(Large) charge convexity in AdS hairy black hole

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A theme of this workshop

We want to understand the structure of

$$E(J) = \mathrm{Min}_{\mathrm{fixed}\, J} \mathrm{Energy}(J)$$

- In this talk (only), I use big J to denote charge

- What is the generic structure of E(J) ?
- Convex? Compatible with thermodynamics?
- Universal scaling behavior at large J: $E(J) \sim J^{\frac{d}{d-1}}$
- Technique to compute it (IPMU group)?

Reissner-Nordstrom black hole in AdS

- Most of the talks will concern field theory (on cylinder)
- This talk will discuss E(J) in AdS gravity in global coordinate
- AdS/CFT? $E(J) = \Delta(J)$
- Compute E(J) in classical GR + Maxwell theory +(cc)
- The natural candidate for the minimal energy with a given charge is extremal (= zero temperature) Reissner-Nordstrom black hole
  - It has non-zero entropy
  - In most other talks, ground state is unique

Reissner-Nordstrom black hole in AdS

- E(J) of extremal AdS-Reissner-Nordstrom BH?
- Simple form in AdS4

$$E(J) = \sqrt{-1 + \sqrt{1 + J^2} + J^2(3 + \sqrt{1 + J^2})}$$

- (It is not a linear function: not BPS!)
- Convex (= second derivative is positive)
- Universal large J behavior $E(J) \sim J^{\frac{3}{2}}$
- First 2 terms in EFT can emulate this (Loukas-Orlando-Reffert-Sarkar)
- 1/J expansion is convergent
- Finite radius of convergence $J = 1$
- What happened to Dyson's argument???

Convexity?

- If you believe that E(J) has a thermodynamic interpretation, we naturally expect E(J) is a convex function with respect to J (see e.g. Landau-Lifshitz)

- → CFT convexity conjecture (Aharony-Palti)
  - Many examples
  - A few counterexamples (Sharon-Watanabe)
  - Only in large charge? Only in averaged sense?
- Can we believe in thermodynamic argument (in gravity)?
  - Small AdS-Schwartzshild BH?
  - Negative specific heat (violation of convexity)
- Convexity or Superadditivity (Antipin et al)?

