origin, we have $\sum_{\beta_i \in l} \beta_i = 0$.

Remark

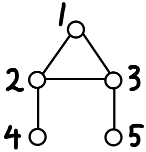
- If R is quasi-symmetric, then R is Gorenstein.
- Quasi-symmetric toric rings have a toric NCCR. (Špenko-Van den Bergh (2017))

Q. When are stable set rings quasi-symmetric?

Stable set rings

For a simple graph G, let

- $V(G) = \{1, \ldots, d\}$ denote the vertex set of G,
- $E(G)$ denote the edge set of G.



Definition

We say that $S \subset V(G)$ is a stable set (resp. a clique) if $\{v, w\} \notin E(G)$ (resp. $\{v, w\} \in E(G)$) for any distinct vertices $v, w \in S$.

Note that the empty set and each singleton are regarded as stable sets.

Definition

We define the stable set ring of G over $\Bbbk$ by setting

$$\Bbbk[\mathrm{Stab}_G] = \Bbbk[\ (\textstyle\prod_{i \in S} t_i)t_0 : S \text{ is a stable set of } G].$$

Perfect graphs

Stable set rings behave well for perfect graphs.

Remark

- A graph G is perfect $\Leftrightarrow$ neither G nor its complement contains an odd cycle of length at least 5 as an induced subgraph. (Chudnovsky-Robertson-Seymour-Thomas (2006))
- Bipartite graphs and their complements are perfect.

If G is perfect,

- $\Bbbk[\mathrm{Stab}_G]$ can be described as the toric ring arising from a rational polyhedral cone.
- $\Bbbk[\mathrm{Stab}_G]$ is Gorenstein if and only if all maximal cliques of G have the same cardinality. (Ohsugi-Hibi (2006))

Characterization of quasi-symmetric stable set rings

Theorem (M.)

Let G be a perfect graph and suppose that $\Bbbk[\mathrm{Stab}_G]$ is not regular but Gorenstein. Then the following are equivalent:

(i) $\Bbbk[\mathrm{Stab}_G]$ is quasi-symmetric;

(ii) G has just 2 maximal cliques;

(iii) $\Bbbk[\mathrm{Stab}_G]$ is isomorphic to the Segre products of two polynomial rings or its polynomial extension.

This class is very limited. Thus, we consider another class of stable set rings, which are similar to quasi-symmetric ones.

The graph $G_{r_1,\dots,r_n}$

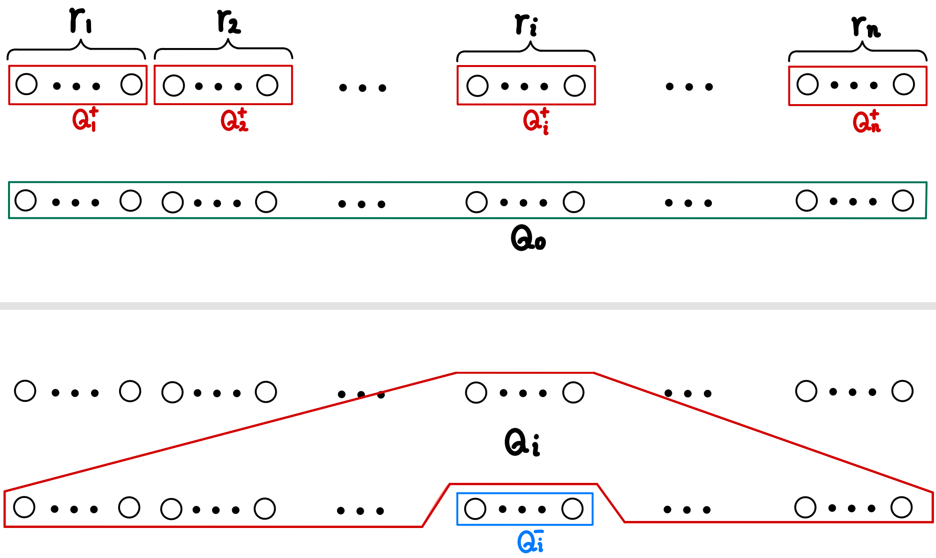
For $n \geq 2$ and $r_1, \dots, r_n \geq 1$, we define the graph $G_{r_1,\dots,r_n}$:

- $V(G_{r_1,\dots,r_n}) = [2d]$, where $d = \sum_{k=1}^{n} r_k$.
- $E(G_{r_1,\dots,r_n}) = \bigcup_{i=0}^{n} \big\{\{v, u\} : v, u \in Q_i\big\}$.
- $Q_0 = \{d + 1, \dots, 2d\}$ and for $i \in [n]$, we let

$$Q_i^+ = \Big\{ \sum_{k=1}^{i-1} r_k + 1, \dots, \sum_{k=1}^{i} r_k \Big\},\ Q_i^- = \Big\{ d + \sum_{k=1}^{i-1} r_k + 1, \dots, d + \sum_{k=1}^{i} r_k \Big\}$$

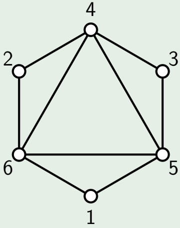
and $Q_i = Q_i^+ \cup (Q_0 \setminus Q_i^-)$.

