Euler characteristics of Fujiki-Oka resolutions via continued fractions

Yusuke Sato (Kogakuin University)

Background

Theorem (Batyrev,99)

Let G be a finite subgroup of $\mathrm{SL}(n,\mathbb{C})$. If a crepant resolution $f : Y \to X = \mathbb{C}^n/G$ exists, then the Euler number of Y equals the number of conjugacy classes of G.

Remark

- If $n = 2, 3$, a crepant resolution is always exist. (Ito, Markushevich, Roan)
- When $n \geq 4$, crepant resolutions do not always exist.

In this talk, we generalize this correspondence to a finite cyclic group of $\mathrm{GL}(n,\mathbb{C})$

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Summary of results

G: a finite semi-isolated cyclic group of $\mathrm{GL}(n, \mathbb{C})$
or a finite cyclic group of $\mathrm{GL}(3, \mathbb{C})$.
For the resolution which is obtained by continued fractions (that is a Fujiki-Oka resolution), the equation

$$\begin{matrix}\text{the Euler characteristic} \\ \text{of the resolution}\end{matrix} = \begin{matrix}\text{the height of} \\ \text{continued fractions}\end{matrix} + \#G$$

holds.

Applying the case of G in $\mathrm{SL}(3, \mathbb{C})$, then the resolution is a crepant. In this case, it is included in the Batyrev's theorem.

Resolutions of toric varieties

$G \subset \mathrm{GL}(n, \mathbb{C})$: a finite cyclic group
We can assume that any $g \in G$ is of the form $\mathrm{diag}(\varepsilon^{a_1}, \ldots, \varepsilon^{a_n})$, where $\varepsilon^r = 1$.
Write $g = \frac{1}{r}(a_1, \ldots, a_n)$.

- $\bar{g} = \frac{1}{r}(a_1, \ldots, a_n) \in \mathbb{R}^n$,
- $N_G := \mathbb{Z}^n + \mathbb{Z}\bar{g}$: lattice
- $e_1, \ldots, e_n$:the canonical basis of $\mathbb{Z}^n$,
- Cone $\sigma = \langle e_1, \ldots, e_n \rangle$.

The toric variety $X(\sigma, N_G) \cong \mathbb{C}^n/G$.

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Multidimensional continued fractions

We construct the resolution for a cyclic group of $\mathrm{GL}(n, \mathbb{C})$.
→ Using multidimensional continued fractions
(defined by Ashikaga,19).

Notation.

- An n-dimensional fraction $\frac{(a_1,a_2,\ldots,a_n)}{r}$ is proper if the positive integer a_i and r satisfies $a_i \leq r$.
- $\mathbb{Q}_n^{prop}$: the set of an n-dimensional proper fraction
- $\overline{\mathbb{Q}_n^{prop}} = \mathbb{Q}_n^{prop} \cup \{\infty\}$

Multidimensional continued fractions

Definition(Ashikaga)

Let $\frac{(a_1,a_2,\ldots,a_n)}{r}$ be a proper fraction.

For $1 \leq i \leq n$, the *i-th remainder map* $R_i : \overline{\mathbb{Q}_n^{prop}} \to \overline{\mathbb{Q}_n^{prop}}$ is define by

$$R_i\left(\frac{(a_1,a_2,\ldots,a_n)}{r}\right) = \begin{cases} \frac{\left(\overline{a_1}^{a_i},\ \ldots,\ \overline{-r}^{a_i},\ \ldots,\overline{a_n}^{a_i}\right)}{a_i} & \text{if } a_i \neq 0 \\ \infty & \text{if } a_i = 0 \end{cases}$$

and $R_i(\infty) = \infty$ where $\overline{a_j}^{a_i}$ is an integer satisfying $0 \leq \overline{a_j}^{a_i} < a_i$ and $\overline{a_j}^{a_i} \equiv a_j$ modulo a_i.

