A variety of partially solvable models: From closed spin chains to open spin chains

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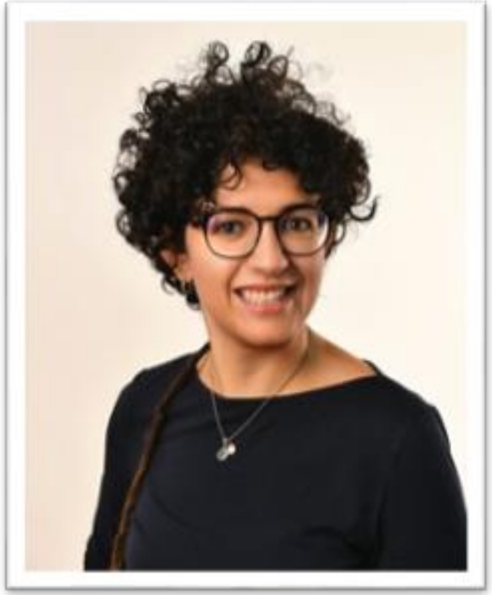
Focus Week on Non-equilibrium Quantum Dynamics

@IPMU, Sep. 30 – Oct. 04

Chihiro Matsui (The University of Tokyo)

Collaborators

Hosho Katsura
(The University of Tokyo)



Chiara Paletta
(Ljubljana University)

Balazs Pozsgay
(Eötvös Loránd University)

Naoto Tsuji
(The University of Tokyo)

Outline

- What is "partial solvability"?
  - Definition of partial solvability
  - Thermalization & quantum many-body scars (QMBS)
- Closed partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Open partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Concluding remarks

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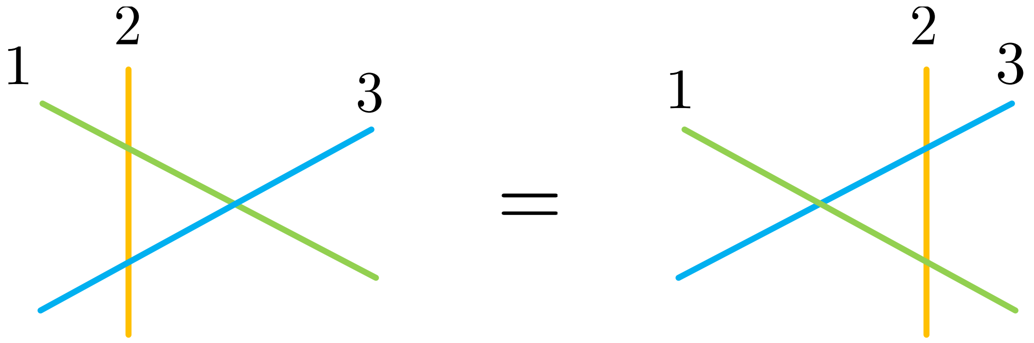
Integrability

- (Quantum) Integrable systems
  - No clear definition.
  - Often said to be “integrable” if the Yang-Baxter structure exists.

$$R_{12}(\lambda_1,\lambda_2)R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3) = R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3)R_{12}(\lambda_1,\lambda_2)$$

$$R_{12}, R_{23}, R_{13} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$$



Decomposing a many-body scattering into a sequence of two-body scatterings does not depend on the way of decomposition.

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$$R_{12}, R_{23}, R_{13} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$$

  - Never thermalize.

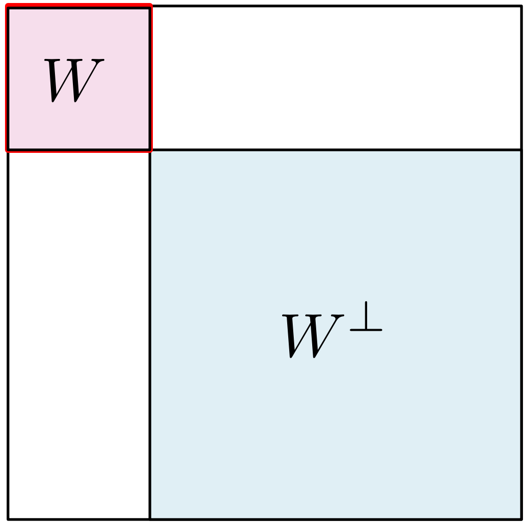
⇒ Violation of (strong) eigenstate thermalization hypothesis (ETH) ≒ Typicality

$$\lim_{N\to\infty} \langle E_a|X_{\mathrm{macro}}|E_a\rangle = \langle X_{\mathrm{macro}}\rangle_{\mathrm{MC}}; \quad \forall E_a \in (E-\delta E, E]$$

[Deutsch (1991), Srednicki (1994),…]
[Biroli et al. (2010), Iyoda et al. (2017)]

Partial Solvability

- Partially solvable systems
  - Hamiltonians with some solvable energy eigenstates (not all).
  - Hamiltonians with the block diagonal structure.



$$\mathcal{H} \simeq W \oplus W^{\perp}$$

Solvable (invariant) subspace

⇒ Solvability does not necessarily come from integrability.

e.g. Projector embeddings [Shiraishi et al. (2017)]

Restricted spectrum generating algebra (rSGA) [Moudgalya et al. (2018)]
[Vefek et al. (2017), Moudgalya et al. (2020)]

Hilbert space fragmentation (HSF) [Pai et al. (2019), Sala et al. (2020), Khemani et al. (2020)]