Note that $Q_i^+ = Q_i \setminus Q_0$ and $Q_i^- = Q_0 \setminus Q_i$.

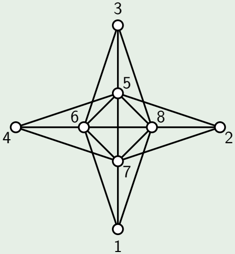

r_1
r_2
r_i
r_n
Q_1^+
Q_2^+
Q_i^+
Q_n^+
Q_0
Q_i
Q_i^-

Example

The graph $G_{1,1}$



The graph $G_{1,1,1}$



The graph $G_{1,1,1,1}$

The graph $G_{r_1,\dots,r_n}$ has the following properties.

Proposition (M.)

(i) The graph $G_{r_1,\dots,r_n}$ is perfect.

(ii) The stable set ring $\mathbb{k}[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ is Gorenstein, and its divisor class group is isomorphic to $\mathbb{Z}^n$.

(iii) Each point in $\widetilde{\mathcal{L}} := \{(z_1, \cdots, z_n) \in \mathbb{Z}^n : -r_i \leq z_i \leq r_i \text{ for } i \in [n]\}$ one-to-one corresponds to the conic divisorial ideal of $\mathbb{k}[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$.

- $\mathbb{k}[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ has the following weights:

$$\mathbf{e}_i \times (r_i + 1) \qquad \text{for } i \in [n],$$

$$-\mathbf{e}_i \times r_i \qquad \text{for } i \in [n],$$

$$-\mathbf{e}_1 - \mathbf{e}_2 - \cdots - \mathbf{e}_n$$

- For $\chi \in \mathbb{Z}^n \cong \mathrm{Cl}(\Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}])$, let M_χ be the divisorial ideal corresponding to χ.

Theorem (M.)

Let $R = \Bbbk[\mathrm{Stab}_{G_{r_1,\dots,r_n}}]$ and

$$\mathcal{L} = \{(z_1, \cdots, z_n) \in \mathbb{Z}^n : 0 \leq z_i \leq r_i \text{ for } i \in [n]\}$$

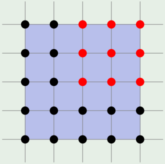
Moreover, let $M_{\mathcal{L}} := \bigoplus_{\chi \in \mathcal{L}} M_\chi$. Then, $E = \mathrm{End}_R(M_{\mathcal{L}})$ is an NCCR of R. In particular, R has a toric NCCR.

Example

Let $n = 2$ and $r_1 = r_2 = 2$. Then we have

$$\widetilde{\mathcal{L}} = \{(z_1, z_2) \in \mathbb{Z}^2 : -2 \leq z_i \leq 2 \text{ for } i \in \{1, 2\}\}$$

$$\mathcal{L} = \{(z_1, z_2) \in \mathbb{Z}^2 : 0 \leq z_i \leq 2 \text{ for } i \in \{1, 2\}\}$$



To prove the theorem...

- gldim $E < \infty$. $\leftarrow$ difficult...

  $\rightsquigarrow$ consider the functor $(-)^G : \mathbf{ref}(G, S) \to \mathbf{ref}(R)$. Then there exists $\Lambda \in \mathbf{ref}(G, S)$ such that $\Lambda^G = E$ (this Λ is the endomorphism ring of a reflexive S-module).

  $\rightsquigarrow$ we can show that gldim $\Lambda < \infty$ by using combinatorial techiques, implying that gldim $E < \infty$.

- E is an MCM. $\leftarrow$ easy!

  $\rightsquigarrow$ $E = \mathrm{End}_R(M_{\mathcal{L}}) \cong \bigoplus_{\chi, \chi' \in \mathcal{L}} M_{\chi - \chi'}$.

  $\rightsquigarrow$ $\chi - \chi' \in \widetilde{\mathcal{L}}$, and hence $M_{\chi - \chi'}$ is a conic divisorial ideal.

Problem

- Does any Gorenstein stable set ring have a toric NCCR?
- Find sufficient conditions for a toric ring to have toric NCCRs.