Conformal graphs as thermal partition functions

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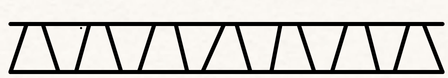
- Thermal two-point functions in CFTs: a perplexing observation.

- Constructing conformal graphs in any dimension from the twisted harmonic oscillator

- Further remarks (extra fun...)



Thermal two-point functions in CFTs

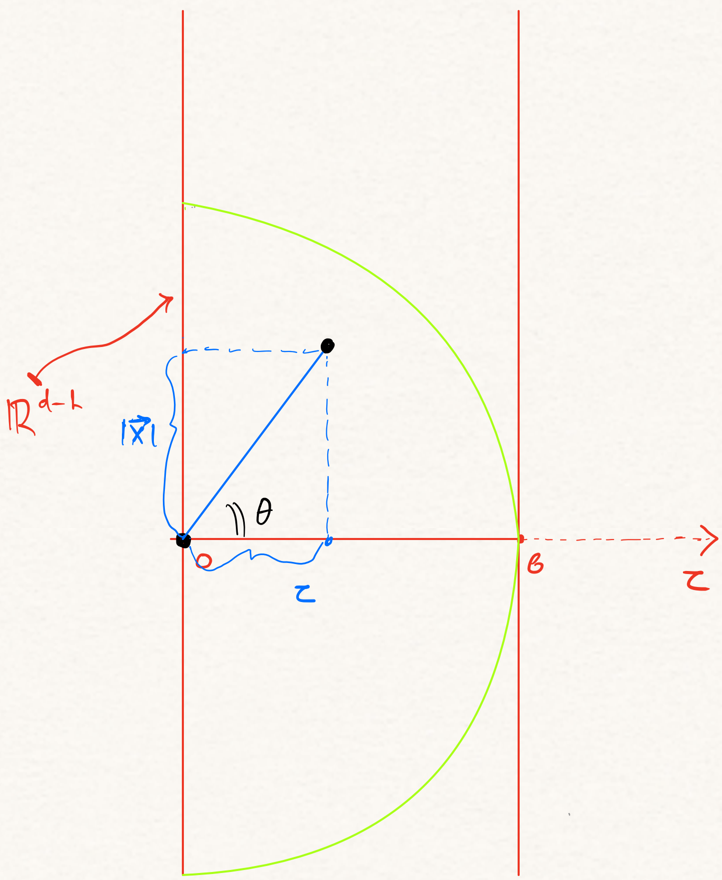
CFTs in $d>2$: $\quad d = 2L+1$

In $\mathbb{R}^d$: $\quad \langle \varphi(x_1)\varphi(x_2)\rangle = \dfrac{C^L_\varphi(\Delta)}{(x_1-x_2)^{2\Delta}}$; $\quad C^L_\varphi(\Delta) = \dfrac{\Gamma(\Delta)}{\Gamma(L+1/2-\Delta)\, 4^{L+1/2-\Delta}\, \pi^{L+1/2}}$

$\varphi(x)$: scalar with scaling dimension Δ.

In $S^1_\beta \times \mathbb{R}^{d-1}$:

* Not conformally related to $\mathbb{R}^d$ for $d>2$.



$$r^2 = \tau^2 + |\vec{x}|^2$$
$$= (\tau + i|\vec{x}|)(\tau - i|\vec{x}|)$$

$$\tau = r\cos\theta$$

$$|\vec{x}| = r\sin\theta$$

The general result is of the form: $(r<\beta)$

$$\langle \varphi(r,\cos\theta)\,\varphi(0,0)\rangle = \frac{1}{\;}\sum a^L \left(\frac{r}{\;}\right)^{\Delta_{\mathcal{O}_s}} C^\nu(\cos\theta)$$

Q_s : All quasiprimaries in the OPE $\varphi \times \varphi$ with dimensions Δ_{Q_s} & spin-s.

$C_s^{\nu}(t)$: Gegenbauer polynomials, $\nu = \frac{d}{2} - 1$

$a^L_{Q_s}$: Combination of 3pt couplings and thermal 1pt functions.

Example: massless free scalar

$$\langle \varphi(r, \cos\theta)\, \varphi(0,\vec{0}) \rangle = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\, e^{-i\omega_n \tau - i\vec{p}\cdot\vec{x}}\, \frac{1}{\omega_n^2 + \vec{p}^{\,2}}$$

- $\omega_n = \frac{2\pi}{\beta} n$
- $\varphi(\tau+\beta, \vec{x}) = \varphi(\tau, \vec{x})$

We obtain:

$$a^L_{Q_s} = C^L_{\varphi}(1)\, J(2L-1+s) \; : \; s = 0, 2, 4, \ldots$$

QUESTION:

Calculate $a^L_{Q_s}$ for non-trivial thermal CFTs

e.g. "thermal bootstrap"

The general picture

$$\begin{tikzcd} \langle \varphi\varphi \rangle_0 \arrow[d, squiggly, "(RG)"'] \arrow[r] & a^L_{Q_s}(g_0) \arrow[d, squiggly] \\ \langle \varphi\varphi \rangle_* \arrow[r] & a^L_{Q_s}(g_*) \end{tikzcd}$$

$$\sum_{\{Q_s\}} \qquad \left(\frac{r}{\beta}\right)^{\Delta_{Q_s}} C^{\nu}_{s}(\cos\theta)$$

Finite temperature extra data

$\wr\wr$

thermal 1pt functions

"thermal blocks" Flat data

e.g.

$$\langle\varphi\varphi\rangle_0 = \mathrm{Tr}[\varphi\varphi\, e^{-\beta \hat{H}_0}] \longrightarrow \langle\varphi\varphi\rangle_* = \mathrm{Tr}[\varphi\varphi\, e^{-\beta(\hat{H}_0 + g_* \hat{H}_*)}]$$

Example: massive free complex scalar

$$S = \int_0^\beta d\tau \int d^{d-1}\vec{x} \left(|D_\mu \varphi|^2 + m^2 |\varphi|^2 \right), \quad \left| \begin{array}{l} D_\mu = \partial_\mu - i A_\mu \\ A_\mu = (\mu, \vec{0}) \end{array} \right.$$

$$\hat{H} = \int d^{d-1}\vec{x} \left(\pi^*_\varphi \pi_\varphi + m^2 |\varphi|^2 + i\mu (\pi_\varphi \varphi - \pi^*_\varphi \varphi^*) \right)$$

$$= \hat{H}_0 + m^2 \hat{O} + i\mu \hat{Q} \quad , \quad \left| \hat{O} = |\varphi|^2 \right.$$

$$\hat{Q} = \Pi_\varphi \varphi - \Pi^*_\varphi \varphi^*$$

* Does the thermal 2pt function have the CFT form?