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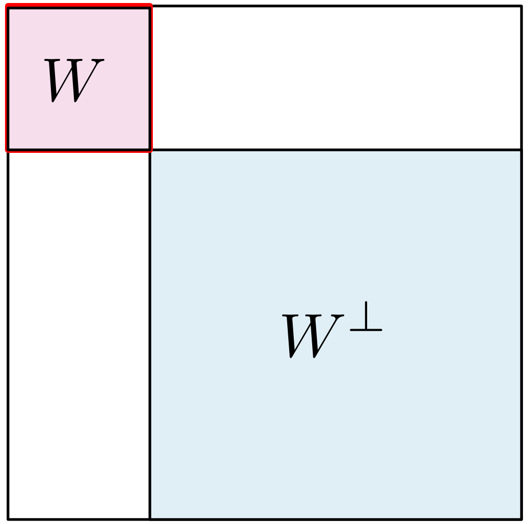
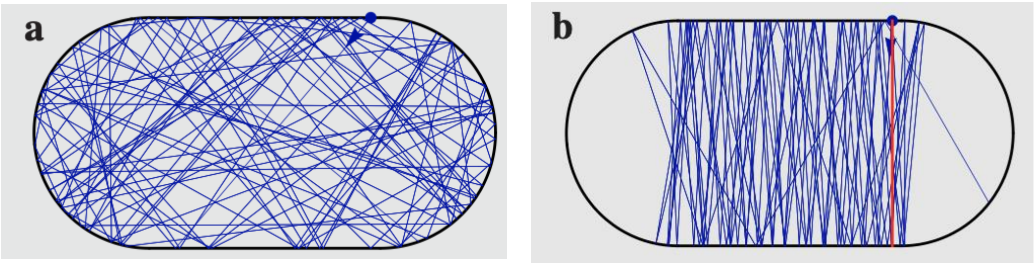


[Shiraishi et al. (2017), Turner et al. (2017), Bernien et al. (2017)]

  - Partially non-thermalize. "Quantum many-body scars (QMBS)"

    ⇒ Weakly violate ergodicity in Hilbert space. ≒ Principle of equal probability

e.g. Scar in stadium billiard [Serbyn et al. (2021)]



a. starting away from unstable periodic trajectory

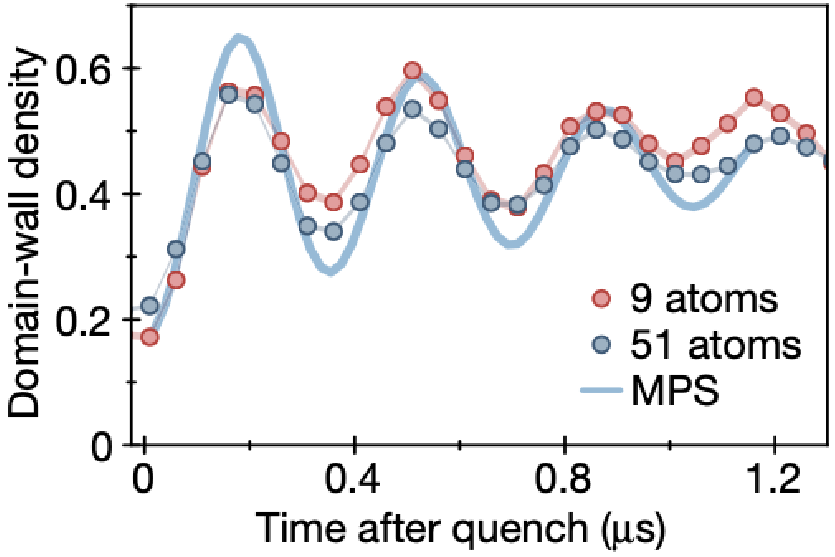
b. starting from near unstable periodic trajectory

Thermalization & QMBS

- QMBS exhibit
  - Persistent oscillation in local observables.
  - Relatively small entanglement entropy $\sim o(V)$ compared to those of thermal states $\sim O(V)$.
- Matrix product states
  - Have entanglement entropy estimated by their bond dimensions χ from above.

$$S_{\mathrm{EE}} := -\mathrm{tr}\left(\rho' \log \rho'\right) \leq \log \chi$$

  - With a finite bond dimension is a good benchmark for finding QMBS.



Domain-wall density after the quench from $|\mathbb{Z}_2\rangle = |\bullet \circ \bullet \circ \cdots\rangle$ on the Rydberg atom chain.

(●: Excited state; ○: Ground state)

[Bernien et al. (2017); Nature 551, 579 (Fig. 6b)]

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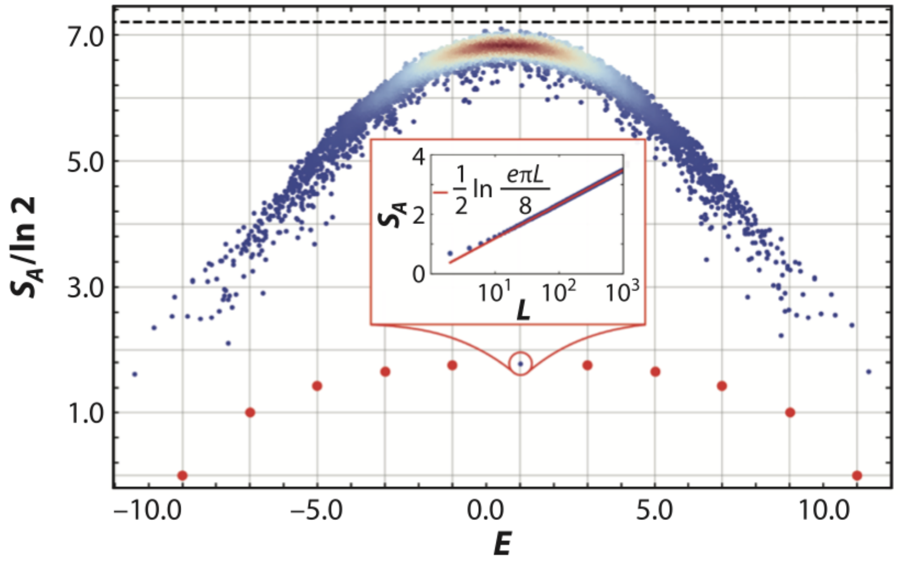

Half-chain entanglement entropy of the spin-1 XY model in zero-magnetization sector.