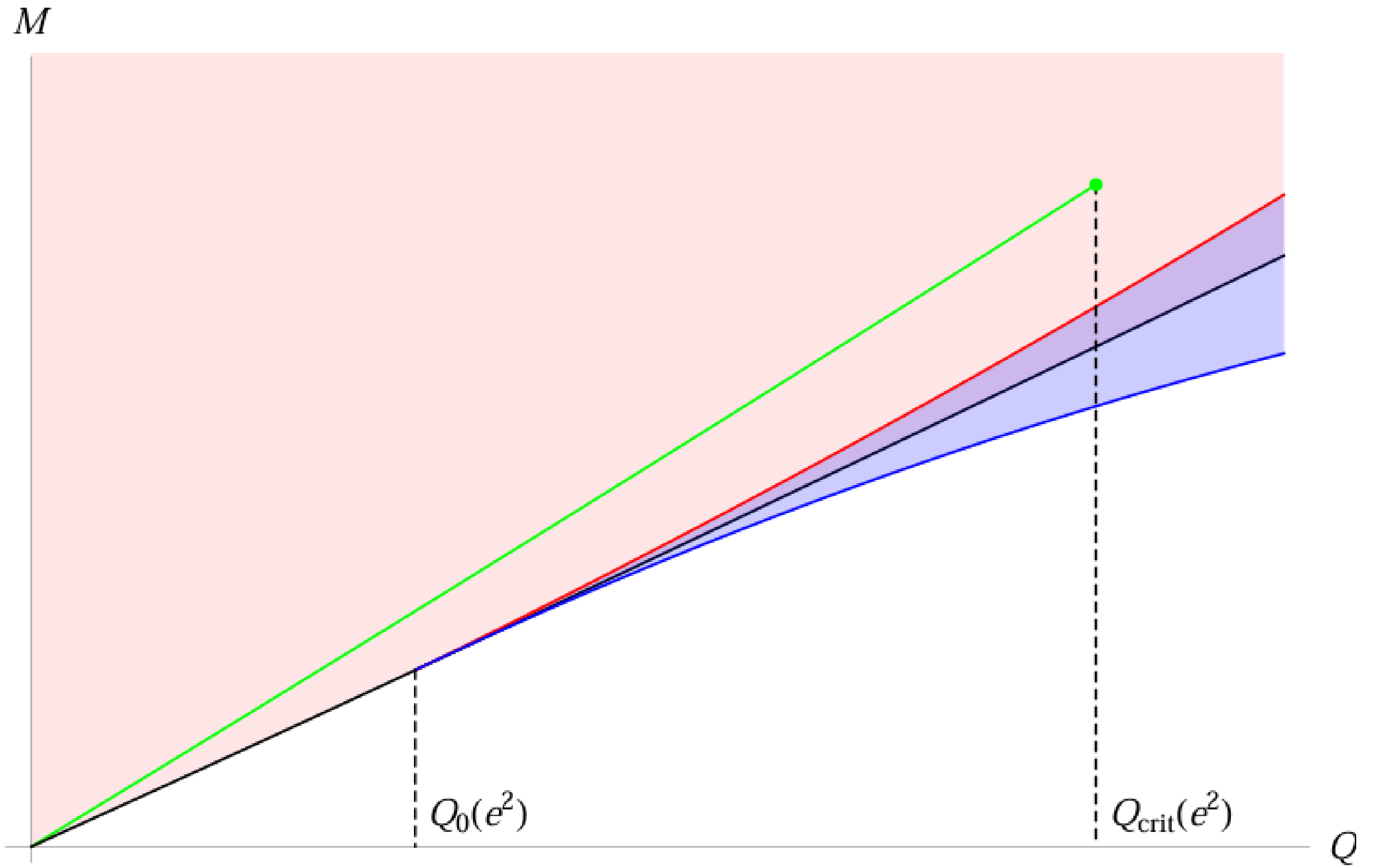
This is not the end of the talk

- Some people believe extremal Reissner-Nordstrom BH is not the lowest energy state with a given charge
- This is essentially what they mean by weak-gravity conjecture Arkani-Hamed-Motl-Nicolis-Vafa
- AdS version is more non-trivial (Nakayama-Nomura)