* How do $a^L_{Q_s}(0,0) \longrightarrow a^L_{Q_s}(m,\mu)$

Result:

$$\boxed{\langle \varphi(r,\cos\theta)\,\varphi(0,0)\rangle = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\, e^{-i(\omega_n-\mu)\tau - i\vec{p}\cdot\vec{x}}\,\frac{1}{(\omega_n-\mu)^2+\vec{p}^{\,2}+m^2}}$$

$$\varphi(\tau+\beta,\vec{x}) = e^{i\beta\mu}\,\varphi(\tau,\vec{x})$$

i.e. "twisted" boundary conditions.

|| Remarkably, the above 2pt function can be expanded in CFT "thermal blocks" with <u>canonical dimensions</u> Δ_{Q_s}.

The "thermal data" $a^L_{Q_s}(0,0) \longrightarrow a^L_{Q_s}(m,\mu)$

$$\boxed{\begin{aligned} a^L_{Q_s}(m,\mu) = {} & \frac{\Gamma(L-1/2)}{\Gamma(L-1/2+s)\,(4\pi)^L\, 2^{2s}} \times \\ & \times \sum_{n=0}^{L-1+s} \frac{z^n}{n!}\,\frac{(\beta m)^n\,(2L-2+s-n)!}{(L-1+s-n)!} \times \Big[\mathrm{Li}_{2L-1+s-n}(z) \\ & \qquad + (-1)^s\,\mathrm{Li}_{2L-1+s-n}(\bar{z})\Big] \end{aligned}}$$

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n} \quad \text{polylogarithms.}$$

$$z = e^{-\beta m - i\beta\mu}$$

Remarks:

- Skipped the interesting calculational details (i.e. inversion formulae, analyticity properties etc)
- $a^L_{Q_s}(m,\mu)$ are <u>single-valued</u> combinations of polylogarithms.
- $a^L_{Q_s}(m,\mu) \xrightarrow[\mu\to 0]{m\to 0} a^L_{Q_s}(0,0) \simeq J(2L-L+s)$
- The deformed theory is <u>NOT</u> a CFT for generic values of m, μ. We need additional info to determine the critical values m_*, μ_*.
  $\simeq$ <u>gap equations</u> (see below).

- The CFT-form of a thermal 2pt function requires a conformal OPE expansion for $\varphi \times \varphi$ which is equivalent with the presence of a basis of conformal quasiprimary operators Q_s in the spectrum.

  These are symmetric, traceless tensors whose 1pt functions yield the $a^L_{Q_s} \simeq \langle Q_s \rangle$

  But in a massive theory, the e.m. tensor is not traceless. So, what is the spin-2 operator whose 1pt function yields $a^L_{Q_2}$? And what about $a^L_{Q_s}$, $s>2$?

Thermal 1pt functions in massive free theories

Since:

$$\hat{H} = \hat{H}_0 + m^2 \hat{O} + i\mu \hat{Q} \,, \quad Z_L = \mathrm{Tr}\, e^{-\beta \hat{H}}$$

$$\Rightarrow \boxed{\langle \hat{O} \rangle_L = -\frac{1}{\beta} \frac{\partial}{\partial m^2} \ln Z_L \,, \quad \langle \hat{Q} \rangle_L = i \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Z_L}$$

Using also:

$$\langle \hat{H} \rangle_L = -\frac{\partial}{\partial \beta} \ln Z_L = -\langle t_{zz} \rangle_L$$

where $t_{\mu\nu}$ is the e.m. tensor of the theory

$$\Rightarrow \boxed{\langle \hat{H} \rangle_L = \frac{d-1}{\beta} \ln Z_L + 2m^2 \langle \hat{O} \rangle_L + i\mu \langle \hat{Q} \rangle_L}$$

generalized virial theorem.

From $t_{\mu\nu}$, $\hat{O}$, $\hat{Q}$ we can construct a spin-2 traceless operator $T_{\mu\nu}$ with:

$$T_{zz} = t_{zz} + 2m^2 \frac{1}{d} \hat{O} + i\mu \hat{Q}$$

$$\Rightarrow \boxed{\langle T_{zz} \rangle_L = -\frac{d-1}{\beta} \ln Z_L - 2m^2 \frac{d-1}{d} \langle \hat{O} \rangle_L}$$

We then find that $a^L_{Q_2}$ does correspond to the thermal 1pt function of $T_{\mu\nu}$:

$$\boxed{a^L_{Q_2} = -\frac{\beta}{(4\pi\alpha^2)^L}\frac{C_q^L(1)\, S_L}{2(d-1)}\langle T_{zz}\rangle_L}$$

$$\alpha^2 = \frac{\ell^2}{4\pi\beta^2} \qquad S_L = \frac{2\pi^{L+1/2}}{\Gamma(L+1/2)}, \quad L+1/2 = \frac{d}{2}$$

Note also that:

$$\boxed{a^L_{Q_0} = \frac{1}{(4\pi)^L\beta\alpha^{2L}}\langle\hat{O}\rangle_L, \quad a^L_{Q_1} = \frac{1}{(4\pi)^L\alpha^{2L}}\frac{1}{2}\langle\hat{Q}\rangle_L}$$

Remarks:

- Thermal 2pt functions of massive free scalars are expanded in CFT "thermal blocks" with coefficients:

$$a^L_{Q_s} \longrightarrow \text{single-valued polylogarithms}$$

- The coefficients $a^L_{Q_s}$ correspond to thermal 1pt functions of "conformal-like" (quasiprimary) operators:

$$a^L_{Q_s} \propto \langle\hat{Q}_s\rangle$$

- A recursive relation: we can show by brute-force that:

$$\boxed{a^{L}_{Q_{s+2}} = \frac{2\pi}{2L-1}\, a^{L+1}_{Q_s} + \frac{(m\beta)^2}{(2L-1+2s)(2L+1+2s)}\, a^{L}_{Q_s}}$$

