Explicit examples of trihedral G-Hilbert schemes

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Classification of finite subgroups $G \subset SL(3,\mathbb{C})$

[Yau-Yu]

- (A) Abelian A
- (B) Isomorphic to Non-Abelian $\bar{G} \subset GL(2,\mathbb{C})$
- (C) $C = A \rtimes \mathbb{Z}_3$ with $\mathbb{Z}_3 \simeq \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$ ←
- (D) $G = C \rtimes \mathbb{Z}_2$ with $\mathbb{Z}_2 \simeq \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$
- + 6 exceptional subgroups of orders 60, 108, 168, 216, 648 and 1080.

$$\boxed{G\text{-Hilb}\,\mathbb{C}^3} = \text{Moduli space of } \underline{G\text{-clusters}} : \left[\begin{array}{c} Z \subset \mathbb{C}^3 \text{ 0-dim}, \\ \mathbb{C}[x,y,z]/I_Z \simeq \underbrace{\mathbb{C}[G]}_{\text{Regular rep.}} = \bigoplus \rho_i^{\dim \rho_i} \end{array} \right] \simeq \boxed{\mathcal{M}_\theta(Q,R)}$$

"scheme theoretical orbits"

Moduli of (θ-gen) θ-stable reps. of the McKay quiver

Universal

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p"'] \arrow[dr] & \\ G\text{-Hilb}\,\mathbb{C}^3 \arrow[d, equal] \arrow[dr] & & \mathbb{C}^3 \arrow[dl] \\ Y & \mathbb{C}^3/G & \end{tikzcd}$$

$$p_* \mathcal{O}_{\mathcal{Z}} = \bigoplus_{\rho \in \mathrm{Irr}\,G} R_k \otimes \rho_k$$

$\underline{R_k \in \mathrm{Pic}(Y)}$

Tautological bundles on $G\text{-Hilb}\,\mathbb{C}^3$

Question: Explicit description of G-Hilb?

Except Abelian case A very few cases are known

⇝ trihedral cases $A \rtimes \mathbb{Z}/3$?

Trihedral groups in $SL(3,\mathbb{C})$

Groups of the form $G = A \rtimes \mathbb{Z}_3$ where A is a diagonal Abelian subgroup and $\mathbb{Z}_3$ is generated by :

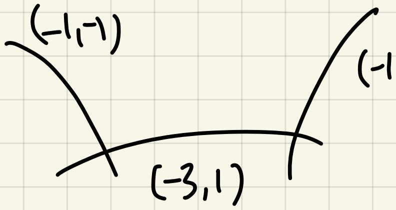
$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{tikzcd} x \arrow[r, bend left] & y \arrow[d, bend left] \\ & z \arrow[ul, bend left] \end{tikzcd}$$

Either $A = \frac{1}{r}(1, s, \overline{s^2}) := \left\langle \begin{pmatrix} \varepsilon & & \\ & \varepsilon^s & \\ & & \varepsilon^{\overline{s^2}} \end{pmatrix} \;\middle|\; \begin{matrix} \varepsilon \text{ prim. } r\text{-th root of } 1 \\ \overline{s^2} \equiv s^2 \pmod r \end{matrix} \right\rangle$ with $r \mid 1+s+s^2$

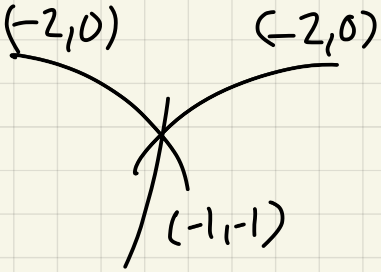
or $A = \mathbb{Z}/r \times \mathbb{Z}/R$ with $r \mid R$.

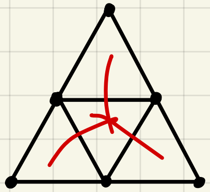
Example. G = Trihedral group of order 12, $A \cong \mathbb{Z}/2 \times \mathbb{Z}/2 := \langle \frac{1}{2}(1,1,0), \frac{1}{2}(1,0,1) \rangle$

Excl. locus $\pi^{-1}(0)$ is 1-dim



$\rightsquigarrow$ 3 flops





A-Hilb

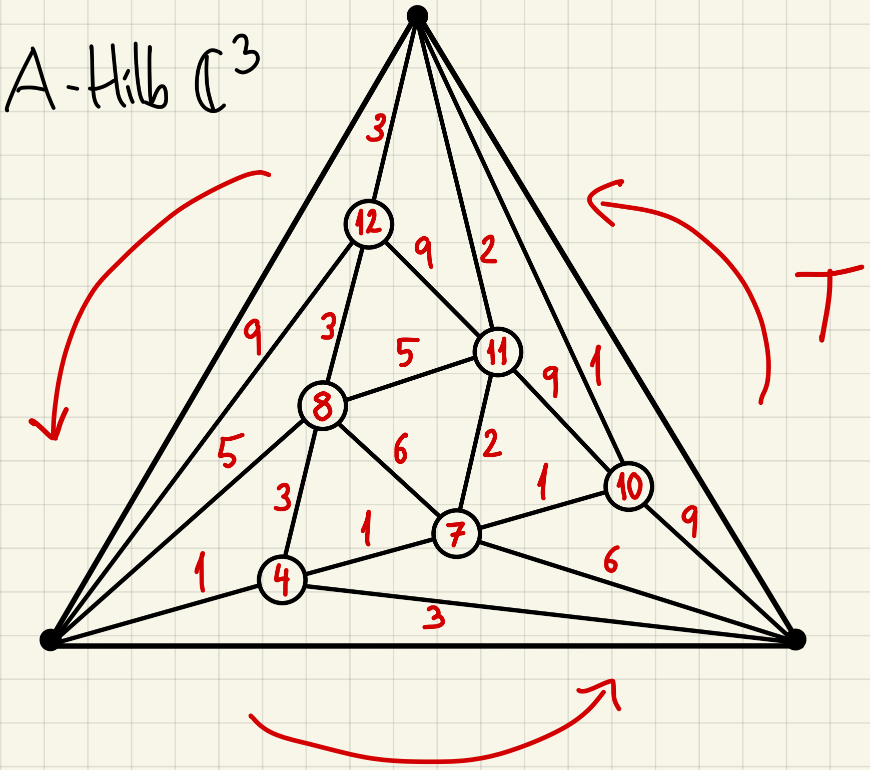
[Ito] [N-Sekiya]