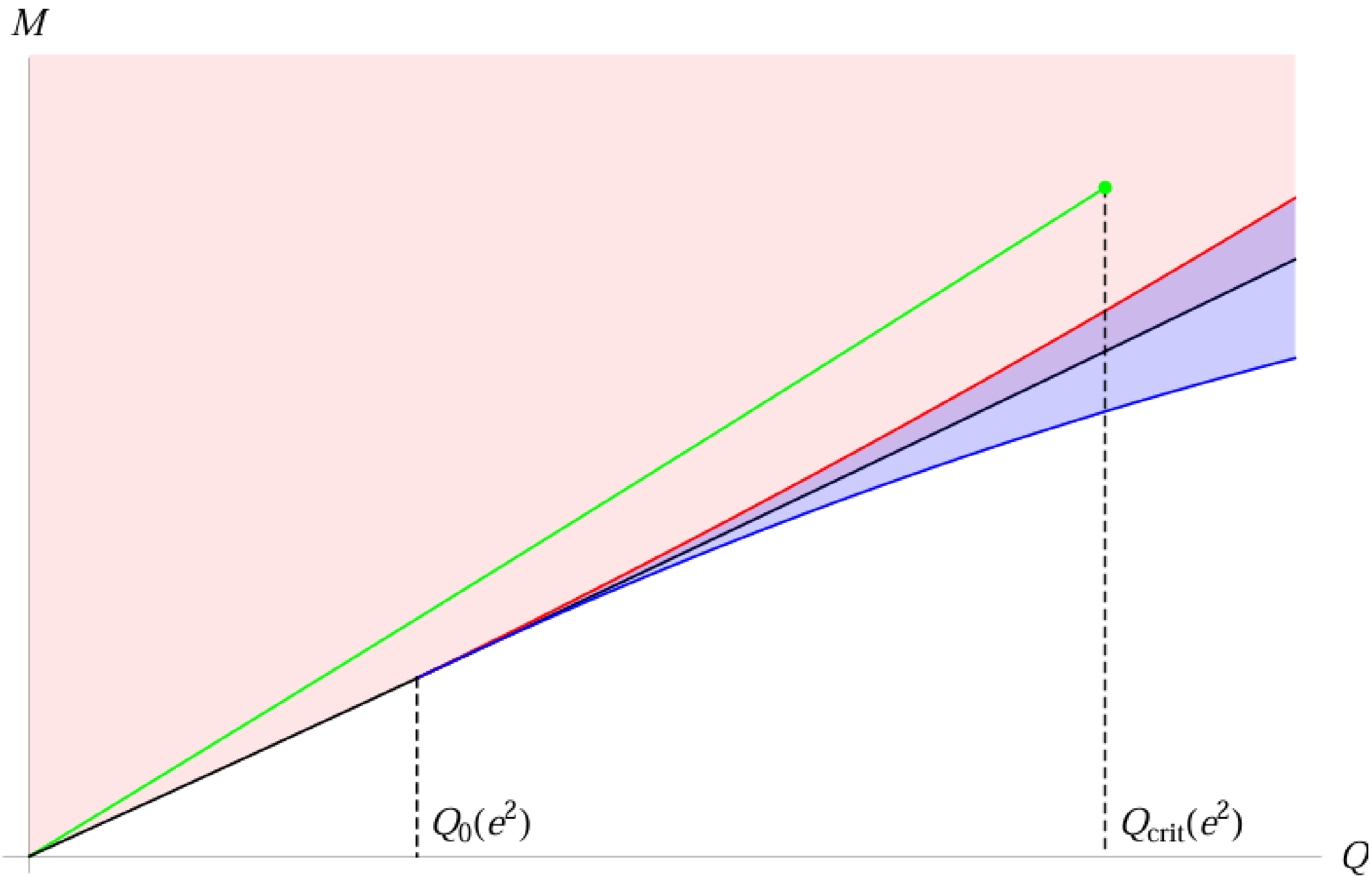
- It seems natural to assume there exists (light) matter to support the charge of RN BH
  - Non-trivial soliton solutions
  - Hairy BH solutions
- Will this change the story?

I found an interesting paper by Dias Figueras Minwalla Mitra Monteiro Santos (2011)

- They seemed to construct a charged hairy BH in AdS
- OK, let's take a look at their E(J)



Really? I should write a PRL paper!



