Exploring Aspects of Large Charge Convexity

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Large Charge Convexity Conjecture

- Consider a modification of the Weak Gravity Conjecture:
  - That some particle should have non-negative binding energy
  - Binding energy also from contact terms

- In AdS, the CFT dual is a constraint on the spectrum:

Aharony, Palti 2108.04594

$$\Delta\left(n_1 q_0 + n_2 q_0\right) \geq \Delta\left(n_1 q_0\right) + \Delta\left(n_2 q_0\right)$$

- For q_0 not necessarily minimal, but still “order one”
- Since discrete, is really superadditivity

Microscopic non-convexity

- If q_0 had to be the minimal charge, would certainly not be true
- Consider the simplest interacting Wess-Zumino model:

Aharony, Palti 2108.04594
Hellerman, Orlando, Reffert, Watanabe 1505.01537

$$W = g\Phi^3$$

- R-charges of fields are:
  - +2/3 for scalar
  - -1/3 for fermion
- In 4d: $\Delta(1/3) = 3/2$, while $\Delta(2/3) = 1$ (not convex) $\rightarrow$ take $q_0 = 2/3$

The meaning of “order one”

Sharon, Watanabe 2301.08262

- Counterexample proposed with “clockwork-like” system of charges $(1, 3, \ldots, 3^N)$
- Due to microscopic non-convexity, convexity does not start until the largest unit of charge
  - $q_0 = 3^N$
- Lesson is that large N can delay convexity

Goal

- Would like to explore the large charge regime of a CFT with microscopic non-convexity due to a boson
  - Need at least two scalars of different charge

- Why:
  - Possible phase transition between semi-classical saddles?
  - Saddles with multiple scalar VEVs?
  - Multiple Goldstone embeddings?
  - Large charge restoration of microscopic convexity?

The simplest model

Model suggested in: Orlando, Palti 2303.02178

- Consider the following (Euclidean) theory in 4-ε dimensions:

$$\mathcal{L} = \sum_{i=1,2} \left(|\partial\phi_i|^2 + \frac{\lambda_i}{4}|\phi_i|^4 \right) + a|\phi_1|^2|\phi_2|^2 + \frac{b}{6}\phi_1\phi_2^{\dagger 3} + h.c.$$

- Will normalize the charges as 1 and 1/3 respectively
- Has a single non-trivial (and non-decoupled) fixed point:

$$16\pi^2\epsilon = 1$$

$$\lambda_1 \approx 0.155$$
$$\lambda_2 \approx 0.147$$
$$a \approx 0.093$$
$$b \approx 0.046$$

Notice the approximate exchange symmetry of this solution!

Preparing the description

- First do a conformal mapping to the cylinder:

$$\mathcal{L} = \sum_{i=1,2} \left(|\partial\phi_i|^2 + m^2|\phi_i|^2 + \frac{\lambda_i}{4}|\phi_i|^4 \right) + a|\phi_1|^2|\phi_2|^2 + \frac{b}{6}\phi_1\phi_2^{\dagger 3} + h.c.$$

- With conformal mass m being the inverse radius R, soon set to 1
- Next write fields with polar variables:

$$\mathcal{L} = \sum_{i=1.2} \left(\frac{1}{2}(\partial\rho_i)^2 + \frac{1}{2}\rho_i^2(\partial\chi_i)^2 + \frac{1}{2}m^2\rho_i^2 + \frac{\lambda_i}{16}\rho_i^4 \right) + \frac{a}{4}\rho_1^2\rho_2^2 + \frac{b}{12}\rho_1\rho_2^3\cos(\chi_1 - 3\chi_2)$$

$$\phi_i = \frac{\rho_i}{\sqrt{2}}e^{i\chi_i}$$

Only non-singular when ρ_i is non-vanishing!

Preparing the description

- Finally perform the following change of variables:

$$n\chi = n_1\chi_1 + n_2\chi_2$$
$$\omega = \chi_1 - 3\chi_2$$

- Where $n = n_1 + n_2/3$
- This ensures that χ has the interpretation of Goldstone:

$$Q = -i\int d\Omega_{d-1}\left(\partial_{\chi_1(\vec{n})} + \frac{1}{3}\partial_{\chi_2(\vec{n})}\right)$$
$$= -i\int d\Omega_{d-1}\,\partial_{\chi(\vec{n})}$$

Calculation of scaling dimension

- For each charge n, we should pick an ansatz wavefunction to evolve in Euclidean time
  - Same method as used for model of single field