$\Rightarrow$ If we know $a^{L}_{Q_0} \sim \langle \hat{O} \rangle_L$ & $a^{L}_{Q_1} \sim \langle \hat{Q} \rangle_L$, for all L

we can construct the thermal 1pt functions

of all higher-spin "conformal" operators $\hat{Q}_s$

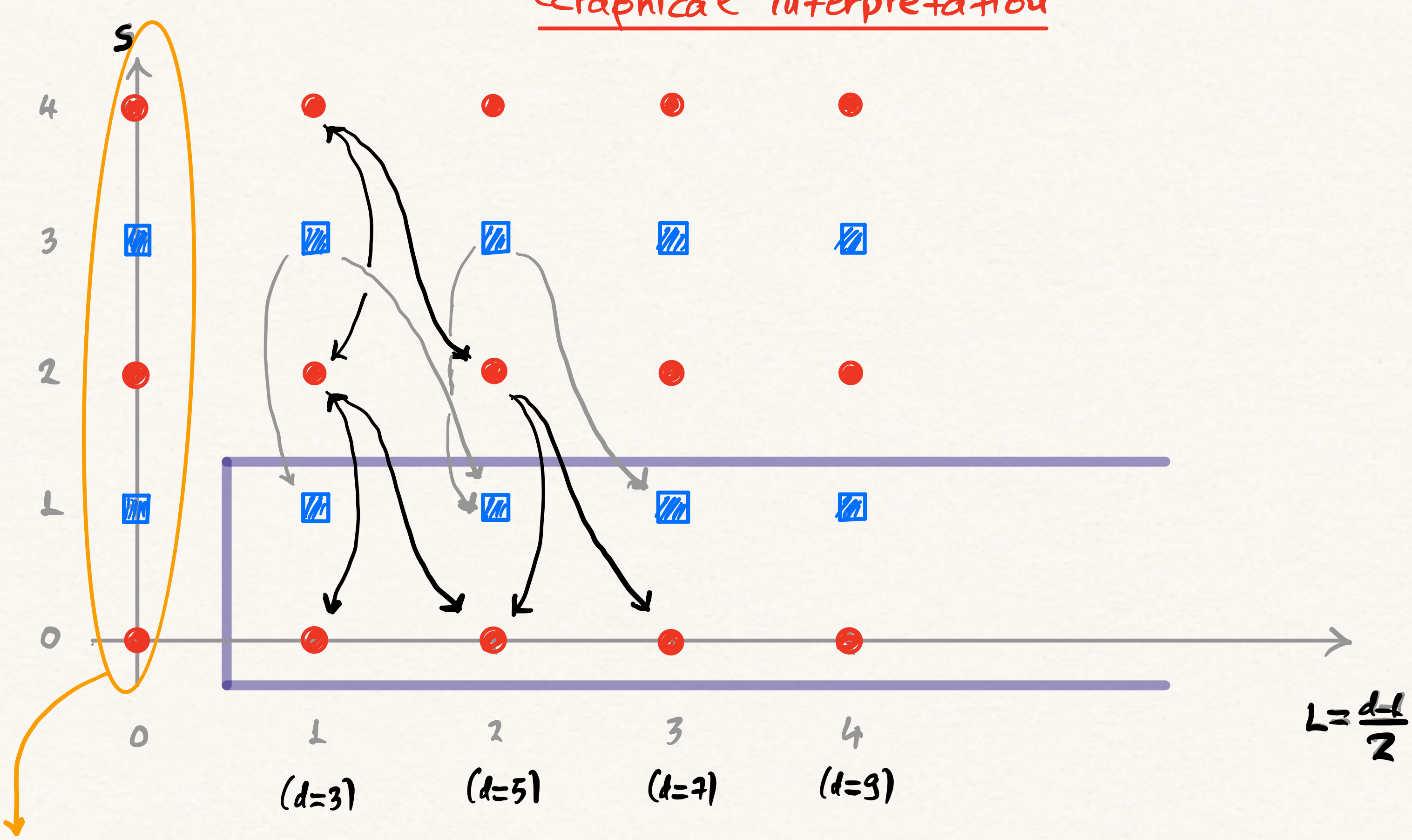
- <u>Gap equations:</u>

  These are conditions on $a^{L}_{Q_0} \sim \langle \hat{O} \rangle_L$ & $a^{L}_{Q_1} \sim \langle \hat{Q} \rangle_L$

  that determite critical values of m_*, μ_*.

  (This opens a huge and unexplored direction...)

Graphical interpretation



$d=1$
Harmonic oscillator

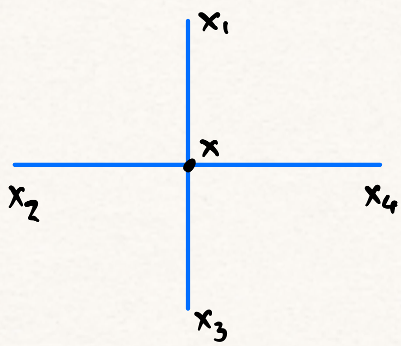
A perplexing observation

Remarkably, the single-valued polylogarithms that correspond to the thermal 1pt functions of the spin-1 (charge) operator $\hat{Q}$, have been seen elsewhere.

Consider the following conformal integral

$$I(x_1,x_2,x_3,x_4) = \frac{1}{\pi^2}\int d^4x \, \frac{1}{(x-x_1)^2(x-x_2)^2(x-x_3)^2(x-x_4)^2}$$

$$=$$



$$= \frac{1}{x_{13}^2 x_{24}^2}\,\Phi^1(v,u)$$

$$v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{14}^2}$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}$$

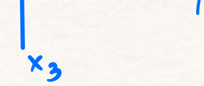
Conformal invariance allows us to take the limit.

$$\lim_{\substack{x_1\to 0\\ x_2\to z}} \left[x_{13}^2\, I(x_1,x_2,x_3,x_4)\right] = \lim \Bigg(x_{13}^2 \times$$

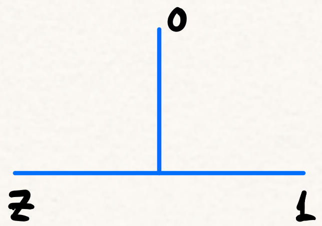
$$\Bigg) =$$

$x_3 \to \infty$

$x_4 \to 1$



$$= \frac{1}{\pi^2} \int d^4x \, \frac{1}{x^2 (x-z)^2 (x-1)^2} =$$



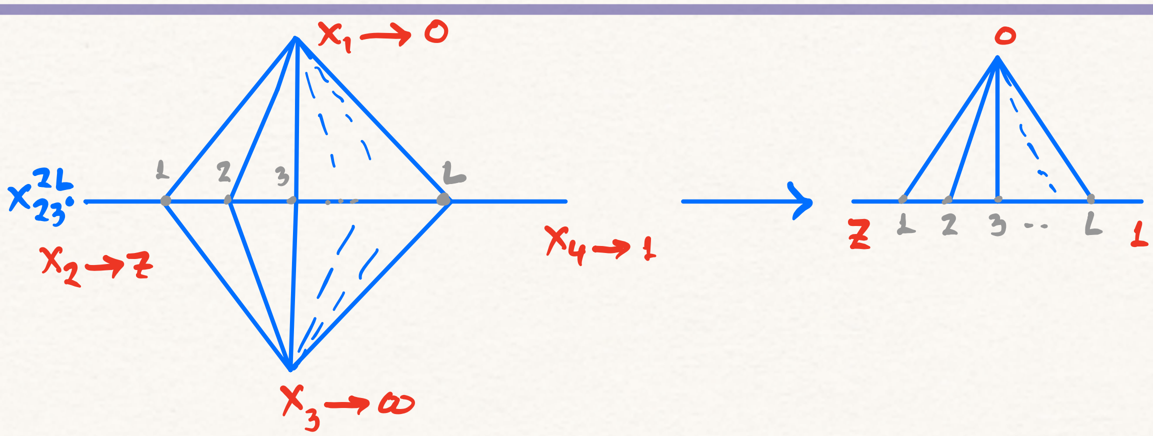
$$= \Phi_4^{(1)}(z, \bar{z}) \quad // \quad v \to z\bar{z}, \quad \frac{v}{u} \to (1-z)(1-\bar{z})$$