Not really…

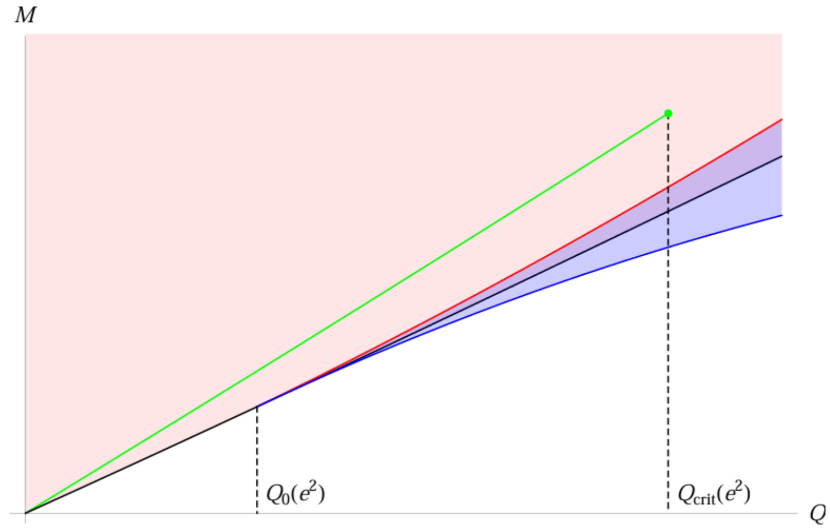
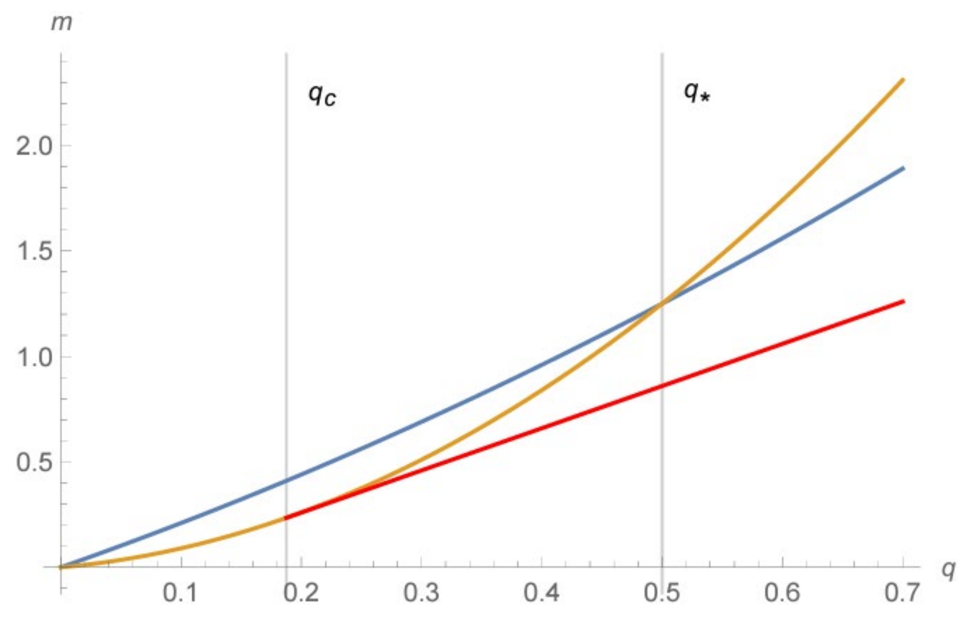


Figure 2: Schematic phase diagram for $3 \leq e^2 \leq \frac{32}{3}$. RN AdS black holes exist (red shaded) for all values of charge and mass above the extremality curve (black). Extremal black holes are unstable for $Q \geq Q_0(e^2)$ and hairy black hole solutions exist for these values of charge. These solutions exist (blue shaded) between the curve of instability of RN AdS black holes (red) and a zero temperature hairy black hole solution (blue). Hairy black holes are the dominant phase whenever they exist. The soliton (green) exists upto a maximum charge $Q = Q_{crit}$ (where a cusp structure, to be discussed only later, appears) and is never the dominant phase. (The soliton curve can be below the extremal RN AdS line for $Q > Q_0(e^2)$ but keeps above the extremal hairy black hole).

The careful examination of their formula shows it is convex!

This was the beginning

- Should study E(J) in hairy BH (with or without WGC) more systematically
- We found in most of the situations, hairy BH satisfy convexity
- Rather the very existence of the hairy BH will restore the convexity (which is violated otherwise)



AdS Solitons and hairy BH

Charged scalar and WGC

- Consider d+1 dim Einstein-Maxwell-cc + charged scalar

$$S = \int d^{d+1}x\sqrt{-g}\left(\frac{1}{16\pi G_N}(R-2\Lambda) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |D_\mu\phi|^2 - m_\phi^2|\phi|^2\right) \ ,$$

- Parametrize mass by conformal dimensions

$$m_\phi^2 = \Delta(\Delta - d)$$

- WGC: gravity is weaker than EM force
- Critical coupling:

$$e_c = \Delta\kappa\sqrt{\frac{d-2}{d-1}}$$

- When $e > e_c$ WGC holds, otherwise violated
- (CFT interpretation may be interesting)
- Without light scalar, the cosmic censorship can be violated (Horowitz et al)