$G\text{-Hilb}\,\mathbb{C}^3 \supseteq \bigcup_{i=1}^{4} \mathbb{C}^3$

$T\text{-Hilb}(A\text{-Hilb}\,\mathbb{C}^3) \supseteq \bigcup_{i=1}^{4} \mathbb{C}^3$

Example. G : Trihedral group of order 39 : $G = \langle \underset{A}{\underset{\shortparallel}{\tfrac{1}{13}(1,3,9)}} , \underset{T \cong \mathbb{Z}/3}{\underset{\shortparallel}{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}}} \rangle \subset SL(3,\mathbb{C})$

① A-Hilb $\mathbb{C}^3$



- Taut. bundles R_i generate $\mathrm{Pic}(A\text{-Hilb}\,\mathbb{C}^3)$ subject to

relations:

$R_1 \otimes R_3 = R_4$ $\quad$ $R_2 \otimes R_9 = R_{11}$

$R_1 \otimes R_6 = R_7$ $\quad$ $R_3 \otimes R_9 = R_{12}$

$R_1 \otimes R_9 = R_{10}$ $\quad$ $R_3 \otimes R_5 = R_8$

Reid's recipe

- $\left\{ \begin{matrix} \text{Irreducible reps.} \\ \text{of } G \end{matrix} \right\} \xleftrightarrow{\text{1-to-1}}$ basis of $H^*(Y,\mathbb{Z})$

(Integral McKay Correspondence)

② $\boxed{T \circlearrowright \mathrm{Irr}\,A}$:

$$\begin{tikzcd}[column sep=small]
 & & & \rho_0 \arrow[loop above] \arrow[ddl] \arrow[dd] \arrow[ddr] & & \rho_1 \arrow[rr] & & \rho_3 \arrow[dl] & \rho_2 \arrow[rr] & & \rho_6 \arrow[dl] & \rho_4 \arrow[rr] & & \rho_{12} \arrow[dl] & \rho_7 \arrow[rr] & & \rho_8 \arrow[dl] \\
 & & & & & & \rho_9 \arrow[ul] \arrow[d] & & & \rho_5 \arrow[ul] \arrow[d] & & & \rho_{10} \arrow[ul] \arrow[d] & & & \rho_{11} \arrow[ul] \arrow[d] & \\
G = A \rtimes T, & \mathrm{Irr}\,G : & V_0 & V_{0'} & V_{0''} & & V(1,3,9) & & & V(2,6,5) & & & V(4,12,10) & & & V(7,8,11) & \\
 & & & \text{1 dim} & & & & & & & \text{3 dim} & & & & & &
\end{tikzcd}$$

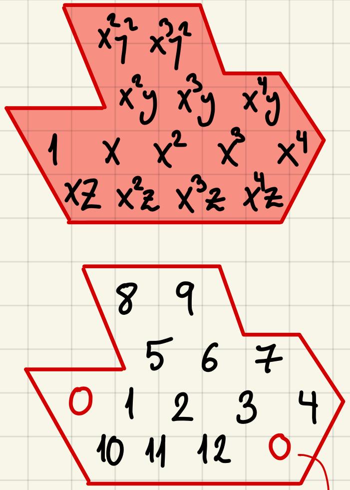
G-graphs are $\mathbb{Z}/_3$-symmetric by the action of T: $$\begin{tikzcd} x \arrow[rr, bend left] & & y \arrow[dl, bend left] \\ & z \arrow[ul, bend left] & \end{tikzcd}$$ They are nicely drawn on the trihedral plane $\mathbb{R}^3/(1,1,1)$ using "boats":

(xyz is G-inv.)

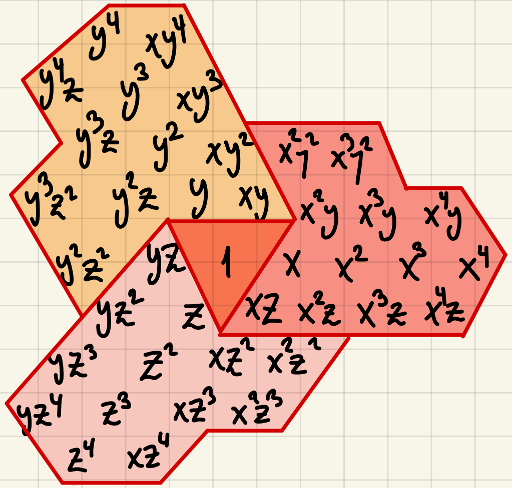
For example, the boat $\mathcal{B}$ represents:

$1 = \dim \rho_i$ element in each $\rho_i \in \operatorname{Irr} A$

$\mathcal{B}$



$\rightsquigarrow$



$\dim V_i$ elements in each $V_i \in \operatorname{Irr} G$

$$\begin{tikzcd} & V_0 \ni 1 \\ \rho_0 \arrow[ur] \arrow[r] \arrow[dr] & \boxed{V_{0'}} \ni x^4y + \omega^2 yz^4 + \omega x^4 z \\ & \boxed{V_{0''}} \ni xy^4 + \omega yz^4 + \omega^2 x^4 z \end{tikzcd}$$