We find:

$$\Phi_4^{(1)}(z,\bar{z}) = \frac{1}{z-\bar{z}} \langle Q \rangle_1 \frac{1}{\alpha^2}$$

$$= \frac{1}{z-\bar{z}} \underbrace{4i \left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \ln|z| \left(\ln(1-z) - \ln(1-\bar{z}) \right) \right]}_{\text{Bloch-Wigner dilogarithm.}}$$

Similarly, higher loop conformal "ladder" graphs yield $\langle Q \rangle_L \sim a^L_{Q_L}$ e.g.



$$= \frac{1}{L!} \frac{1}{z-\bar{z}} \langle \hat{Q} \rangle_L$$

Thermal 1pt functions of spin-ℓ (charge) operators in massive free scalar theories in $d=2L+1$ dimensions, are given by L-loop conformal "ladder" graphs in $D=4$.

Remarks

- Among others, the "ladder" graphs appear in fishnet models:

$$\mathcal{L}_D = N_c \, \mathrm{Tr}\left[\Phi_1^\dagger (-\partial^2)^{\omega} \Phi_1 + \Phi_2^\dagger (-\partial^2)^{\frac{D}{2}-\omega} \Phi_2 + \alpha^2_{D,\omega} \Phi_1^\dagger \Phi_1 \Phi_2^\dagger \Phi_2\right]$$

$$\Phi_{1,2} \longrightarrow \mathrm{Ad}\, SU(N_c) \quad , \quad \omega \in \left(0, \tfrac{D}{2}\right) \quad , \quad D \in [2,4]$$

We consider the 4pt function:

$$G^L_{D,\omega}(x_i) = \langle \mathrm{Tr}\left[\Phi_2^L(x_1)\Phi_1(x_2)\Phi_2^{\dagger L}(x_3)\Phi_1^\dagger(x_4)\right]\rangle$$

it leading-N_c contribution are the D-dimensional conformal "ladder" graphs.

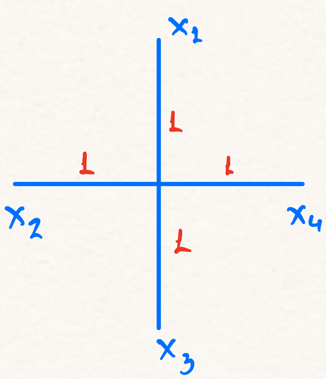
We have: $$G^L_{4,1}(z,\bar{z}) = \frac{1}{L!}\frac{1}{z-\bar{z}}\langle \hat{Q}^\ell \rangle_L$$

- Notice:

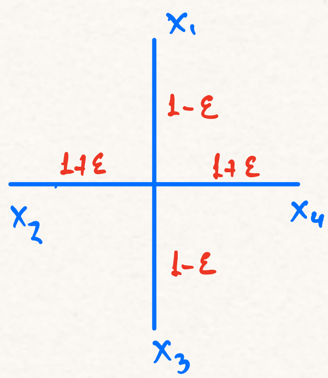
Graphs	$\longleftrightarrow$	Thermal 4pt functions
D	$\longleftrightarrow$	?? $\to$ later
L-loops	$\longleftrightarrow$	$d = 2L+L$
$x_i \to z, \bar{z}$	$\longleftrightarrow$	$m, \mu \to z = e^{-\beta m - i\beta\mu}$
$\alpha^2_{4,1}$	$\longleftrightarrow$	$\alpha^2 = \frac{e^2}{4\pi\beta^2}$

- Why the $D=4$ conformal ladder graphs have spin-1?

Hint: a little known fact



$= \lim\limits_{\varepsilon \to 0}$



$$\propto \lim_{\varepsilon \to 0} \left[\Delta^4_{2+\varepsilon}(z, \bar{z}) + C(\varepsilon)\, \Delta^4_{2-\varepsilon}(z, \bar{z}) \right]$$

conformal blocks of scalar operators with dimensions $2 \pm \varepsilon$ — conformal partial wave

They are singular as $\varepsilon \to 0$.

The singularity cancels in the sum.

in the sum

[redacted] What about the $\langle \hat{O} \rangle_L \simeq \alpha^L_{Q_s}$ spin-0 1pt functions?

Remarkably they correspond to 4pt functions of the singular $D=2$ fishnet model.

$$\boxed{G^{L}_{2,1}(z,\bar{z}) = \tilde{a}^{2L}_{2,1} \sum_{m\in\mathbb{Z}} \int d\nu \, \frac{(z\bar{z})^{i\nu}\left(\frac{z}{\bar{z}}\right)^{m/2}}{\left(\frac{m^2}{4}+\nu^2\right)^{L+1}}} \, , \quad \tilde{a}_{D,\omega} = \frac{a_{D,\omega}}{\Gamma\left(\frac{D}{2}-\omega\right)}$$

The result of the contour integration is:

$$\boxed{G^{L}_{2,1}(z,\bar{z}) = \frac{2\pi}{L!}\,\frac{1}{8\alpha^2}\,\langle \hat{O} \rangle_{L+1}}$$

Let us tie the above observations....

Constructing conformal graphs in any dimension from the twisted harmonic oscillator

The partition functions Z_L, and the thermal 1pt functions $\langle \hat{O} \rangle_L$, $\langle \hat{Q} \rangle_L$ can be constructed from the p.f. Z_0 of a twisted harmonic oscillator

$$Z_0 = \mathrm{Tr}\, e^{-\beta \hat{H}}$$

$$\hat{H} = \hat{H}_0 + m^2 \hat{O} + i\mu \hat{Q}\,, \quad \hat{H}_0 = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} \qquad [\hat{x}_i, \hat{p}_i] = i\delta_{ij}$$

$$\hat{O} = \frac{1}{2}(\hat{x}_1^2 + \hat{x}_2^2)\,, \quad \hat{Q} = \hat{p}_2 \hat{x}_1 - \hat{p}_1 \hat{x}_2$$