Badel, Cuomo, Monin, Rattazzi 1909.01269

$$\langle\psi_n|e^{-HT}|\psi_n\rangle$$

- The lowest energy eigenvalue is then Δ/R
- We will make the convenient choice:

Constants to be fixed later

$$|\psi_{n_1,n_2}\rangle = \int \mathcal{D}\chi(\vec{n}) \exp\left[i\frac{n}{R^{d-1}\Omega_{d-1}}\int d\Omega_{d-1}\,\chi(\vec{n})\right]|\rho_1 = f_1, \rho_2 = f_2, \chi(\vec{n}), \omega = 0\rangle$$

- Saddle point Neumann and Dirichlet boundary conditions

Calculation of scaling dimension

- Look for constant solutions $\rho_1 = f_1$, $\rho_2 = f_2$, $\omega = 0$
- Bulk EOMs for χ_i are:

$$\ddot{\chi}_i = 0$$

- The χ Neumann condition from wavefunction fixes:

$$\chi_1 = -i\mu\tau\,, \qquad \chi_2 = -\frac{1}{3}i\mu\tau$$

$$\mu\left(\rho_1^2 + \frac{\rho_2^2}{9}\right) R^{d-1}\Omega_{d-1} = n$$

- Familiar superfluid form for χ

Calculation of scaling dimension

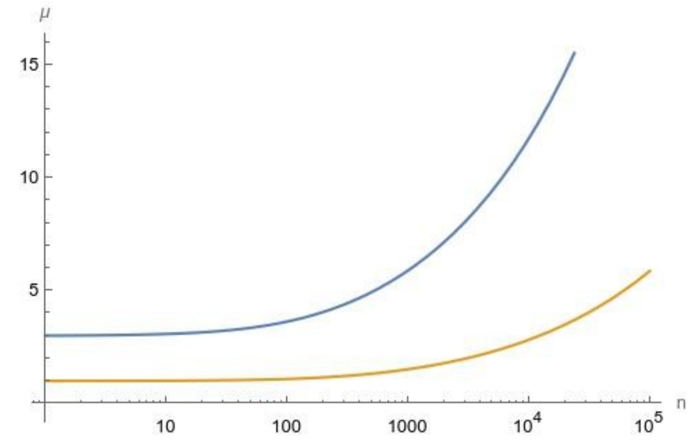
- Finally the bulk EOMs for ρ_i are:

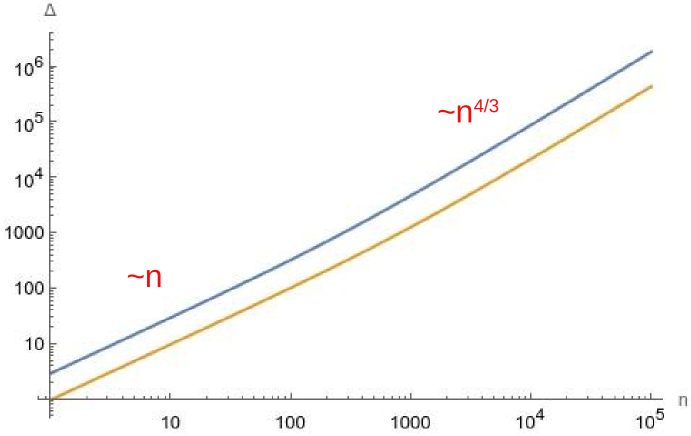
$$0 = \left(-\mu^2 + m^2 + \frac{a}{2} f_2^2\right) f_1 + \frac{\lambda_1}{4} f_1^3 + \frac{b}{12} f_2^3$$

$$0 = \left(-\frac{\mu^2}{9} + m^2 + \frac{a}{2} f_1^2\right) f_2 + \frac{\lambda_2}{4} f_2^3 + \frac{b}{4} f_1 f_2^2$$

- Second equation implies a solution with $f_2 = 0$ (the previously found one-field solution for φ_1)
- In the exchange symmetric limit ($b = 0$, $\lambda_1 = \lambda_2$), this leads to a similar solution for just φ_2 (with $\mu \rightarrow \mu/3$)
  - Broken so solution is $f_2 \neq 0$ and $f_1 \neq 0$ but small (by $\sim 10^{-2}$ at large charge)

Results





- Expected transition to large regime at $\varepsilon n \approx 1$
- φ_2 action is larger than φ_1 action by factor of 3 at small charge → really map onto φ_1^n and φ_2^n series!
  - Makes sense because require more φ_2's to get same charge
- Factor becomes $3^{4/3}$ (in exchange symmetric limit) → φ_2 saddle interpretation not so clear now

Interpretation

- So φ_1 saddle is always dominant (no phase transition)
  - But here $f_2 = 0 \rightarrow$ polar change of variables singular
- Thus only applicable to integer-charge states
  - Created by acting on vacuum with φ_1^n

- What about third-integer-charge states?