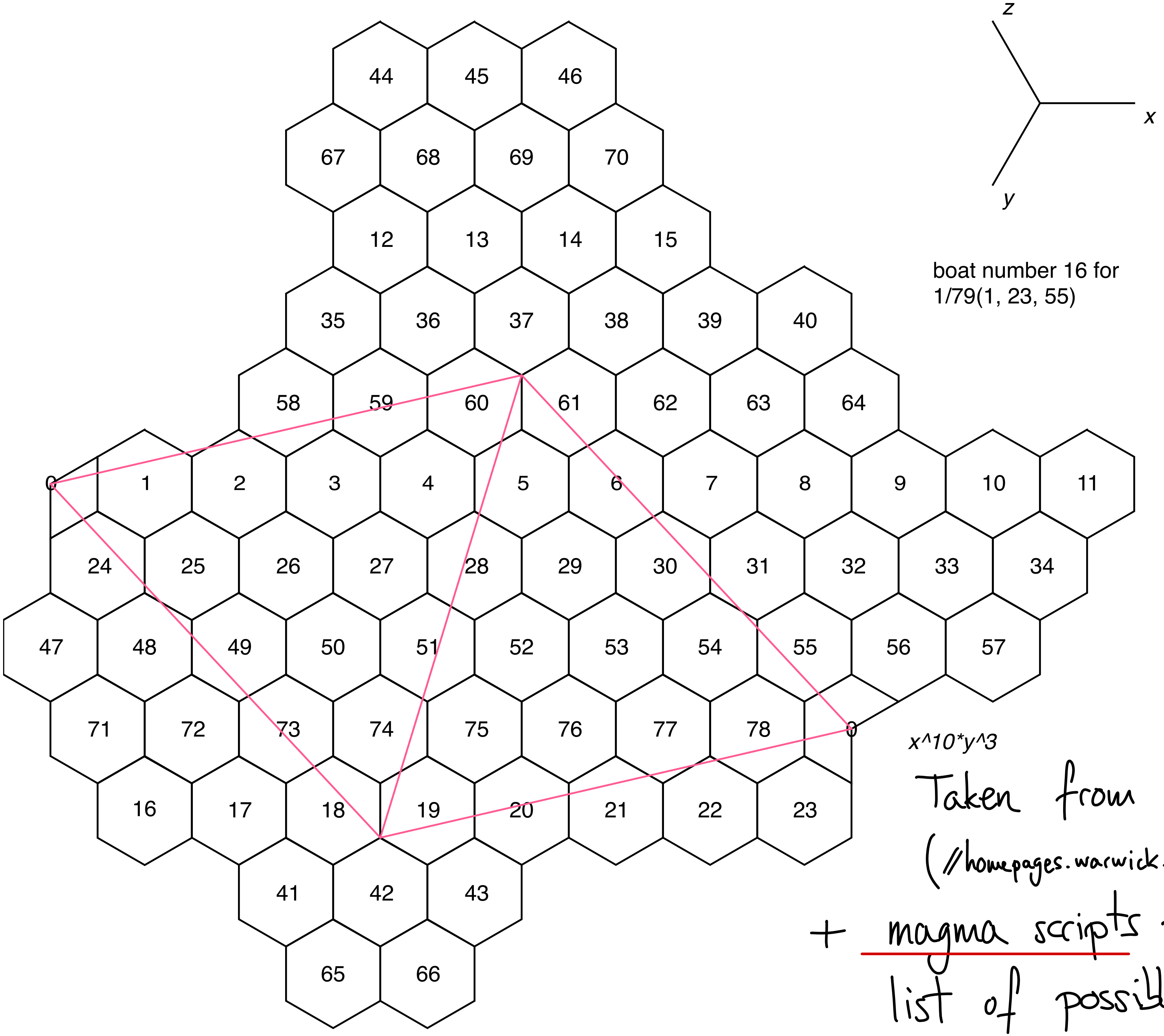
- $\mathcal{B}$ tesselates the plane
  (better with hexagons!)

- $$\left(\overset{\Uparrow}{\underset{2}{}}\,x^2, y^2, z^2\right) \in V(2,6,5)$$

  $$(x^2, y^2, z^2) \in V(2,6,5)$$
  $$\underset{2}{\Uparrow}\quad \underset{6}{\Uparrow}\quad \underset{5}{\Uparrow}$$