One obtains:

$$Z_0 = \mathrm{Tr}\, e^{-\beta[m(\hat{N}_1 + \hat{N}_2 + 1) + i\mu(\hat{N}_1 - \hat{N}_2)]}$$

$$\left(\hat{N}_i = \hat{a}_i^{\dagger} \hat{a}_i \qquad i = 1,2\right)$$

$$\Rightarrow \quad Z_0 = \frac{(z\bar{z})^{1/2}}{(1-z)(1-\bar{z})}\,, \qquad z = e^{-\beta m - i\beta\mu}$$

Hence:

$$\ln Z_0 = \ln|z| - \ln(1-z) - \ln(1-\bar{z})$$

- Define the differential operators:

$$\hat{D} = \frac{1}{\beta^2}\frac{\partial}{\partial m^2} = \frac{1}{2\ln|z|}(z\partial_z + \bar{z}\partial_{\bar{z}})$$

$$\hat{L} = \frac{i}{\beta}\frac{\partial}{\partial\mu} = z\partial_z - \bar{z}\partial_{\bar{z}}$$

Such that:

$$\boxed{\begin{aligned}\langle \hat{O}\rangle_0 &= -\beta\,\hat{D}*\ln Z_0 = \frac{1}{2m}\langle \hat{N}_1+\hat{N}_2+1\rangle_0 = \frac{\beta}{2\ln|z|}\,\frac{|z|^2-1}{(1-z)(1-\bar{z})}\\ \langle \hat{Q}\rangle_0 &= \hat{L}*\ln Z_0 = \langle \hat{N}_1-\hat{N}_2\rangle_0 = \frac{z-\bar{z}}{(1-z)(1-\bar{z})}\end{aligned}}$$

- From Z_0 to Z_L: <u>the relativistic gas</u>

Given $\ln Z_0$ we calculate $\ln Z_L$ as

$$\ln Z_L = \int d\omega\, \rho_L(\omega; m)\, \ln Z_0$$

where $\rho_L(\omega;m)$ is the one-particle density-of-states

→ Consider the system (i.e. the relativistic thermal gas) in a spatial volume $V_{d-1} = \ell^{d-1} \equiv \ell^{2L}$ with quantized momentum

$$\vec{p} = \left(\frac{2\pi}{\ell}n_1, \dots, \frac{2\pi}{\ell}n_{d-1}\right) = \frac{2\pi}{\ell}\vec{n}$$

The number of modes with momenta inside the spherical shell with radii $|\vec{p}|$ & $|\vec{p}|+d|\vec{p}|$ is

$$dn = \frac{2\pi^L}{\Gamma(L)}\left(\frac{\ell^2}{4\pi^2}\right)^L |\vec{p}|^{2L-1}\, d|\vec{p}|$$

From $\omega^2 = |\vec{p}|^2 + m^2$, we obtain

$$\rho_L(\omega; m) = \frac{2\alpha^2\beta^2}{\dots}\,\omega(\omega^2-m^2)^{L-1}$$

$$(L-1)!$$

$$\Rightarrow \boxed{\ln Z_L = \frac{2\alpha^{2L} b^{2L}}{(L-1)!} \int_m^\infty \omega\, d\omega\, (\omega^2 - m^2)^{L-1} \ln Z_0}$$

Acting on $\ln Z_L$ with $\hat{D}$, $\hat{L}$ we obtain expressions for $\langle \hat{O} \rangle_L$ & $\langle \hat{O} \rangle_L$ as:

$$\boxed{\begin{pmatrix} \langle \hat{O} \rangle_L \\ \langle \hat{Q} \rangle_L \end{pmatrix} = 2\alpha^{2L} b^{2L} \int_m^\infty \omega\, d\omega\, \frac{(\omega^2 - m^2)^{L-1}}{(L-1)!} \begin{pmatrix} \langle \hat{O} \rangle_0 \\ \langle \hat{Q} \rangle_0 \end{pmatrix}}$$

- This coincides with known integral representations of conformal ladder graphs.

[redacted] Properties of ladder graphs

$$\boxed{\begin{aligned} \langle \hat{O} \rangle_L &= -b\, \hat{D} * \ln Z_L = b\alpha^2 \ln Z_{L-1} \\ \langle \hat{Q} \rangle_L &= \hat{L} * \ln Z_L = -\frac{1}{\alpha^2} \hat{D} * \langle \hat{Q} \rangle_{L+1} \end{aligned}}$$

— A second-order equation:

$$\hat{\Delta} = 4b^2 z\bar{z}\, \partial_z \partial_{\bar{z}} = \frac{\partial^2}{\partial m^2} + \frac{\partial^2}{\partial \mu^2}$$

$$\hat{\Delta} * \begin{pmatrix} \langle \hat{O} \rangle_L \\ \langle \hat{Q} \rangle_L \end{pmatrix} = -4L\alpha^2 b^2 \begin{pmatrix} \langle \hat{O} \rangle_{L-1} \\ \langle \hat{Q} \rangle_{L-1} \end{pmatrix}$$

$$\langle Q\rangle_L \qquad\qquad \langle Q\rangle_{L-1}$$

The above are $1^{\underline{st}}$ and $2^{\underline{nd}}$ order novel recursive relations for conformal ladder graphs in $D=2,4$

They can be combined into a unique equation:

$$\boxed{\left(\hat{\Delta} - 4L^2 6^2 \hat{D}\right) * \begin{pmatrix} \langle \hat{O}\rangle_L \\ \langle \hat{Q}\rangle_L \end{pmatrix} = 0}$$

$\rightarrow$ This may be interpreted as the Laplace-Beltrami equation on AdS_{2L+2} with metric:

$$ds^2 = \frac{1}{m^2}\left(dm^2 + d\mu^2 + \sum_{i=1}^{2L} dx^i dx^i\right)$$

for functions of m, μ only.

■ Pause- Recap

- Starting from $\ln Z_0$ we constructed $\ln Z_L$ and then $\langle \hat{O}\rangle_L$, $\langle \hat{Q}\rangle_L$: the conformal ladder graphs in $D=2,4$.

- We have found novel differential recursive relations among the conformal ladder graphs
We have found a novel $2^{\underline{nd}}$ order eq. for them.

- The operator $\hat{D}$ <u>lowers</u> L. We can define its inverse that <u>raises</u> L.

$$\hat{D} = \frac{1}{\beta^2}\frac{\partial}{\partial m^2} \longrightarrow \hat{D}^{-1} \equiv \hat{d} = 2\beta^2 \int_m^\infty \omega \, d\omega$$

$$\Rightarrow \qquad \hat{d} * \langle \hat{O} \rangle_L = -\frac{1}{\alpha^2} \langle \hat{O} \rangle_{L+1}$$

$\Rightarrow$ Starting from $\ln Z_0$ we construct $\ln Z_L$ by repeating actions of the integral operator $\hat{d}$

$$\frac{1}{\beta\alpha^2}\langle \hat{O} \rangle_{L+1} = \ln Z_L = -\alpha^2 \hat{d} * \ln Z_{L-1}$$

$$= (-\alpha^2)^L (\hat{d})^L * \ln Z_0$$

■ We constructed the L-loop, $D=2$ conformal graphs from $\ln Z_0$.