[Chandran et al. (2003); Ann. Rev. 14, 443 (Fig. 1c)]

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Restricted Spectrum Generating Algebra

- Restricted spectrum-generating algebra (rSGA) [Arno et al. (1988), Yang (1989)] [Moudgalya et al. (2018)]
  - Partial dynamical symmetry

$$\exists Q, \quad \text{s.t.}\ [H, Q] - \mathcal{E}Q\big|_W = 0, \quad W \subset \mathcal{H}, \ \ Q: \text{Local operator}$$

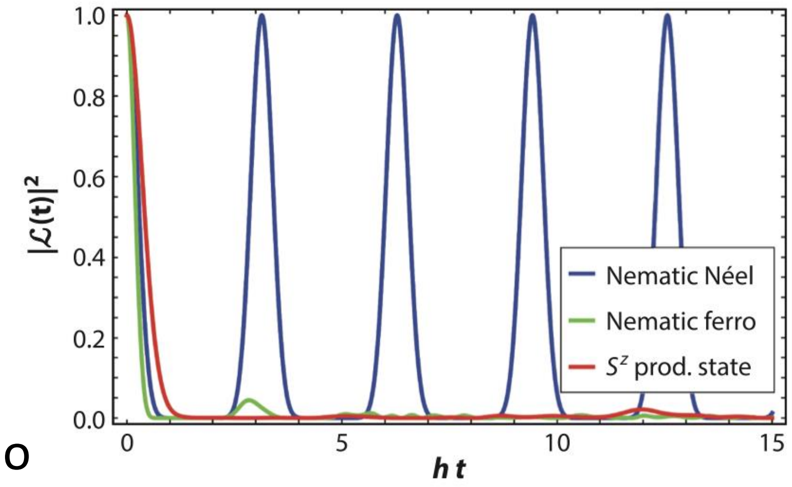
  - The solvable subspace is systematically constructed if $|\psi_0\rangle$ is an energy eigenstate:

$$H|\psi_0\rangle = \mathcal{E}_0|\psi_0\rangle \Rightarrow HQ^n|\psi_0\rangle = (\mathcal{E}_0 + n\mathcal{E})|\psi_0\rangle$$

$$|\psi(0)\rangle = \sum_n c_n Q^n|\psi_0\rangle, \quad c_n \in \mathbb{R}$$

$$\langle\psi(t)|\mathcal{O}_{\text{local}}|\psi(t)\rangle$$

$$= \sum_{m,n} c_m c_n e^{i\mathcal{E}(m-n)t}\langle\psi(0)|\mathcal{O}_{\text{local}}|\psi(0)\rangle$$



Strong revivals observed in dynamics of Loschmidt echo for the spin-1 XY from each initial state. [Chandran et al. (2023); Ann. Rev. 14, 443 (Fig. 1d)]

Restricted Spectrum Generating Algebra

- Simple example: free fermion model

$$H = \sum_k \Lambda_k \eta_k^\dagger \eta_k$$

$$\{\eta_k,\, \eta_\ell^\dagger\} = \delta_{k.\ell}, \quad \{\eta_k,\, \eta_\ell\} = \{\eta_k^\dagger,\, \eta_\ell^\dagger\} = 0$$

- "Spectrum generating algebra" (SGA)

$$[H,\, \eta_k^\dagger] = \Lambda_k \eta_k^\dagger$$

⇒ Provides the spectrum for a tower of states

$$\{|\mathrm{vac}\rangle,\, \eta_{k_1}^\dagger|\mathrm{vac}\rangle,\, \eta_{k_2}^\dagger\eta_{k_1}^\dagger|\mathrm{vac}\rangle, \ldots\}$$

- Energy eigenstates

$$H\eta_{k_1}^\dagger \cdots \eta_{k_n}^\dagger|\mathrm{vac}\rangle = (\Lambda_{k_1} + \cdots + \Lambda_{k_n})\eta_{k_1}^\dagger \cdots \eta_{k_n}^\dagger|\mathrm{vac}\rangle$$

Restricted Spectrum Generating Algebra

- Example of rSGA: perturbed spin-1 XY model [Schecter et al. (2019)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1} \ \in \mathrm{End}((\mathbb{C}^3)^N), \quad \mathbb{C}^3 = \mathrm{span}\{|0\rangle, |1\rangle, |2\rangle\}$$

$$h = \frac{J}{2}(S^x \otimes S^x + S^y \otimes S^y) + \frac{m}{2}(S^z \otimes \mathbf{1} + \mathbf{1} \otimes S^z)$$

  - Spin-1 operators

$$S^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^y = \frac{1}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

  - Non-integrable spin-1 chain. (Not all energy eigenstates are solvable.)

Restricted Spectrum Generating Algebra

- Example of rSGA: perturbed spin-1 XY model

  [Schecter et al. (2019)]

  - Trivial energy eigenstate

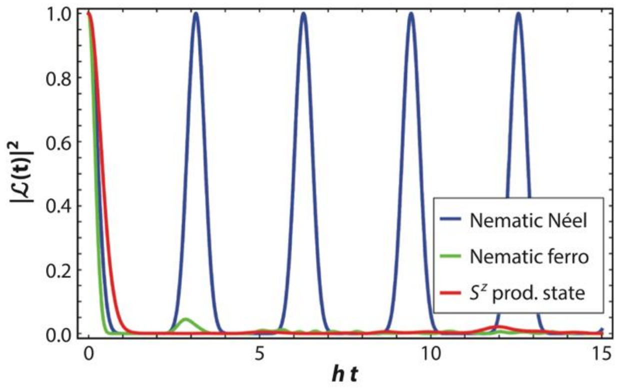
    $$H|\Omega\rangle = -hN|\Omega\rangle, \quad |\Omega\rangle = |22\ldots2\rangle$$

  - Subspace of quasiparticle (bimagnon) excitations

    $$W = \mathrm{span}\{(Q^\dagger)^n|\Omega\rangle\}_n, \quad Q^\dagger = \sum_{x=1}^{N}(-1)^x(S_x^+)^2$$

    is the solvable subspace due to the spectrum generating algebra

    $$[H, Q^\dagger] - 2mQ^\dagger\Big|_W = 0.$$



[Chandran et al. (2023); Ann. Rev. 14, 443 (Fig. 1d)]

Energy

$\vdots$

$$H(Q^\dagger)^3|\Omega\rangle = (6m - hN)(Q^\dagger)^3|\Omega\rangle$$

$$H(Q^\dagger)^2|\Omega\rangle = (4m - hN)(Q^\dagger)^2|\Omega\rangle$$

$$HQ^\dagger|\Omega\rangle = (2m - hN)Q^\dagger|\Omega\rangle$$

$$H|\Omega\rangle = -hN|\Omega\rangle$$

Restricted Spectrum Generating Algebra

- Example of rSGA: AKLT-type model [Moudgalya et al. (2018), CM (2024)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1} \ \in \mathrm{End}((\mathbb{C}^3)^N), \quad \mathbb{C}^3 = \mathrm{span}\{|0\rangle, |1\rangle, |2\rangle\}$$

$$\begin{aligned} h = {} & \alpha(|00\rangle\langle 00| + |22\rangle\langle 22|) + \beta|11\rangle\langle 11| \\ & + \frac{\alpha}{2} \sum_{a\in\{0,2\}} (\gamma|a1\rangle\langle a1| + |a1\rangle\langle 1a| + |1a\rangle\langle a1| + \gamma|1a\rangle\langle 1a|) \\ & + \omega^2\beta(|02\rangle\langle 02| + |02\rangle\langle 20| + |20\rangle\langle 02| + |20\rangle\langle 20|) \\ & - \omega\beta(|02\rangle\langle 11| + |11\rangle\langle 02| + |11\rangle\langle 20| + |20\rangle\langle 11|) \end{aligned}$$