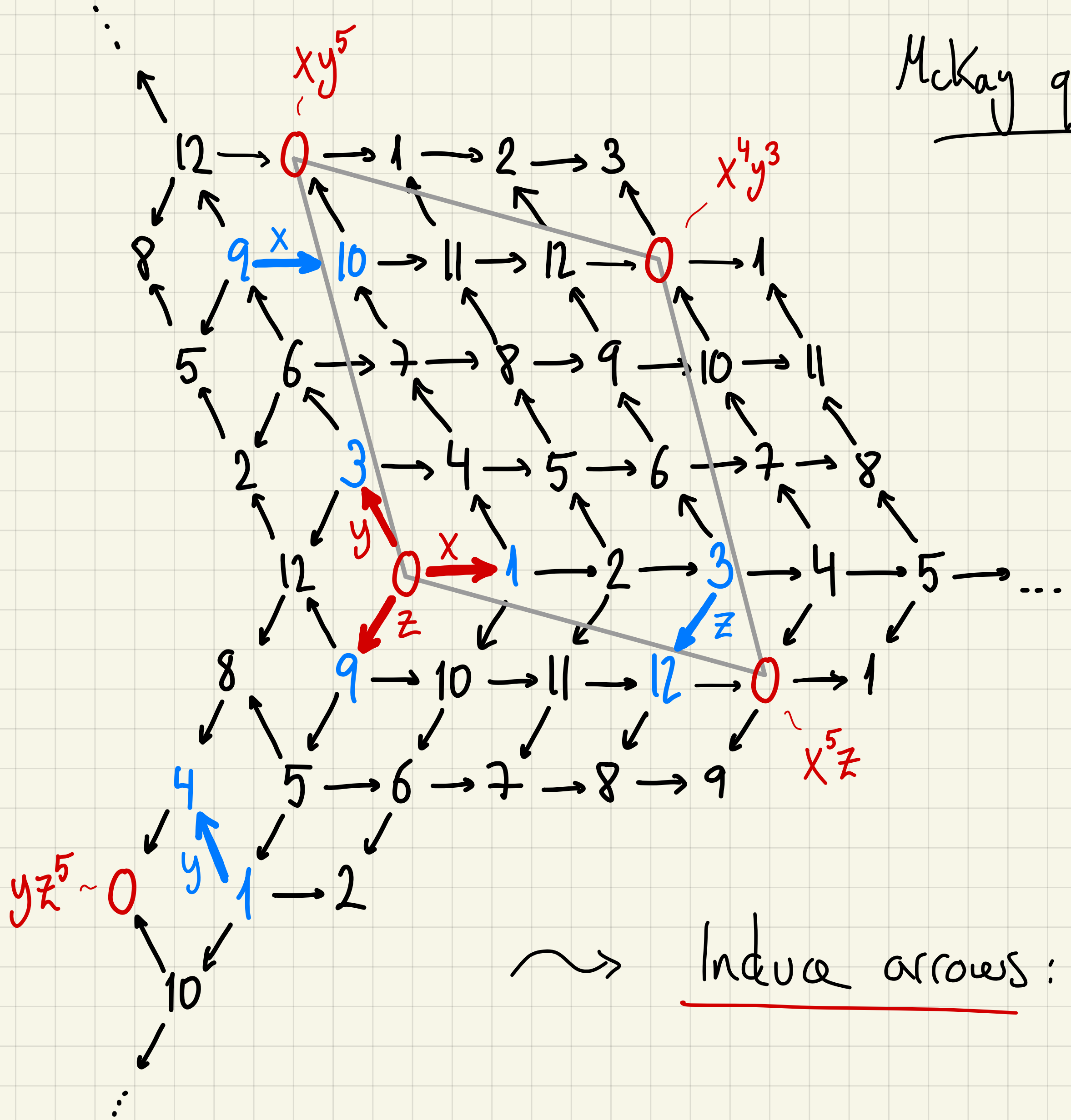
- Forms a quasi-monomial basis of a G-cluste

- Boat $\mathcal{B} \rightsquigarrow$ open $\mathcal{U}_{\mathcal{B}} \subset G\text{-Hilb}\,\mathbb{C}^3$



boat number 16 for
1/79(1, 23, 55)

Taken from M. Reid's webpage
(//homepages.warwick.ac.uk/~masda/McKay/tri/)
+ magma scripts to calculate the
list of possible boats.



McKay quiver of A

$\circlearrowleft T$: $$\begin{tikzcd} x \arrow[rr, bend left] & & y \arrow[dl, bend left] \\ & z \arrow[ul, bend left] & \end{tikzcd}$$