Consider:

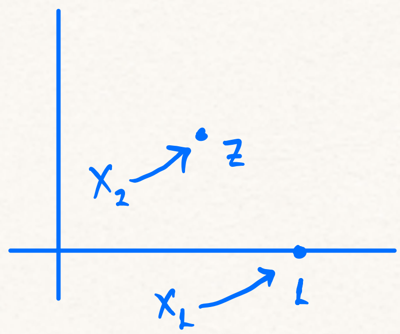
$$\hat{L} * \ln Z_0 = \langle \hat{Q} \rangle_0 = \langle \hat{N}_1 - \hat{N}_2 \rangle = \frac{z - \bar{z}}{(1-z)(1-\bar{z})}$$

$$= (z - \bar{z}) \cdot q_0^{(1)}$$

with $\boxed{q_0^{(1)} = \frac{1}{|1-z|^2}}$

This "is" the massless free 2pt function in $D=4$

$$\langle \varphi(x_1)\varphi(x_2)\rangle = \frac{1}{(x_1-x_2|^{D/2-1}} \equiv \frac{1}{|x_1-x_2|^2} \longrightarrow \frac{1}{|1-z|^2}$$



$\longrightarrow$ Better think of it as a "singular" 4pt function

$$\langle \Phi_1^{+,L}(x_1)\,\Phi_2^{+}(x_2)\,\Phi_1^{L}(x_3)\,\Phi_2(x_4)\rangle\Big|_{L=0} \xrightarrow[x_3\to\infty \;\; x_4\to 1]{x_1\to 0 \;\; x_2\to z} \langle \Phi_2^{+}(z)\,\Phi_2(1)\rangle$$

$$= \frac{1}{|1-z|^{D-2}} \qquad (D=4)$$

[redacted] We can raise L acting with $\hat{d}$ as:

$$\hat{D} * \langle \hat{Q}\rangle_L = -\alpha^2 \langle \hat{Q}\rangle_{L-1} = -\alpha^2 \hat{L} * \ln Z_{L-1}$$

$$= -\alpha^2 \hat{L} * \left[(-\alpha^2)^{L-1} (\hat{d})^{L-1} * \ln Z_0\right]$$

$$= (-\alpha^2)^L \cdot (\hat{d})^{L-1} * \hat{L} * \ln Z_0$$

$$\Rightarrow \quad \boxed{\langle \hat{Q} \rangle_L = (-\alpha^2)^L (\hat{d})^L * \left[(z-\bar{z}) \cdot q_0^{(1)}\right]}$$

Thus, we construct the L-loop, $D=4$ ladder graphs from $\ln Z_0$.

Reflect:

Having at hand $\hat{D}$, $\hat{d}$, $\hat{L}$ we can generalize!

We find:

$$\left(\frac{1}{z-\bar{z}} \hat{L}\right)^k * \ln Z_0 = \frac{1}{(z-\bar{z})^k} (\hat{L}-k+1)(\hat{L}-k+2)\cdots \hat{L} * \ln Z_0$$

$$\langle \hat{S} \rangle_0 = \langle \hat{N}_1 - \hat{N}_2 \rangle_0$$

$$= \frac{1}{(z-\bar{z})^k} \langle \hat{S}(\hat{S}-1)\cdots(\hat{S}-k+1) \rangle_0$$

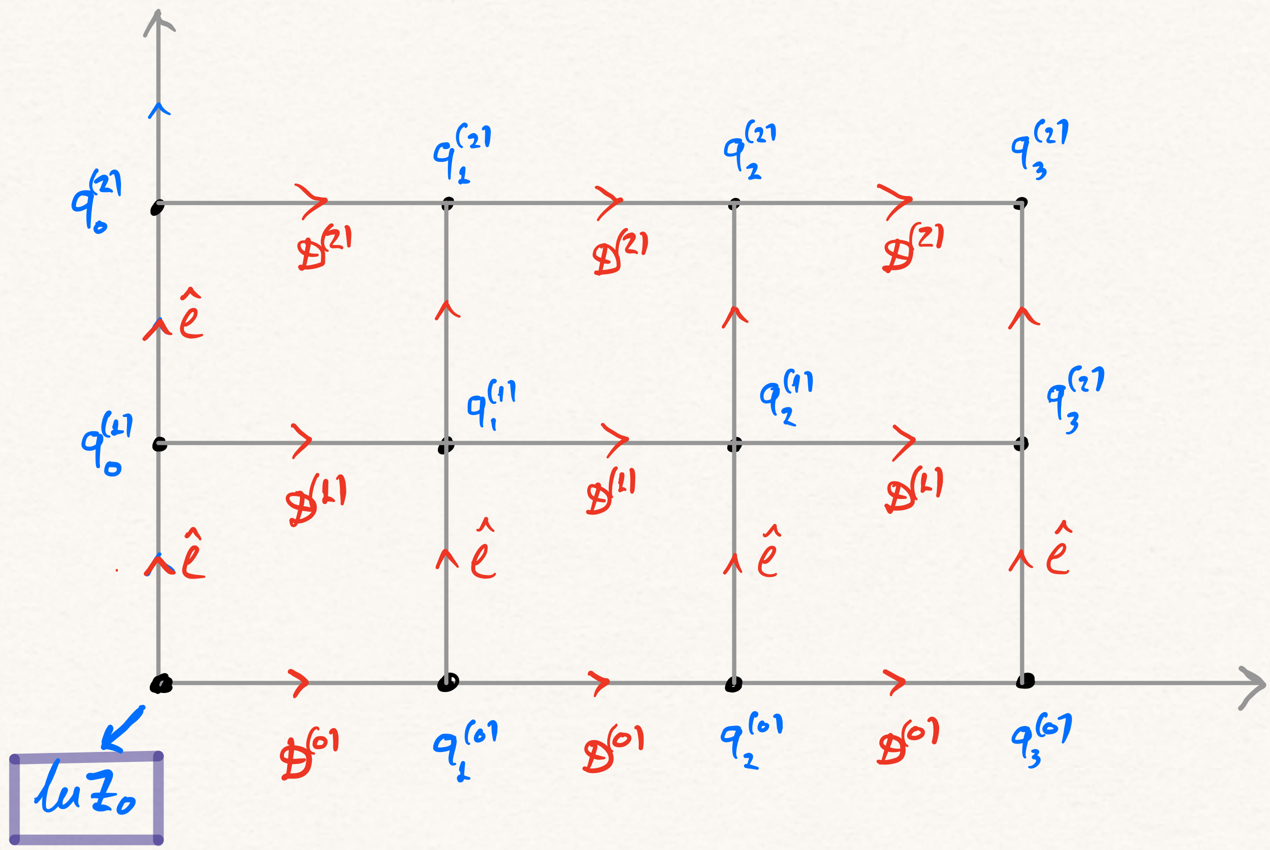
$$= \frac{1}{|1-z|^{2k}} \equiv q_0^{(k)}$$

Setting $2k = D-2 \Rightarrow \boxed{k = \frac{D}{2} - 1}$

we can identify $q_0^{(k)}$ with the "singular" 4pt function in D-dimensions!