: ALKT at $\frac{\beta}{\alpha} = \frac{2}{3},\ \gamma = 1,\ \omega = -\frac{1}{2}.$

- Non-integrable spin-1 chain. (Not all energy eigenstates are solvable.)
- The ground state and some excitation states were known to be solvable for AKLT.

[Affleck et al. (1987), Arovas (1989)]

Restricted Spectrum Generating Algebra

- Example of rSGA: AKLT-type model [Moudgalya et al. (2018), CM (2024)]
  - The exact zero-energy state is written by the matrix product state:

$$|\psi_A\rangle = \sum_{\{m_1,\ldots,m_N\}\in\{0,1,2\}^N} \mathrm{tr}_a(K_a A_{m_1} A_{m_2} \cdots A_{m_N})\,|m_1, m_2, \ldots, m_N\rangle \in (\mathbb{C}^3)^N$$

$$= \mathrm{tr}_a(K_a \vec{A} \otimes_p \vec{A} \otimes_p \cdots \otimes_p \vec{A})$$

$$\vec{A} = \begin{pmatrix} a_0\sigma^+ \\ a_1\sigma^z \\ a_2\sigma^- \end{pmatrix}, \quad a_1^2/a_0a_2 = \omega,\ a_0, a_1, a_2 \in \mathbb{C}, \quad \sigma^+, \sigma^z, \sigma^-$$: Pauli matrices

  - $K_a \in \mathrm{End}(\mathbb{C}^2)$ is determined by the boundary condition.
    ($K_a = 1_a$ for the periodic boundary; $\mathrm{rank}\, K_a = 1$ for an open boundary)

Restricted Spectrum Generating Algebra

- Example of rSGA: AKLT-type model [Moudgalya et al. (2018), CM (2024)]
  - The exact zero-energy state is written by the matrix product state:

$$|\psi_A\rangle = \mathrm{tr}_a(\vec{A} \otimes_p \vec{A} \otimes_p \cdots \otimes_p \vec{A})$$

$$\Leftarrow h(\vec{A} \otimes_p \vec{A}) = \vec{A}' \otimes_p \vec{A} - \vec{A} \otimes_p \vec{A}'$$: Local divergence condition/ Frustration-free condition for $\vec{A}' = 0$

  - Quasiparticle-picture for the excitation states:

$$|\psi_{A,B^n}\rangle = (Q^\dagger)^n |\psi_A\rangle, \quad Q^\dagger := \sum_{x=1}^N e^{i\pi x}(S_x^+)^2$$: Creates a quasiparticle with momentum π.

$$= \sum_{x_1,\dots,x_n} e^{i\pi \sum_{j=1}^n x_j} \mathrm{tr}_a(\vec{A} \otimes_p \cdots \otimes_p \underset{x_1}{\vec{B}} \otimes_p \cdots \otimes_p \underset{x_n}{\vec{B}} \otimes_p \cdots \otimes_p \vec{A}), \quad \vec{B} := (S^+)^2 \vec{A}$$

$$\Leftarrow h(\vec{B} \otimes_p \vec{A} + e^{i\pi} \vec{A} \otimes_p \vec{B}) = \frac{\mathcal{E}}{2}(\vec{B} \otimes_p \vec{A} + e^{i\pi} \vec{A} \otimes_p \vec{B}) - (\vec{B} \otimes \vec{A}' - e^{i\pi} \vec{A}' \otimes_p \vec{B})$$

$$\vec{B} \otimes_p \vec{B} = 0, \quad (S^+)^2 \vec{B} = 0$$: No double/adjacent occupations are allowed.

Restricted Spectrum Generating Algebra

- Example of rSGA: AKLT-type model [Moudgalya et al. (2018), CM (2024)]
  - Restricted spectrum-generating algebra:

$$[H, Q^\dagger] - \mathcal{E}Q^\dagger\big|_W = 0$$

$$W = \mathrm{span}\left\{|\psi_A\rangle, Q^\dagger|\psi_A\rangle, (Q^\dagger)^2|\psi_A\rangle, \ldots, (Q^\dagger)^{\lfloor \frac{N}{2} \rfloor}|\psi_A\rangle\right\}$$

  - Embedded equally-spaced energy spectrum

$$H|\psi_A\rangle = 0$$

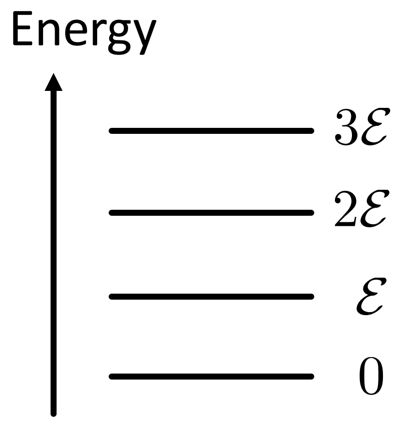
$$HQ^\dagger|\psi_A\rangle = \mathcal{E}Q^\dagger|\psi_A\rangle$$

: Embedded equally-spaced spectrum

⇒ Identical & non-interacting quasiparticles

$$\vdots$$

$$H(Q^\dagger)^n|\psi_A\rangle = n\mathcal{E}(Q^\dagger)^n|\psi_A\rangle$$



Beyond rSGA

- Generalization of the AKLT-type model [CM (2024)]
  - Quasiparticle-excitation states

$$(Q^\dagger)^n|\psi_A\rangle, \quad Q^\dagger := \sum_{x=1}^{N} e^{i\pi x}(S_x^+)^2$$