written on $\mathbb{Z}^3/(1,1,1)$

xyz G-inv

Irr G

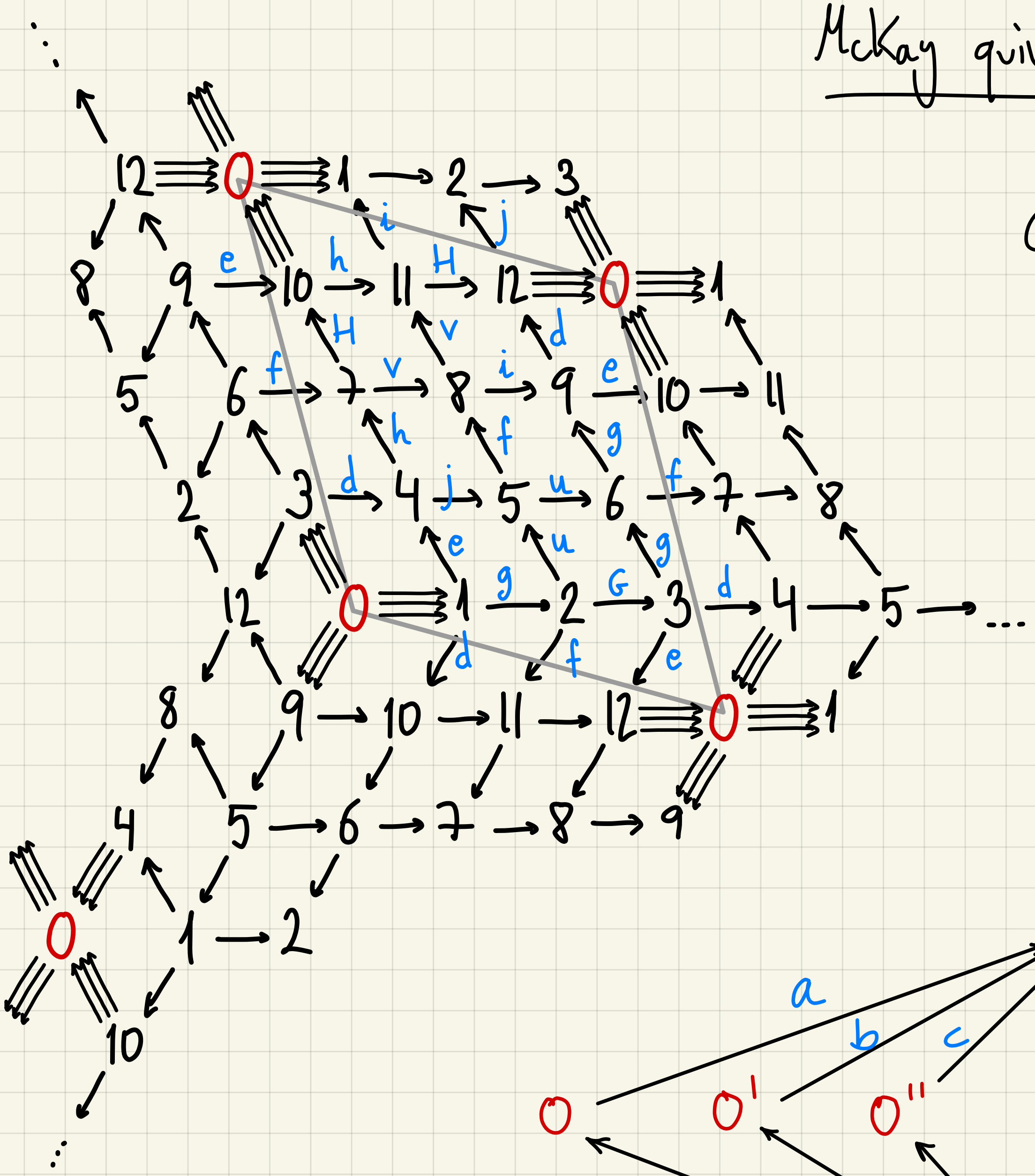
1 dim V_0 $V_{0'}$ $V_{0''}$

3 dim $V(1,3,9)$ $V(2,6,5)$ $V(4,12,10)$ $V(7,8,11)$

⇝ Induce arrows: $V(1,3,9) \xrightarrow{\begin{bmatrix} y & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & x \end{bmatrix}} V(4,12,10)$

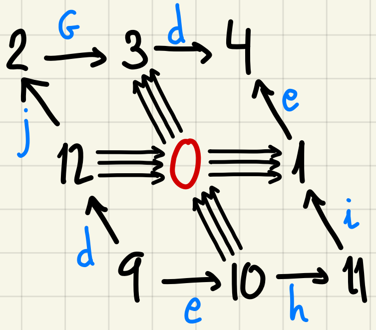
$$\begin{tikzcd} & V_0 \arrow[dr, "{[x,y,z]}"] & \\ V(4,12,10) \arrow[ur, "{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}"] \arrow[r, "{\begin{bmatrix} \omega z \\ x \\ \omega^2 y \end{bmatrix}}"] \arrow[dr, "{\begin{bmatrix} \omega^2 z \\ x \\ \omega y \end{bmatrix}}"'] & V_{0'} \arrow[r, "{[x,\omega y,\omega^2 z]}"] & V(1,3,9) \\ & V_{0''} \arrow[ur, "{[x,\omega^2 y,\omega z]}"'] & \end{tikzcd}$$

McKay quiver of $G = A \rtimes T$ on $\mathbb{Z}^3/(1,1,1)$



Commutativity Relations

+



$$Aa + \omega Bb + \omega^2 Cc = 3hi$$

$$Aa + \omega^2 Bb + \omega Cc = 3jG$$

$dA = eA$

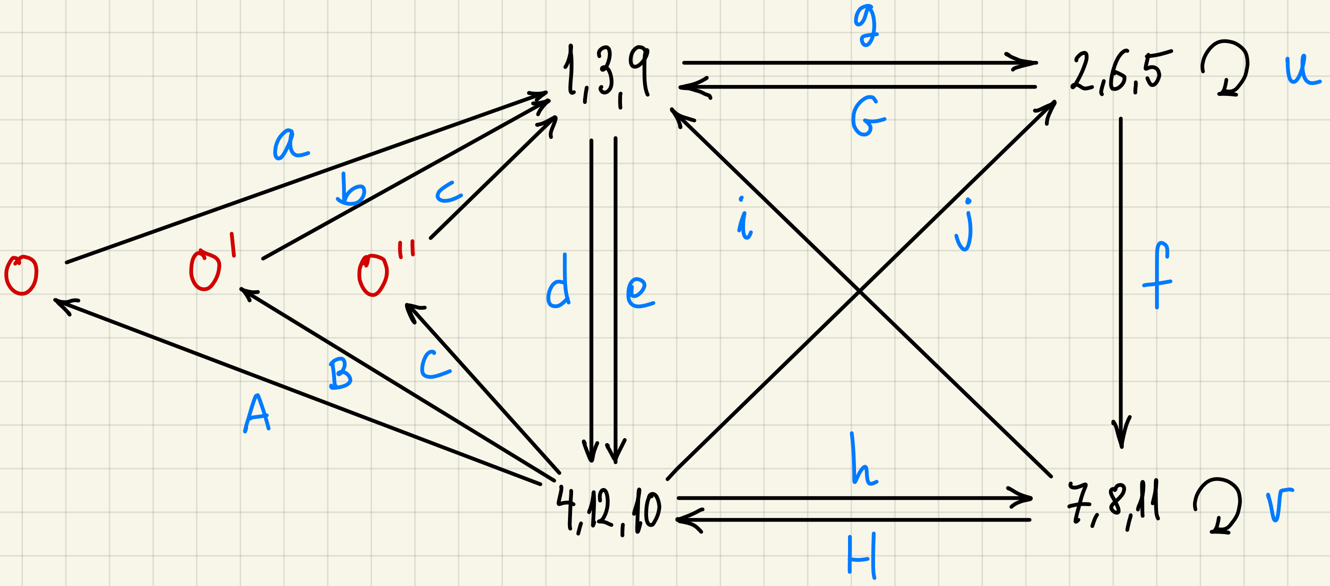
$dB = \omega eB$

$dC = \omega^2 eC$

$ad = ae$

$bd = \omega be$

$cd = \omega^2 ce$

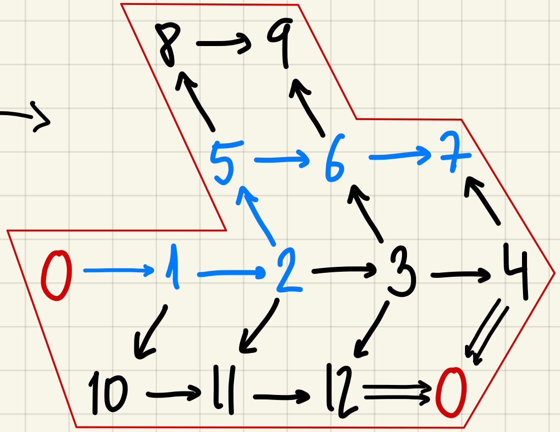


From boats to representations :