Interpretation

- Rather than being associated with the large-action φ_2 saddle, act on φ_1 saddle with φ_2 (quantum, no purely semi-classical description)
- On the large charge saddle, the mixing term acts as a mass

$$a|\phi_1|^2|\phi_2|^2$$

- One obtains:

$$m_2 = \sqrt{\frac{2a}{\lambda_1}}\mu \approx 1.095\,\mu$$

Compare:

$$\Delta(n) = \frac{6\pi^2}{\lambda_1}\left(\frac{\lambda_1 n}{8\pi^2}\right)^{4/3}$$

$$\Delta(n+1) - \Delta(n) \approx \left(\frac{\lambda_1 n}{8\pi^2}\right)^{1/3} = \mu R$$

Interpretation

- Since the φ_2 mass is larger than the gap:

$$|n+1/3\rangle = \phi_2|n\rangle \qquad |n+2/3\rangle = \phi_2^\dagger|n+1\rangle \neq \phi_2^2|n\rangle$$

- Spectrum not monotonic, let alone convex
  - But notice it could have been both if at large charge $m_2 < \mu/2$
  - Also must have $m_2 > \mu/3$ for φ_1 saddle to be dominant
- Does some other theory lie in this window? Interpretation?

Summary

- Within convex envelope, bosons can also lead to microscopic non-convexity
  - Restoration seems possible (though no examples known)
- The saddle of the largest-charge field is always dominant
  - Should think of fractional-charge states as quantum fluctuations around saddle
  - No phase transition
  - Not a proof but appears universal

Convexity: Lessons from the EFT

- Now only focus on semiclassical envelope
- When EFT is gapped, convexity guaranteed:
  - $\Delta \sim Q^{d/(d-1)}$
- Less straight forward when there is a light mode:

Orlando, Palti 2303.02178

$$\mathcal{L}_{\pi_0, r_0} = \frac{1}{2}\alpha\dot{\pi}_0^2 + 2m\beta r_0\dot{\pi}_0 + \frac{1}{2}\dot{r}_0^2 - \frac{1}{2}\gamma r_0^2$$

Goldstone Light mode

- Theory can still be stable for γ < 0 due to "magnetic" cross term
  - In this case, power of Q is sub-unity (concave)

Building superfluid EFTs

- Since Weyl + Diff → Conformal, first build Weyl invariant metric:

$$\hat{g}_{\mu\nu} = (g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi)g_{\mu\nu}, \text{ with } \chi = \mu t + \pi$$

- Then write down everything allowed:

$$\begin{aligned} S &= \int d^d x\, \sqrt{\hat{g}} \left(c_1 + c_2 \hat{R} + \cdots\right) \\ &= \int d^d x\, \sqrt{g} |\partial\chi|^d \left(c_1 + c_2 \frac{R}{(\partial\chi)^2} + \cdots\right) \end{aligned}$$

- Legendre transform makes leading $Q^{d/(d-1)}$ behavior clear

Marginally convex case

- First look at theories with exactly massless mode
  - Due to this relation, should have constant μ:

$$\frac{\partial^2 \Delta}{\partial Q^2} = \frac{\partial \mu}{\partial Q}$$

$$\mathcal{L} = (\partial\chi)^d \left[c_1 - \phi^2 \left(1 - c_2 \frac{m^2}{(\partial\chi)^2} \right) + \cdots \right]$$

Kinetic terms, etc.

- Notice how the massless φ acts as a Lagrange multiplier

$$\begin{aligned} \Delta &= \mu Q - L \\ &= \mu Q - c_1 \Omega_{d-1}\, \mu^d \end{aligned}$$

- Subleading c_1 unconstrained: if negative → subadditive

(Line from origin to a point on the graph is above the graph to the right)

Using same idea, Gabriel has constructed EFTs with negative mass for light mode

Conclusions

- These conjecture-violating EFTs are peculiar (non-generic), and likely have no UV completion

- Nevertheless, they are internally consistent
  - It does not seem possible to prove the conjecture from within the EFT