Then we can use $\hat{d}$ to obtain the L-loop ladder graphs in D-dimensions.

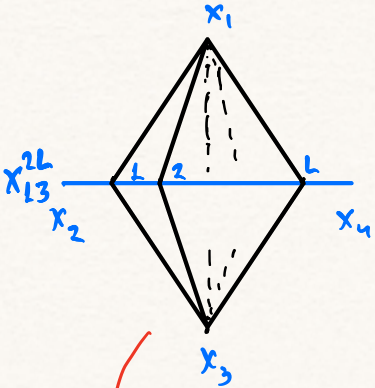
The general construction



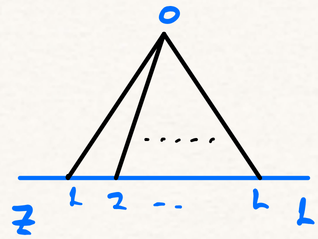
$$\boxed{\hat{e} = \frac{\hat{L}}{z-\bar{z}}\,, \qquad \mathcal{D}^{(k)} = \frac{1}{(z-\bar{z})^k}\left[\hat{d}\,(z-\bar{z})^k\right]}$$

Remarks:

- We have constructed all L-loop conformal ladder graphs in $D = 2k+2, \quad k = 0, 1, 2, \dots$



$\longrightarrow$



e.g. $L=2$

$$\int d^Dx\, d^Dy \, \frac{x_{13}^4}{(x_1-x)^2(x_1-y)^2(x_3-x)^2(x_3-y)^2(x_2-x)^{D-2}(x-y)^{D-2}(x_4-y)^{D-2}} \xrightarrow{\lim}$$

$$\longrightarrow \int d^Dx\, d^Dy \, \frac{1}{x^2\, y^2\, (z-x)^{D-2}(x-y)^{D-2}(y-1)^{D-2}}$$

- We learn that the conformal ladder graphs have can be assigned <u>two indices</u>

Dimension $D=2k+2$ of the integral $\longleftarrow$ $g^{(k)}_L$ $\longrightarrow$ "twist"

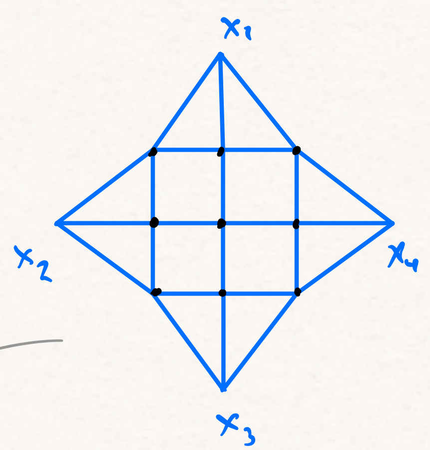
Loop order $\longleftarrow$ $g^{(k)}_L$ $\longrightarrow$ dimension $d=2L+1$ of the underlying massive free theory

Conformal 4pt graphs ⟵ Correspondence ⟶ "Thermal" partition functions

Further remarks (extra fun...)

☒ Conformal fishnet graphs (Basso-Dixon)

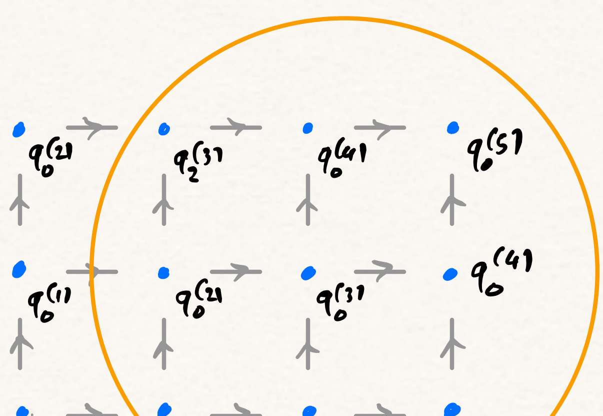
e.g.



⟶ order $(a^2)^9$ is given as a product of ladder graphs.

"to be confirmed": corresponds to subdeterminants of a cubic lattice.

e.g. in $D=2$



$\rightarrow \; = \hat{D}^{(0)}$

$q_0^{(k)} \sim \alpha^{2k}$

$\ln z_0 \qquad q_0^{(1)} \qquad q_0^{(2)} \qquad q_0^{(3)}$

☒ Resummation of infinite loops

Consider the quantity

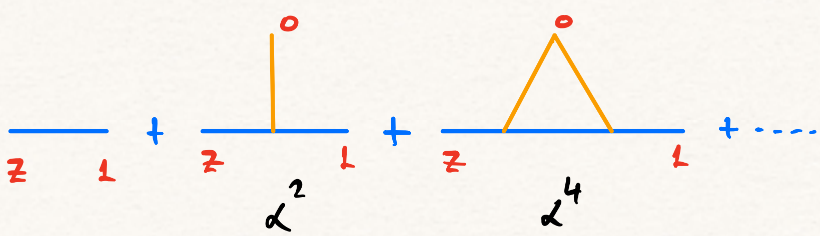
$$\boxed{\begin{aligned}\Phi(x_1,x_2,x_3,x_4) &= \sum_{L=0}^{\infty} \mathrm{Tr}\left[\langle (\varphi_1^{\dagger})^L(x_1)\, \varphi_2^{\dagger}(x_2)\, (\varphi_1)^L(x_3)\, \varphi_2(x_4)\rangle\right] x_{13}^{2L} \\ &= \mathrm{Tr}\langle \varphi_2^{\dagger}(x_2)\, \varphi_2(x_4)\rangle + \mathrm{Tr}\langle \varphi_1^{\dagger}(x_1)\, \varphi_2^{\dagger}(x_2)\, \varphi_1(x_3)\, \varphi_2(x_4)\rangle\, x_{13}^2 + \cdots\end{aligned}}$$