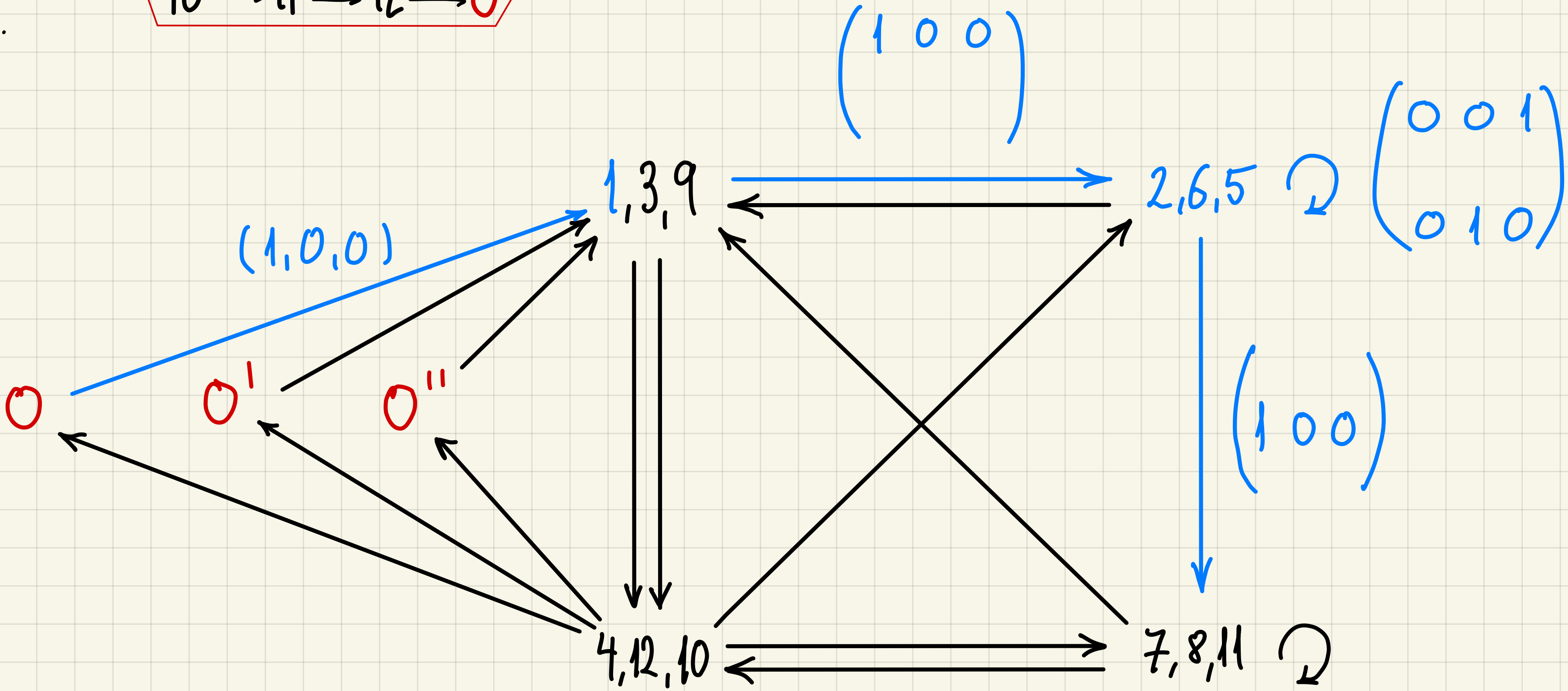
non-zero arrows

+

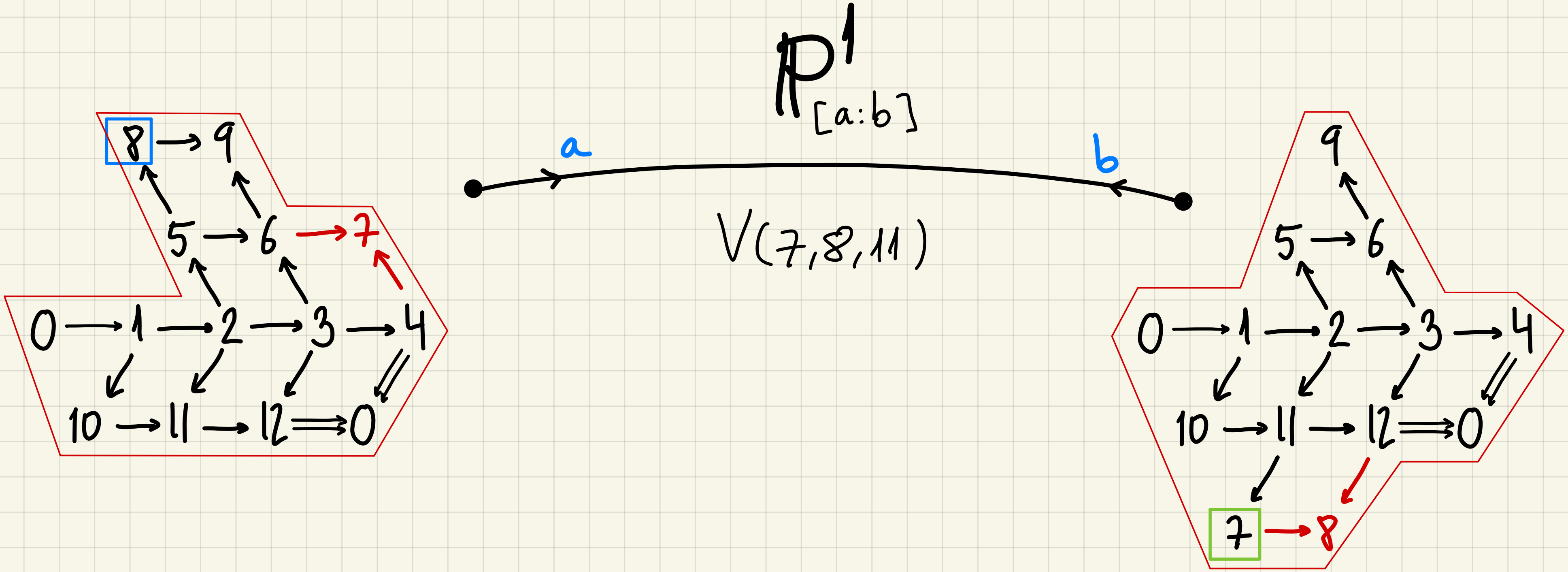
their corresp. $\mathbb{Z}/3$-sym.