: Carrying momentum π

⇩

$$Q^\dagger(k) := \sum_{x=1}^{N} e^{ikx}(S_x^+)^2$$

: Carrying momentum k

  - Repulsive property is lost.

$$\vec{B} \otimes_p \vec{B} = 0, \quad (S^+)^2\vec{B} \neq 0$$

Beyond rSGA

- Perturbed XXC model [CM (2024)]

$$H=\sum_{x=1}^{N}\mathbf{1}\otimes\cdots\otimes\underset{x,x+1}{h}\otimes\cdots\otimes\mathbf{1}$$

$$h=\sum_{a\in\{0,2\}}\underbrace{(|a1\rangle\langle a1|+|1a\rangle\langle 1a|+|a1\rangle\langle 1a|+|1a\rangle\langle a1|)}_{\text{XXC model (integrable)}}+\alpha(|00\rangle\langle 00|+|22\rangle\langle 22|)$$

- $W=\mathrm{span}\{|\psi_{A,B^n}\rangle\}_n$ is solvable subspace of H.

$$|\psi_{A,B^n}\rangle=\sum_{1\leq x_1<\cdots<x_n\leq N}\sum_{P\in\mathfrak{S}_n}A_n(P)\,e^{i\sum_{j=1}^{n}k_{P(j)}x_j}\,\mathrm{tr}_a(\vec{A}\otimes_p\cdots\otimes_p\underset{x_1}{\vec{B}}\otimes_p\cdots\otimes_p\underset{x_n}{\vec{B}}\otimes_p\cdots\otimes_p\vec{A})$$

: Bethe-like state

$$\neq\prod_{j=1}^{n}Q^\dagger(k_j)|\psi_A\rangle\quad\Leftarrow\vec{B}=\begin{pmatrix}b_0&0&0\\0&b_1&0\\0&0&b_0\end{pmatrix}\vec{A},\quad\begin{pmatrix}b_0&0&0\\0&b_1&0\\0&0&b_0\end{pmatrix}\vec{B}\neq0$$

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XXC model (integrable)

- $W = \mathrm{span}\,\{|\psi_{A,B^n}\rangle\}_n$ is solvable subspace of H.

$|\uparrow\rangle$ $|\downarrow\rangle$ $|\downarrow\rangle$ $|\uparrow\rangle$

$$|\psi_{A,B^n}\rangle = \sum_{1\le x_1<\cdots<x_n\le N} \sum_{P\in\mathfrak{S}_n} A_n(P)\, e^{i\sum_{j=1}^n k_{P(j)}x_j}\, \mathrm{tr}_a(\vec{A}\otimes_p \cdots \otimes_p \underset{x_1}{\vec{B}} \otimes_p \cdots \otimes_p \underset{x_n}{\vec{B}} \otimes_p \cdots \otimes_p \vec{A})$$

: Bethe-like state

$$e^{ik_jN} = (-1)^{n-1} \prod_{l=1,l\neq j}^{n} \frac{e^{i(k_j+k_l)}+1-2e^{ik_j}}{e^{i(k_j+k_l)}+1-2e^{ik_l}}, \quad \forall j = 1,\ldots,n$$

: Bethe-ansatz equations for s=1/2 XXX

Beyond rSGA

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XXC model (integrable)

- $W = \mathrm{span}\,\{|\psi_{A,B^n}\rangle\}_n$ is solvable subspace of H.

$$H|\psi_{A,B^n}\rangle = \mathcal{E}_n(\{k_j\})|\psi_{A,B^n}\rangle, \qquad \mathcal{E}_n(\{k_j\}) = \left(2\sum_{j=1}^{n} \cos k_j - \frac{n}{2}\right)$$

$$\Leftarrow h\vec{A} \otimes_p \vec{A} = h\vec{B} \otimes_p \vec{B} = 0$$

$$h\vec{A} \otimes_p \vec{B} = -\vec{A} \otimes_p \vec{B} + \vec{B} \otimes_p \vec{A}$$

$$h\vec{B} \otimes_p \vec{A} = \vec{A} \otimes \vec{B} - \vec{B} \otimes_p \vec{A}$$

Embedded s=1/2 XXX spectrum (not equally-spaced)
⇒ Interacting quasiparticles

Beyond rSGA

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Embedded s=1/2 XXX spectrum (not equally-spaced)
⇒ Interacting quasiparticles

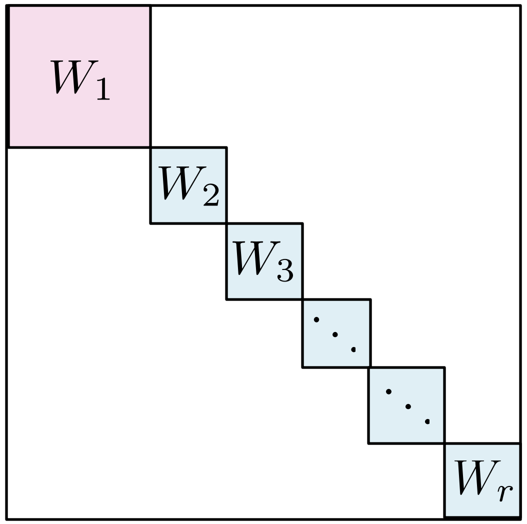
Why? ⇒ Hilbert-space fragmentation

Hilbert space fragmentation

- Hilbert-space fragmentation (HSF; Krylov restricted thermalization)

[Retort et al. (2003), Pai et al. (2019)]

$$\mathcal{H} = \bigoplus_{\alpha=1}^{r} W_\alpha\,, \quad W_\alpha = \mathrm{span}\,\{H^{n_\alpha}|\psi_\alpha\rangle\}_{n_\alpha}$$



  - Exponentially-many block diagonal structure.
  - Fragmented subspaces are not distinguished by obvious local symmetries of H.
  - Solvable subspaces are sometimes embedded (not always).