Example:If $v = \frac{(1,2,7)}{12}$, then

$$R_2(v) = \frac{(1,0,1)}{2} \text{ and } R_3(v) = \frac{(1,2,2)}{7}.$$

Definition(Ashikaga)

Let $\frac{\mathbf{a}}{r}$ be an n-dimensional proper fraction. The *remainder polynomial* $\mathcal{R}_*\left(\frac{\mathbf{a}}{r}\right) \in \overline{\mathbb{Q}_n^{prop}}[x_1, x_2, \ldots, x_n]$ is defined by

$$\mathcal{R}_*\left(\frac{\mathbf{a}}{r}\right) = \frac{\mathbf{a}}{r} + \sum_{(i_1, i_2, \ldots, i_l) \in \mathbf{I}^l,\ l \geq 1} (R_{i_l} \cdots R_{i_2} R_{i_1}) \left(\frac{\mathbf{a}}{r}\right) \cdot x_{i_1} x_{i_2} \cdots x_{i_l}$$

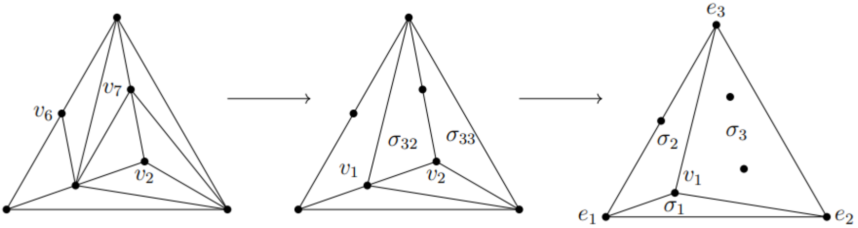
where we exclude terms with coefficients ∞ or $\frac{(0,0,\ldots,0)}{1}$.

Example :Let $v = \frac{(1,2,7)}{12}$, then the remainder polynomial is

$$\begin{aligned} \mathcal{R}_*\left(\frac{(1,2,7)}{12}\right) &= \frac{(1,2,7)}{12} + \frac{(1,0,1)}{2} x_2 + \frac{(1,2,2)}{7} x_3 \\ &+ \frac{(1,1,0)}{2} x_3 x_2 + \frac{(1,0,1)}{2} x_3 x_3. \end{aligned}$$

$$\mathcal{R}_*\left(\frac{(1,2,7)}{12}\right) = \frac{(1,2,7)}{12} + \frac{(1,0,1)}{2}x_2 + \frac{(1,2,2)}{7}x_3 + \frac{(1,1,0)}{2}x_3x_2 + \frac{(1,0,1)}{2}x_3x_3.$$

Let $G = \langle \frac{1}{12}(1,2,7) \rangle$. Subdivide by the lattice point $v_i = \bar{g}^i \in N_G$.



We call the induced toric morphism Fujiki-Oka resolution.

Definition

For a proper fraction (or a lattice point in N_G) $b = (b_1, \ldots, b_n)/r$, we define the *height* as follows:

$$\text{height}(b) = \sum_{i=1}^{n} b_i - r.$$

In addition, the *height* of a remainder polynomial $\mathcal{R}_*$ is defined the sum of heights of all coefficients in a remainder polynomial.

Example: $\text{height}\left(\frac{(1,2,7)}{12}\right) = 1 + 2 + 7 - 12 = -2.$

$\text{height}\left(\mathcal{R}_*\left(\frac{(1,2,7)}{12}\right)\right) = -2 + 0 + -2 + 0 + 0 = -4.$

height vs age

For the subdivison $\Sigma \to \sigma$, we have a birational map $f : X(N_G, \Sigma) \to X(N_G, \sigma)$ and

$$K_{X(N_G,\Sigma)} = f^*(K_{X(N_G,\sigma)}) + \sum_{\tau \in \Sigma(1)} a_\tau D_\tau,$$

where D_τ is an exceptional divisor corresponding to the one dimensional cone $\tau \in \Sigma(1)$ in Σ and a_τ is a discrepancy.