Using boats one can define a "consistent" choice for the non-zero arrows



Walking along the exceptional divisor :



Only 1 relation along the curve : $b\,(y^2z^3, \underset{\overset{\uparrow}{8}}{x^3z^2}, x^2y^3) = a\,(\underset{\overset{\uparrow}{7}}{x^4y}, y^4z, xz^4)$ in V(7,8,11)

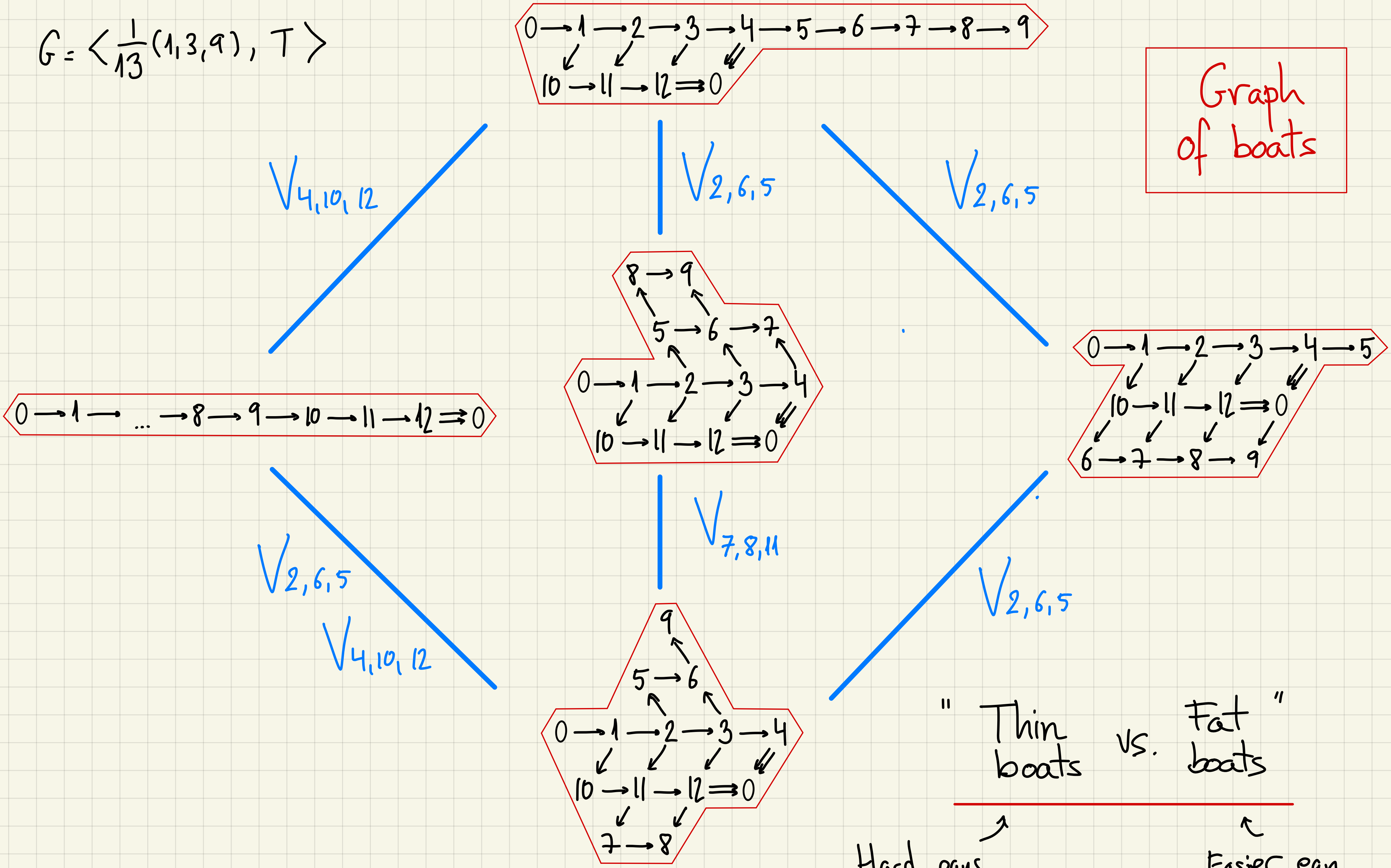
marks the curve !