Hilbert space fragmentation

- Simple example: Sato's model [Sato (1995)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1}$$

$$h = \sum_{\substack{a,b=0 \\ a \neq b}}^{2s} f(a,b)|ab\rangle\langle ba| + \sum_{r=0}^{2s} c(r)|rr\rangle\langle rr| + \sum_{\substack{a,b=0 \\ a \neq b}}^{2s} g(a,b)|ab\rangle\langle ab|$$

  - Non-integrable arbitrary spin-s chain.
  - The interactions only exchange the neighboring configurations.

$$\begin{aligned} H : |a_1, a_2, a_3, a_4\rangle \mapsto & f(a_1, a_2)|\underline{a_2, a_1}, a_3, a_4\rangle + f(a_2, a_3)|a_1, \underline{a_3, a_2}, a_4\rangle \\ & + f(a_3, a_4)|a_1, a_2, \underline{a_4, a_3}\rangle + f(a_4, a_1)|\underline{a_4}, a_2, a_3, \underline{a_1}\rangle \\ & + (g(a_1, a_2) + g(a_2, a_3) + g(a_3, a_4) + g(a_4, a_1))|a_1, a_2, a_3, a_4\rangle \end{aligned}$$

Hilbert space fragmentation

- Simple example: Sato's model [Sato (1995)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1}$$

$$h = \sum_{\substack{a,b=0 \\ a\neq b}}^{2s} f(a,b)|ab\rangle\langle ba| + \sum_{r=0}^{2s} c(r)|rr\rangle\langle rr| + \sum_{\substack{a,b=0 \\ a\neq b}}^{2s} g(a,b)|ab\rangle\langle ab|$$

  - Non-integrable arbitrary spin-s chain.
  - The entries in each configuration never change by the interactions
    ⇒ Hilbert-space fragmentation (according to multisets of configuration entries).

$$|m_1, m_2, \ldots, m_N\rangle$$
$$|m_2, m_1, \ldots, m_N\rangle$$
$$\vdots$$
$$|\sigma(m_1), \sigma(m_2), \ldots, \sigma(m_N)\rangle, \quad \sigma \in \mathfrak{S}_N$$

: All in the same invariant subspace.

Hilbert space fragmentation

- Simple example: Sato's model [Sato (1995)]

$$H = \sum_{x=1}^{N} \mathbf{1} \otimes \cdots \otimes \underset{x,x+1}{h} \otimes \cdots \otimes \mathbf{1}$$

$$h = \sum_{\substack{a,b=0 \\ a\neq b}}^{2s} f(a,b)|ab\rangle\langle ba| + \sum_{r=0}^{2s} c(r)|rr\rangle\langle rr| + \sum_{\substack{a,b=0 \\ a\neq b}}^{2s} g(a,b)|ab\rangle\langle ab|$$

  - Non-integrable arbitrary spin-s chain.
  - The model is integrable in the subspaces given by

$$W^{\sigma} = (\mathrm{span}\,\{|0\rangle, |\sigma\rangle\})^{\otimes N}, \quad \sigma = 1, \ldots, 2s.$$

$|0\rangle \leftrightarrow |\uparrow\rangle$: Vacuum

$|\sigma\rangle \leftrightarrow |\downarrow\rangle$: Particles

$\Rightarrow$ $H\big|_W \sim H_{\mathrm{XXZ}}$ with $\Delta_{\sigma,0} = (g(0,\sigma) + g(\sigma,0) - c(\sigma))/f(\sigma,0)$

Anisotropy depending on σ.

Embedded Integrable Models in HSF

- Example: XXC model [Maassarani (1997, 1999), de Leeuw et al. (2023)]

$$H_{XXC} = \sum_{j=1}^{N} \mathbf{1} \otimes \cdots \otimes h_{j,j+1}^{XXC} \otimes \cdots \otimes \mathbf{1}$$

$$h^{XXC} = \cosh\eta \left(\sum_{s,s' \in \{0,2\}} |ss'\rangle\langle ss'| + |11\rangle\langle 11| \right) + \sum_{s\in\{0,2\}} \left(|s1\rangle\langle 1s| + |1s\rangle\langle s1| \right)$$

  - Integrable spin-1 chain.

anisotropy parameter (→ η)

$$R(\lambda) = \sum_{a,a'=0,2} \Big\{ (|aa'\rangle\langle a'a| + |11\rangle\langle 11|)\sinh(\lambda+\eta) + (|a1\rangle\langle 1a| + |1a\rangle\langle a1|)\sinh\eta + (x_a|a1\rangle\langle a1| + x_a^{-1}|1a\rangle\langle 1a|)\sinh\lambda \Big\}$$

$$(\lambda, x_a \in \mathbb{C})$$

$$H_{XXC} \xrightarrow[\mathcal{H}\backslash\{|0\rangle,|2\rangle\}^N]{} H_{XXZ}$$

becomes XXZ model by identifying $|0\rangle$ & $|2\rangle$.

Embedded Integrable Models in HSF

- Example: XXC model [Maassarani (1997, 1999), de Leeuw et al. (2023)]

$$H_{XXC} = \sum_{j=1}^{N} \mathbf{1} \otimes \cdots \otimes h^{XXC}_{j,j+1} \otimes \cdots \otimes \mathbf{1}$$

$$h^{XXC} = \cosh\eta \left(\sum_{s,s' \in \{0,2\}} |ss'\rangle\langle ss'| + |11\rangle\langle 11| \right) + \sum_{s\in\{0,2\}} \underbrace{(|s1\rangle\langle 1s| + |1s\rangle\langle s1|)}_{\text{exchange terms}}$$

- The configuration of 0 & 2 never changes by the interactions.

  "Irreducible string (IS)" [Dhar et al. (1993), Barma et al (1994), Menon et al. (1997), Dhar (1997)]

- Hilbert-space fragmentation occurs according to the IS.

e.g. $H : |10211012\rangle \mapsto (h^{10}_{10} + h^{21}_{21} + h^{12}_{12} + h^{21}_{21} + h^{10}_{10})|10211012\rangle$
$+ h^{10}_{01}|01211012\rangle + h^{21}_{12}|10121012\rangle + h^{12}_{21}|10210112\rangle$ (IS = 0 2 0 2)
$+ h^{21}_{12}|10211102\rangle + h^{10}_{01}|10211021\rangle$

Embedded Integrable Models in HSF

- Example: perturbed XXC model [CM (2024), in preparation with KPP]

$$H_{XXC}^{\rm pol} = \sum_{j=1}^{N} \left(h_{j,j+1}^{XXC} + h_{j,j+1}^{\rm pol} \right)$$

$$h^{\rm pol} = \alpha_{\rm d1} |02\rangle\langle 02| + \alpha_{\rm d2} |20\rangle\langle 20| + \alpha_{\rm o1} |02\rangle\langle 20| + \alpha_{\rm o2} |20\rangle\langle 02| \xrightarrow{P_{\rm pol}^{(0)},\ P_{\rm pol}^{(2)}} 0$$

: vanishing terms

$$h_{j,j+1}^{XXC} + h_{j,j+1}^{\rm pol} \underset{P_{\rm pol}\mathcal{H}\setminus\{|0\rangle,|2\rangle\}^N}{\longmapsto} \sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{1}{2}(\cosh\eta)\left(\sigma_j^z \sigma_{j+1}^z\right) + \frac{1}{2}\cosh\eta$$

s=1/2 XXZ model!