Remark

- age $-$ 1 $=$ height$/r$ $=$ discrepancy.
- Since the height of the remainder polynomial is a local value, it does not match the sum of the discrepancies.

Main result

Theorem (S)

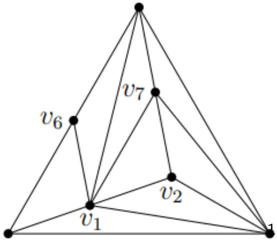
Let G be a cyclic group of type $\frac{1}{r}(1, a_2, \ldots, a_n)$. For the Fujiki-Oka resolution $f : Y \to \mathbb{C}^n/G$, the following holds

$$e(Y) = \text{height}\left(\mathcal{R}_*\left(\frac{(1, a_2, \ldots, a_n)}{r}\right)\right) + \#G,$$

where $e(Y)$ is a topological Eular characteristic of Y.

Example: Let $G = \frac{1}{12}(1, 2, 7)$, then height $\left(\mathcal{R}_*\left(\frac{1}{12}(1, 2, 7)\right)\right) = -4$. Since the Euler characteristic of F.O.R is a number of three dimensional cone, we have

$$8 = -4 + 12.$$



Two dimensional case

In dimension two, a Fujiki-Oka resolution is a minimal resolution.

Let G be a cyclic group of type $\frac{1}{7}(1,3)$. Then the remainder polynomial is

$$\mathcal{R}_*\left(\frac{(1,3)}{7}\right) = \frac{(1,3)}{7} + \frac{(1,2)}{3}x_2 + \frac{(1,1)}{2}x_2x_2$$

The height of $\mathcal{R}_*$ is $-3+0+0=-3$.
The Euler characteristic of the minimal resolution Y of $\mathbb{C}^2/G$ is

$$e(Y) = -3+7 = 4.$$

Theorem(K.Sato, S)

The Fujiki-Oka resolution is crepant if and only if the height of all coefficients of the remainder polynomial are 0.
Especially, if $G \subset \mathrm{SL}(3, \mathbb{C})$ then a Fujiki-Oka resolution is crepant.

Corollary

For a Gorenstein cyclic quotient singularity and its Fujiki-Oka crepant resolution, we have $e(Y) = \#G$.

There are many examples where the height of the remainder polynomial equals to 0 even though it is not Gorenstein.
Let $G = \frac{1}{11}(1, 3, 4)$.Then the remainder polynomial is

$$\begin{aligned} \mathcal{R}_* \left(\frac{(1,3,4)}{11} \right) &= \frac{(1,3,4)}{11} + \frac{(1,1,1)}{3} x_2 + \frac{(1,3,1)}{4} x_3 \\ &+ \frac{(1,2,1)}{3} x_3 x_3 + \frac{(1,1,1)}{2} x_3 x_3 x_3. \end{aligned}$$

Theorem (S)

Let G be a cyclic group of $\mathrm{GL}(3, \mathbb{C})$. For the Hilbert basis resolution $f : Y \to \mathbb{C}^3/G$, the following holds

$$e(Y) = \mathrm{height}\,(f) + \#G,$$

where $e(Y)$ is a topological Eular characteristic of Y.

Remark

- The exceptional divisors of the Hilbert basis resolution are BGS-essential divisors.
- In dimension three, a Hilbert basis resolution is always exist (Bouvier, Gonzalez-Sprinberg).
  It is obtained by repeated star subdivisions (Aguzzoli, Mundici).
- G is not necessarily semi-isolated (including the type of $\frac{1}{30}(2, 3, 5)$).

Future tasks

- Classify the group where the height of the remainder polynomial equals 0.
- Construct the McKay correspondence for $GL(3, \mathbb{C})$ as an analogy of Reid's recipe, that is the correspondence between the exceptional divisors and the special representations.
  (joint work with Y.Ito and K.Sato)