+

1 rotation : $(\underset{\overset{\uparrow}{7}}{\boxed{x^2z^2}}, \underset{\overset{\uparrow}{8}}{\boxed{x^2y^2}}, y^2z^2) \in V(7,8,11)$

$G = \langle \frac{1}{13}(1,3,9),\ T \rangle$

Graph of boats

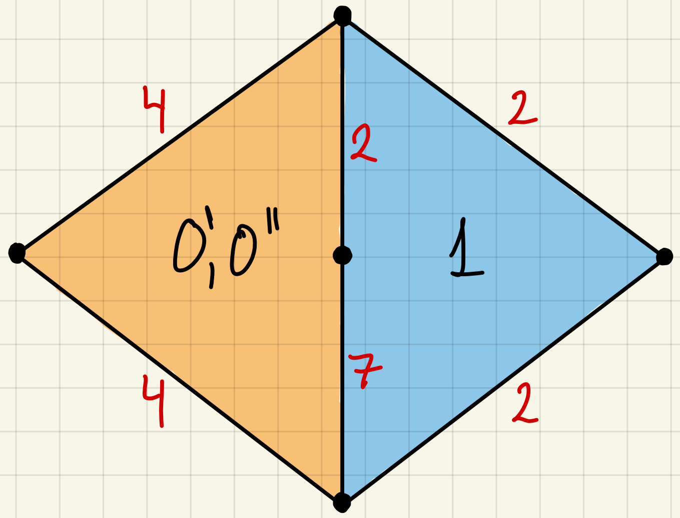


" Thin boats vs. Fat boats "

Hard eqns
Easy J-igsaw moves

Easier eqn
+
Harder J-igsaw
(∃ rotations)

The exceptional divisor of G-Hilb $\mathbb{C}^3$ in this case is:



Proposition

① G-Hilb $\mathbb{C}^3 = \bigcup_{i=1}^{5} U_i$ open cover with 2 compact excl. divisors

② Every non-trivial $\rho \in \mathrm{Irr}\, G$ markes either a curve or a divisor

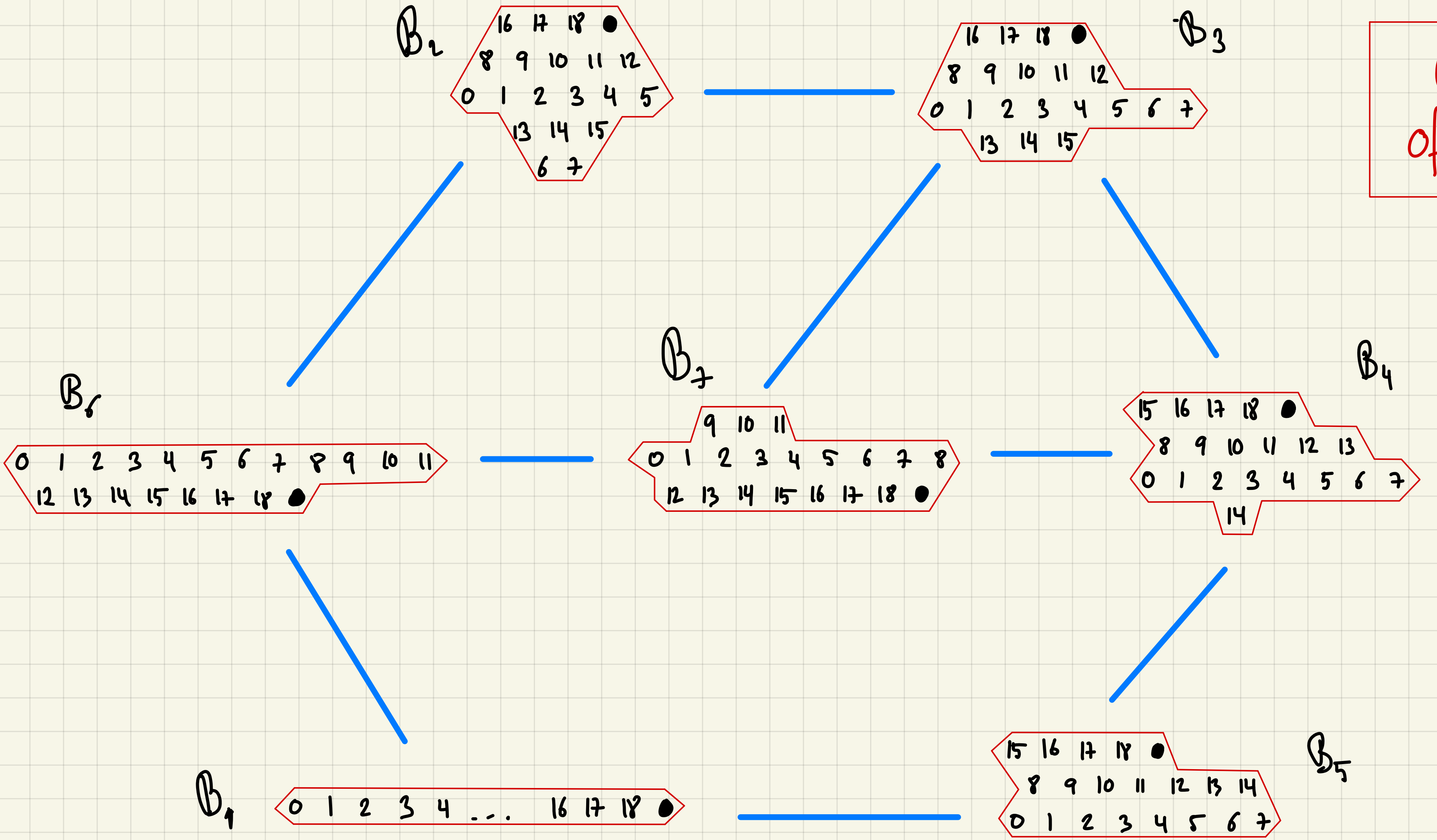
③ Taut. bundles $\{\det R_i\}$ generates Pic Y subjet to relations:

$$\boxed{\begin{aligned} R_{0'} \otimes R_{0''} &= \det R_4 \\ \det R_1 &= \det R_2 \end{aligned}}$$

and the Integral McKay correspondence holds.

④ [Donten-Bury, Grab] There are 28 different crepant resolutions in this case.

Example. G : Trihedral group of order 57 : $G = \langle \underset{\overset{\shortparallel}{A}}{\frac{1}{19}(1,7,11)} , \underset{\overset{\shortparallel}{T}}{\begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}} \rangle \subset SL(3,\mathbb{C})$



Graph of boats

Thank you!