Projective crepant resolutions of toric cDV singularities and dimer combinatorics

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§1. Introduction

Thm [Craw-Ishii, Yamagishi]

$G \overset{\text{finite}}{\subset} SL(3,\mathbb{C})$

Any projective crepant resolution of $\mathbb{C}^3/G$ can be described as

$\mathcal{M}_\theta(Q_G, d\mathbb{1})$ for some stability parameter θ

The moduli space of θ-stable rep's of the Mckay quiver Q_G

Thm [Ishii-Ueda]

Any projective crepant resolution of a 3-dim. Gor. toric singularity can be described as $\mathcal{M}_\theta(Q, \mathbb{1})$ for some stability parameter θ

the quiver ass. with a consistent dimer model.

- The space $\Theta(Q)$ of stability parameters has "the wall-and-chamber structure".

  - $\theta, \theta' \in C$: chamber $\Rightarrow \mathcal{M}_\theta \cong \mathcal{M}_{\theta'}$ (write $\mathcal{M}_C$)

  - Some walls induce flops :

$C \mid C'$

$\mathcal{M}_C \mid \mathcal{M}_{C'}$

flop

- It is difficult to describe the whole wall-and-chamber str. in general.

Today

$R_{a,b} := \mathbb{C}[x,y,z,w]/(xy - z^a w^b)$

where $a \geq 1$, $a \geq b \geq 0$.

toric cDV sing. of type A_{n-1} $(n = a+b)$

$Q_{a,b}$: the dimer quiver Ass. to $R_{a,b}$

⇝ Describe the wall-and-chamber str. of $\Theta(Q_{a,b})$ using the dimer combinatorics.

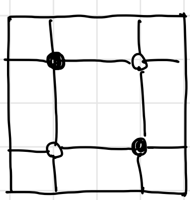
Note: Homological MMP (M. Wemyss) provides a method to describe the wall-and-chamber str. for any cDV sing.
But, my method uses the combinatorics of dimer models.

§2. Dimer models

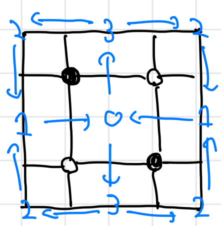
· A dimer model (brane tiling) Γ is a finite bipartite graph described on the real 2-torus T, which induces a polygonal cell decomposition of T.

Assume Γ is "consistent"

e.g.



dimer model Γ



quiver $Q = Q_\Gamma$

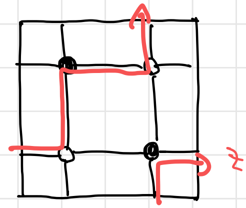
(with relations)

- A <u>zigzag path</u> is a path on Γ s.t.
  it <u>makes</u> a <u>maximum</u> turn to the left on $\circ$
  to the right on $\bullet$

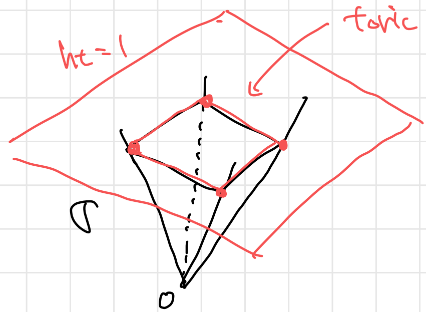
z : zigzag path $\rightsquigarrow$ $[z] \in H_1(T) \cong \mathbb{Z}^2$

the slope of z

e.g.



- $R = \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^3]$
  3-dim. Gor. toric ring.



<u>Thm</u> [Gulotta, Ishii-Ueda, etc]

not unique

For any 3-dim. Gor. toric ring R, $\exists$ consistent DM Γ s.t.

$$\{\text{outer normal vectors of primitive line segments of } \Delta(R)\}$$

$$\updownarrow 1:1$$

$$\{\text{slopes of zigzag paths of } \Gamma\}$$

- Γ : consistent DM ass. to R $\xleftrightarrow{\text{dual}}$ Q

Consider rep's of Q with dim. vector $\mathbb{1} = (1, \cdots, 1)$

Let $\Theta(Q) := \left\{ \theta = (\theta_i)_{i\in Q_0} \in \mathbb{R}^{\# Q_0} \;\middle|\; \sum_{i\in Q_0} \theta_i = 0 \right\}$.

King's stability condition induces the wall-and-chamber str. of $\Theta(Q_{a,b})$.

Thm [IU]

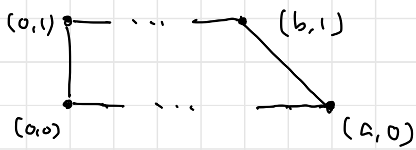
For any chamber C, $\mathcal{M}_C$ is a proj. crepant resol. of Spec R.

§3. Toric cDV singularities

$R_{a,b} := \mathbb{C}[x,y,z,w]/(xy - z^a w^b)$, $\quad n := a+b$

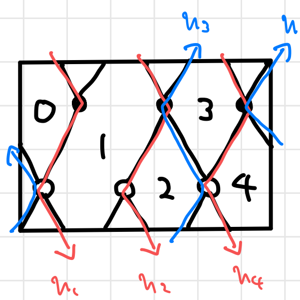
$\rightsquigarrow \exists\ \Gamma_{a,b}$: consistent DM (not unique, choose one of them)

$Q_{a,b}$: the quiver dual to $\Gamma_{a,b}$

$\Delta(R_{a,b}) =$ 

- $\mathcal{U}_1, \cdots, \mathcal{U}_n$: zigzag paths of $\Gamma_{a,b}$ s.t. $[\mathcal{U}_i] = (0,-1)$ or $(0,1)$

e.g. $a=3$, $b=2$



Thm [N.]

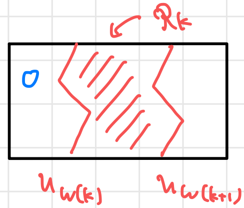
Seq. of zigzag paths

$$\{\text{chambers of } \Theta(Q_{a,b})\} \overset{1:1}{\longleftrightarrow} \{ Z_w := (u_{w(1)}, \cdots, u_{w(n)}) \mid w \in S_n \}$$

Satisfying the following :

(1) If $C_w \subset \Theta(Q_{a,b})$ corresponds to Z_w, then the walls of C_w are given by

$W_k : \sum_{i \in R_k} \theta_i = 0 \quad (k = 1, \cdots, n-1)$ where



(2) A wall-crossing of W_k $\longleftrightarrow$ the action of $(k, k+1) \in S_n$ on Z_w

$\Rightarrow$ The chambers in $\Theta(Q_{a,b})$ are identified with the Weyl chamber of type A_{n-1}

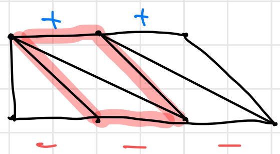
(3) The triangulation of $\Delta(R_{a,b})$ corresponding to $\mathcal{M}_{C_w}$ is given by the "sign" of Z_w.

e.g. $a = 3$, $b = 2$.

C_w

$$\underset{-}{(u_1},\ \underset{-}{u_2},\ \underset{+}{u_5},\ \underset{-}{u_4},\ \underset{+}{u_3})$$

$\rightsquigarrow$



$\cong \mathcal{M}_{C_w}$

flopping wall

$\theta_2 + \theta_3 + \theta_4 = 0$

Thus, W_k induces a flop if $[u_{w(k)}] \neq [u_{w(k+1)}]$

: a divisor-to-curve contraction if $[u_{w(k)}] = [u_{w(k+1)}]$.