Soliton solution

- There exists a horizonless soliton solution (self-gravitating boson star)
- EOM can be solved perturbatively in J
- (Since it is not a GR talk, I'm not showing explicit solutions)
- E(J) can be computed in small J expansion

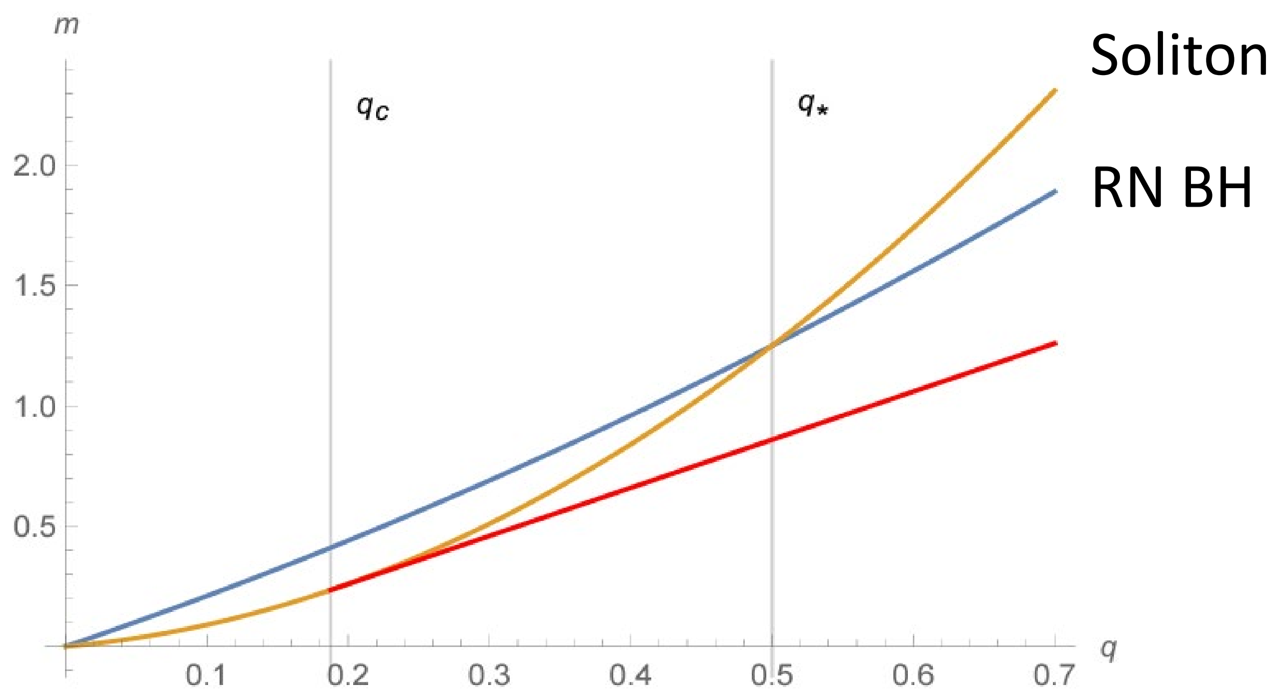
$$E = (d-2)\omega_{d-1}\left[\frac{\Delta}{e}J + \frac{\Gamma(\Delta)^2\,\Gamma(2\Delta+1-d/2)}{2\,\Gamma(d/2)\,\Gamma(2\Delta)\,\Gamma(\Delta+1-d/2)^2}\left(1 - \frac{(d-2)\Delta^2(\Delta-d/2)}{(d-1)(\Delta-d/4)}\frac{\kappa^2}{e^2}\right)J^2\right]$$

$$\mu = \frac{\Delta}{e} + \frac{\Gamma(\Delta)^2\,\Gamma(2\Delta+1-d/2)}{\Gamma(d/2)\,\Gamma(2\Delta)\,\Gamma(\Delta+1-d/2)^2}\left(1 - \frac{(d-2)\Delta^2(\Delta-d/2)}{(d-1)(\Delta-d/4)}\frac{\kappa^2}{e^2}\right)J\ ,$$