- Projector onto the fully-polarized IS $\ldots 0000 \ldots$ (resp. $\ldots 2222 \ldots$)

$$P_{\rm pol}^{(0)} = \bigotimes_{j=1}^{N} (|0\rangle\langle 0| + |1\rangle\langle 1|)_j \quad \text{(resp. } P_{\rm pol}^{(2)} = \bigotimes_{j=1}^{N} (|2\rangle\langle 2| + |1\rangle\langle 1|))_j \text{)}$$

Embedded Integrable Models in HSF

- Example: perturbed XXC model [CM-Tsuji (2024), in preparation with KPP]

$$H_{XXC}^{\text{alt}} = \sum_{j=1}^{N} \left(h_{j,j+1}^{XXC} + h_{j,j+1}^{\text{alt}} \right)$$

$$h^{\text{alt}} = \beta_{\text{d1}}|00\rangle\langle 00| + \beta_{\text{d2}}|22\rangle\langle 22| + \beta_{\text{o1}}|00\rangle\langle 22| + \beta_{\text{o2}}|22\rangle\langle 00| \overset{P_{\text{alt}}}{\longmapsto} 0$$

: vanishing terms

$$h_{j,j+1}^{\text{XXC}} + h_{j,j+1}^{\text{alt}} \underset{P_{\text{alt}}\mathcal{H}\backslash\{|0\rangle,|2\rangle\}^N}{\longmapsto} \sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+ + \frac{1}{2}\cosh\eta\,(\sigma_j^z\sigma_{j+1}^z) + \frac{1}{2}\cosh\eta$$

s=1/2 XXZ model!

- Projector onto the alternating IS $\ldots 0202 \ldots$

$$P_{\text{alt}} = \text{tr}_a \otimes_{j=1}^{N} \text{diag.}(\sigma_a^+, \mathbf{1}_a, \sigma_a^-)_j - \otimes_{j=1}^{N}(|1\rangle\langle 1|)_j$$

Partial solvability in open quantum systems?

Outline

- What is "partial solvability"?
  - Definition of partial solvability
  - Thermalization & quantum many-body scars (QMBS)
- Closed partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Open partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Concluding remarks

Beyond Isolated Quantum Systems

- Lindblad master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t))\,, \quad \mathcal{L}(\rho) = -i[H,\,\rho] + \sum_\mu \varepsilon_\mu \mathcal{D}_\mu(\rho)$$

⇩

Steady state
as the fixed point of $\mathcal{L}$

$$\mathcal{D}_\mu(\rho) = 2A_\mu\rho A_\mu^\dagger - \{A_\mu^\dagger A_\mu,\,\rho\}$$: Dissipation terms

Quantum jump operator (A_μ)

  - Is the most general CPTP map under the assumptions:
    Markovian time evolution & Direct product initial state.

Beyond Isolated Quantum Systems

- Lindblad master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t))\,, \quad \mathcal{L}(\rho) = -i[H,\,\rho] + \sum_\mu \varepsilon_\mu \mathcal{D}_\mu(\rho)$$

⇩

Steady state
as the fixed point of $\mathcal{L}$

$$\mathcal{D}_\mu(\rho) = 2A_\mu \rho A_\mu^\dagger - \{A_\mu^\dagger A_\mu,\,\rho\}$$: Dissipation terms

(A_μ:) Quantum jump operator

- Sometimes has the exactly solvable steady state.

	bulk dissipators	boundary dissipators
completely solvable	✓ [Prosen (2008) et al.]	✓ [Paletta et al. (2024)]
steady state solvable	✓	✓ [Prosen (2013)]
partially solvable	✓ [Tindall et al. (2020)]	our model [CM-Tsuji (2024)]

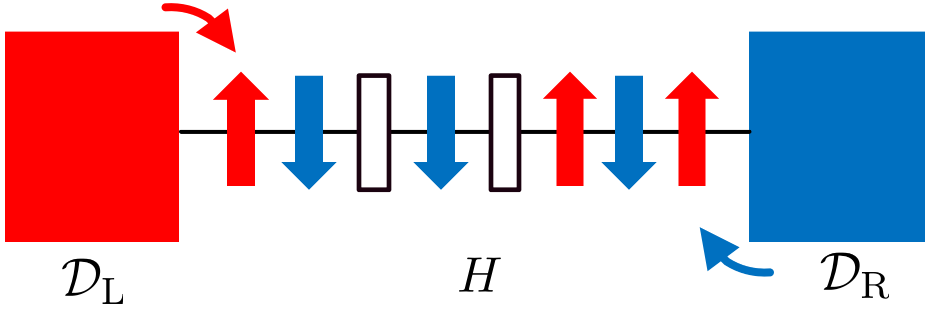
- Can partial solvability be robust against the boundary dissipators?