In the limit $x_1 \to 0,\ x_2 \to z,\ x_3 \to \infty,\ x_4 \to 1$

this resums the infinite set of conformal ladder graphs

for any $D = 2k+2$

i.e. $\langle \varphi_1^{\dagger}(x_1)\, \varphi_1(x_2)\rangle = \dfrac{1}{|x_{12}|^2}$, $\langle \varphi_2^{\dagger}(x_1)\, \varphi_2(x_2)\rangle = \dfrac{1}{|x_{12}|^{D-2}}$

$\Phi(x_1,\dots,x_4) =$ 

e.g. $D=4$

$$= \frac{1}{z-\bar{z}}\left[q_0^{(1)} + q_1^{(1)} + \frac{1}{2!} q_2^{(1)} + \frac{1}{3!} q_2^{(1)} + \cdots\right]$$

But

$$q_L^{(1)} = 2\alpha^{2L}\beta^{2L}\int^{\infty} w\, dw\, \frac{(w^2-m^2)^{L-1}}{(L-1)!}\, q_0^{(1)}$$

M

$$\Rightarrow \boxed{Q^{(1)} = \sum_{L=0}^{\infty} \frac{1}{L!} q_0^{(L)} = q_0^{(1)} + 2\lambda^2\sigma^2 \int_M^{\infty} w\,dw\, \frac{I_1\left[2\sqrt{\lambda^2\sigma^2 w^2 - \lambda^2\sigma^2 \omega^2}\right]}{\sqrt{\lambda^2\sigma^2 w^2 - \lambda^2\sigma^2 m^2}}\, q_0^{(1)}}$$

$$\sum_{L=1}^{\infty} \frac{z^{L-1}}{L!\,(L-1)!} = \frac{1}{\sqrt{z}} I_1[2\sqrt{z}] \rightsquigarrow \text{modified Bessel}$$

☒ Relation to large-charge results:

[Cuomo and Hyman (20)]

The calculation of the 2pt function of large-charge operators gives schematically.

$$\langle Q(x_1) Q(x_2) \rangle \simeq \int [\mathcal{D}\sigma]\, e^{-N\left(\frac{1}{2}\mathrm{Tr}\ln(-\partial^2+\sigma) - c\ln G(x_1,x_2;\sigma)\right)}$$

[Essentially one uses the Hubbard–Stratonovich effective action $S_{eff} \sim \int \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}\sigma\varphi^2$]

The Green's function $G(x_1, x_2; \sigma)$ is the solution to:

$$\left(-\partial_1^2 + \sigma(x_1)\right) G(x_1, x_2; \sigma) = \delta^d(x_1 - x_2)$$

Using the Born expansion:

$$G(x_1, x_2; \sigma) = G^{(0)}(x_1, x_2) + G^{(1)}(x_1, x_2; \sigma) + \cdots$$

$$-\partial^2 G^{(0)}(x_1,x_2) = \delta^d(x_1-x_2)$$

$$-\partial^2 G^{(L+1)}(x_1,x_2) = -6(x_1)\, G^{(L)}(x_1,x_2;6)$$

we find: $\left[\partial_x^2 \frac{1}{|x-y|^{d-2}} = -\frac{4\pi^{d/2}}{\Gamma(d/2-1)}\,\delta^d(x-y)\right]$

$$G^{(0)}(x_1,x_2) = C(d)\,\frac{1}{|x_1-x_2|^{d-2}}$$

$$G^{(1)}(x_1,x_2;6) = \int d^dx\, \frac{C(d)}{|x-x_1|^{d-2}}\cdot 6(x_1)\,\frac{C(d)}{|x_1-x_2|^{d-2}}$$

$$G^{(2)}(x_1,x_2;6) = \int d^dx\, \frac{C(d)}{|x-x_1|^{d-2}}\; G^{(2)}(x_1,x_2;6)$$

$\vdots$

Remarkably if one chooses

$$6(x) = \frac{6}{x^2} \qquad x_1 \to z,\quad x_2 \to 1$$

then $G(x_1,x_2;6) \longrightarrow G(z,\bar z) \propto \Phi(x_1,x_2,x_3,x_4)$

with $\alpha^2 = C(d)\cdot 6$

The result of the summation coincides with our result.

This is also supposed to give a HHLL 4pt function

$\Rightarrow$ Conjecture:

Since HHLL 4pt functions are presumably expanded in thermal expectation values, we suggest that our results provide a working example of the ETH.

☒ Hyperbolic-hyper-geometry ?

$$\langle Q\rangle_1 = 4i\alpha^2 \underbrace{D(z)}$$

→ Bloch-Wigner dilogarithm.
It gives the volume of an ideal tetrahedron in 3d $\mathbb{H}^3$ hyperbolic space with vertices on $\partial\mathbb{H}^3$.

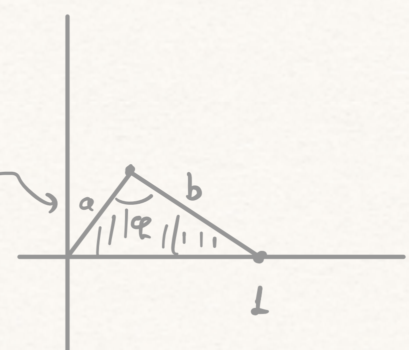
Amusingly $\langle Q\rangle_0$ also has a geometric interpretation

$$\langle Q\rangle_0 = \frac{z-\bar{z}}{(1-z)(1-\bar{z})} \quad : \quad z = \frac{b}{a}e^{i\varphi}$$

$$\cos\varphi = \frac{a^2+b^2-1}{2ab}$$

$$\Rightarrow \langle Q\rangle_0 = 4i\left(\frac{1}{2}ab\sin\varphi\right)$$

Area.



⇒ We find that the hyperbolic tetrahedron volume comes from the integration of the above triangle. known ?

What about interpreting $\langle Q\rangle_L$ as volumes on hyper-hyperbolic spaces ?

☒ Phase space of relativistic particle decay.

$$\langle Q\rangle_0 = -8\pi \underbrace{\rho^{D=4}_{m_0 \to m_1 + m_2}}$$

D-dimensional $1 \to 2$ decay phase space of relativistic massive particles

$$\rho^{D=4}_{m_0 \to m_1 + m_2} = \frac{1}{8\pi}\sqrt{\lambda\left(1, \frac{m_1^2}{m_0^2}, \frac{m_2^2}{m_0^2}\right)}$$

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

(Heron's formula)

$\downarrow$

Källen function

For $a = \frac{m_1}{m_0}$, $b = \frac{m_2}{m_0}$

$\langle Q\rangle_0$ represents a virtual process with $\underline{\lambda < 0}$.

Then,

$$\langle Q\rangle_L \sim \int_m^\infty \omega\, d\omega \, \frac{(\omega^2 - m^2)^{L-1}}{(L-1)!} \langle Q\rangle_0$$

is closely related to a recurrence formula for relativistic phase spaces in different dimensions

[Delbourgo and Roberts (03)]

ありがとう