- In most cases (with unitarity bound), E(J) is convex

- For specific d and m, large J expansion was studied in the literature (de la Fuente and Zosso)

Soliton vs RN BH



- Soliton and RN BH can switch if we increase J
- If this first-order phase transition occurred, the convexity would be violated (large J expansion may also fail)

Hairy BH from thermodynamics

- Suppose WGC holds
- For small J, soliton is the lowest energy states
- If we increase J, at certain point, due to the non-linear nature of E(J), it costs less to create small AdS RN BH rather than make the soliton fatter
- This should lead to hairy BH

- Suppose WGC is violated
- For small J, RN BH is the lowest energy states.
- If we increase J, it costs less to create small soliton rather than make the RN BH fatter
- This should lead to hairy BH

Hairy BH from thermodynamics

- Assumption: the gravitational system satisfies thermodynamics
- Hairy BH = thermodynamic mixture of soliton and RN BH

(many other examples studied by Minwalla et al)

- Equilibrium condition:

$$\mu(J = J_{sol} + J_{RN}) = \mu_{sol}(J_{sol}) = \mu_{RN}(J_{RN})$$

- E is determined by the distributions of Js

Sample examples

- Hairy BH = thermodynamic mixture of soliton and RN BH
- Solve the equilibrium conditions: $Q_{hairy} = Q_{sol} + Q_{RN}$

$$E_{hairy} = E_{sol} + E_{RN}$$

$$\mu_{sol} = \mu_{RN}$$

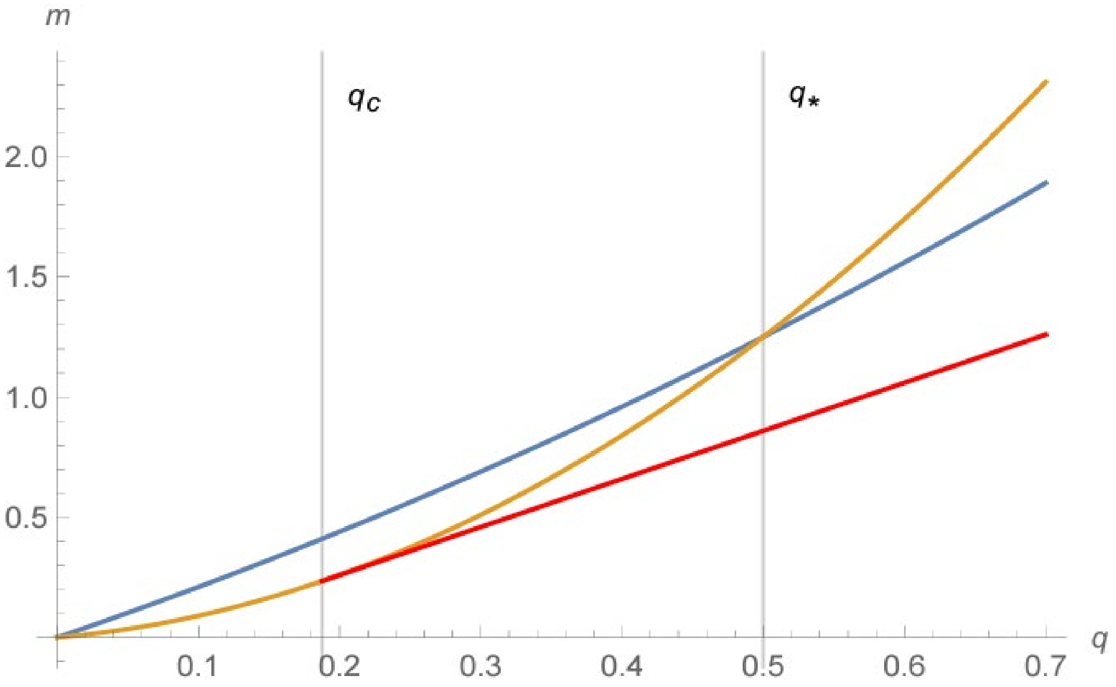
- Predictions for hairy BH thermodynamics
- Example: $e < e_c$ in AdS4

$$E_{hairy} = E_{RN} - \frac{3 \times 2^{2\Delta-5}\,\Gamma(\Delta-1/2)^2\,\Gamma(\Delta+1/2)}{\Gamma(\Delta)\,\Gamma(2\Delta-3/2)}\kappa^4(J^2 - J_c^2)^2 \ ,$$

$$E_{RN} = \sqrt{2}\kappa J + \frac{\kappa^3 J^3}{2\sqrt{2}} + O(J^5) \ . \qquad J_c = \sqrt{e_c^2 - e^2}$$

- Careful study shows it is convex (when $\Delta \geq \frac{3}{4}$, $e < e_c$)

Convexity?



- Hairy BH makes E(J) convex
- Second order phase transition at J=Jc
- Derivative is continuous but E(J) is non-analytic

Discussions

When convexity does not hold…

- Look at our E(J) again

$$E = (d-2)\omega_{d-1}\left[\frac{\Delta}{e}J + \frac{\Gamma(\Delta)^2\,\Gamma(2\Delta+1-d/2)}{2\,\Gamma(d/2)\,\Gamma(2\Delta)\,\Gamma(\Delta+1-d/2)^2}\left(1-\frac{(d-2)\Delta^2(\Delta-d/2)}{(d-1)(\Delta-d/4)}\frac{\kappa^2}{e^2}\right)J^2\right]$$

$$\mu = \frac{\Delta}{e} + \frac{\Gamma(\Delta)^2\,\Gamma(2\Delta+1-d/2)}{\Gamma(d/2)\,\Gamma(2\Delta)\,\Gamma(\Delta+1-d/2)^2}\left(1-\frac{(d-2)\Delta^2(\Delta-d/2)}{(d-1)(\Delta-d/4)}\frac{\kappa^2}{e^2}\right)J\ ,$$

- It violates the convexity when $\Delta < \frac{d}{4}$
- Occurs only in d=3
- Very non-intuitive: gravity becomes infinite at $\Delta = \frac{3}{4}$ (negative mass^2 in the alternative quantization)
- Strong violation of (effective) WGC? Not realized in string landscape?
- $g|\phi|^4$ can also change the story: large negative g will violate the convexity (violation of unitarity?)

What I still don't understand

- Weak gravity conjecture says extremal RN black hole must be unstable
- Why?
- Lowest energy state with a given charge has a large entropy (Remnant)
- Soliton has zero entropy
- But do we accept hairy BH? Why don't we demand extremal hairy BH be unstable? Non-zero entropy?
- Coincidentally or not, hairy BH solution at zero temperature looks singular (at the "horizon")

Large charge expansion?

- Our gravity construction is small charge expansion

- Is Large J expansion possible?
- Connection to EFT analysis (through AdS/CFT?)

- Large J limit looks like a planar hairy black brane studied in the context of holographic superconductor (with the Lifshitz scaling?, no horizon?)

- 1/J expansion = curvature expansion?

Summary

- (Large) J convexity in gravitational system is interesting
- Is there fundamental reason why it should hold?
- More conceptual understanding WGC (in AdS)
- Large J EFT in gravity is highly welcome
- Can be very different in d=2 (3D gravity, 2D CFT)

Convexity vs Superadditivity

- Convex function

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2)$$

- When twice differentiable $f''(x) \geq 0$

- Supreadditive (Aharony-Palti)

$$f(a) + f(b) \leq f(a+b)$$

- Convex + f(0) = 0 → superadditive
- Converse may not be true