HSF-induced solvable eigenmodes

- System coupled to boundary dissipators

$$\mathcal{L}(\rho) = -i[H,\, \rho] + \sum_\mu \gamma_\mu \mathcal{D}_\mu(\rho)$$

$$\mathcal{D}_\mu(\rho) = 2A_\mu \rho A_\mu^\dagger - \{A_\mu^\dagger A_\mu,\, \rho\}$$



- Thermofield double vector expression

$$\rho = \sum_{m,n} \rho_{m,n} |m\rangle\langle n| \mapsto \sum_{m,n} \rho_{m,n} |m\rangle \otimes |n\rangle = |\rho\rangle\rangle$$

  - Evolution of the density matrix

$$\frac{d}{dt}|\rho(t)\rangle\rangle = -i\widetilde{H}|\rho(t)\rangle\rangle \in \mathcal{H} \otimes \mathcal{H}^*$$

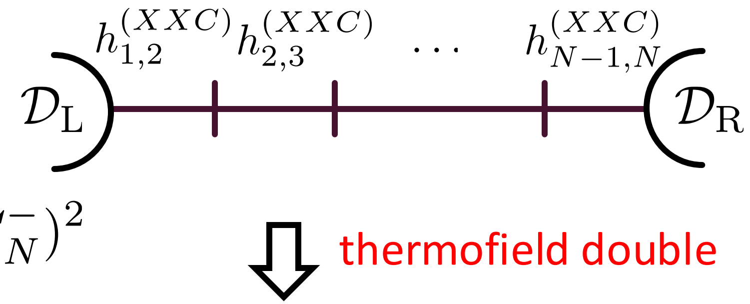
Can $\widetilde{H}$ be an integrable Hamiltonian in a certain subspace?

$$\widetilde{H} = H \otimes \mathbf{1} - \mathbf{1} \otimes {}^tH + i \sum_\alpha \gamma_\alpha \left((A_\alpha \otimes A_\alpha^*) - \frac{1}{2}(A_\alpha^\dagger A_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes {}^tA_\alpha A_\alpha^*) \right)$$

: Non-Hermitian effective Hamiltonian

HSF-induced partially solvable Liouvillian

- Example: XXC Hamiltonian coupled to boundary dissipators [CM-Tsuji (2024)]
  - Spin-1 XXC Hamiltonian ⇒ HSF by config. of 0 & 2
  - Boundary dissipators ⇒ violate integrability

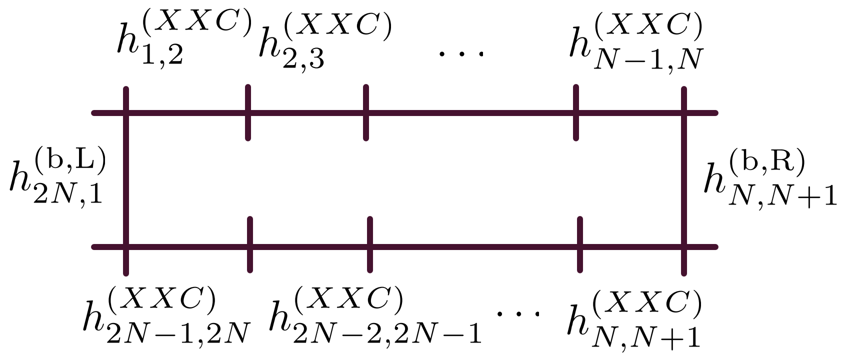


$$A_{\mathrm{L},+} = (S_1^+)^2, \quad A_{\mathrm{L},-} = (S_1^-)^2, \quad A_{\mathrm{R},+} = (S_N^+)^2, \quad A_{\mathrm{R},-} = (S_N^-)^2$$

  - Effective non-Hermitian Hamiltonian

$$\widetilde{H}_{XXC} = \sum_{n=1}^{N-1} h_{n,n+1}^{(XXC)} + h_{N,N+1}^{(\mathrm{b,R})} - \sum_{n=N+1}^{2N} h_{n,n+1}^{(XXC)} + h_{2N,1}^{(\mathrm{b,L})}$$

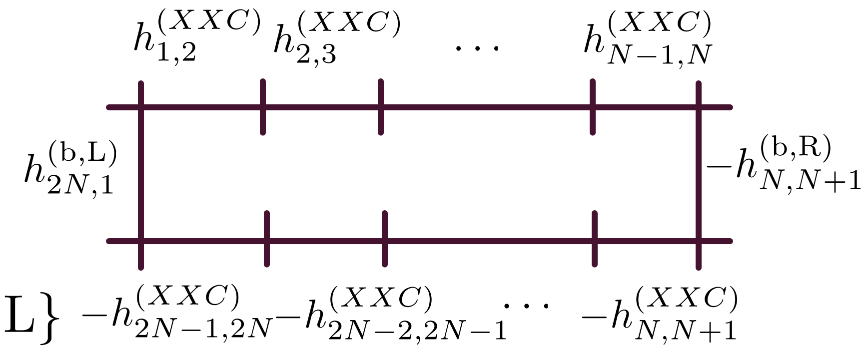
⇒ Two XXC chains coupled at the boundaries.



HSF-induced partially solvable Liouvillian

- Example: XXC Hamiltonian coupled to boundary dissipators [CM-Tsuji (2024)]
  - Effective non-Hermitian Hamiltonian

$$h^{(\mathrm{b},\alpha)} = i\gamma_{\alpha,+}\Big(|00\rangle\langle 22| - \frac{1}{2}(|2\rangle\langle 2|\otimes \mathbf{1} + \mathbf{1}\otimes|2\rangle\langle 2|)\Big) + i\gamma_{\alpha,-}\Big(|22\rangle\langle 00| - \frac{1}{2}(|0\rangle\langle 0|\otimes \mathbf{1} + \mathbf{1}\otimes|0\rangle\langle 0|)\Big), \quad \alpha\in\{\mathrm{R},\mathrm{L}\}$$

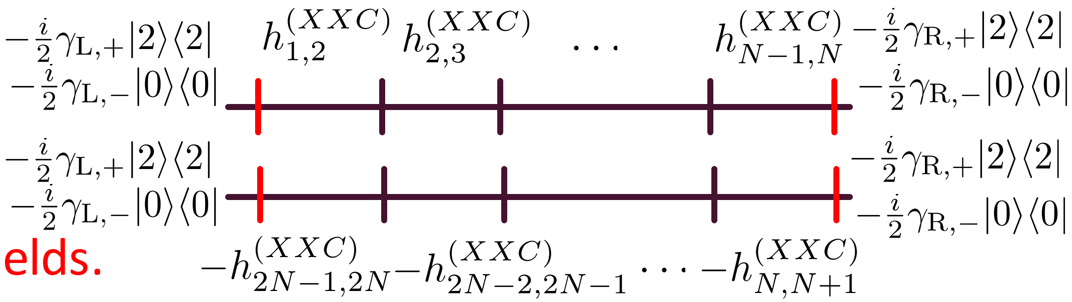


Two terms irrelevant in the subspace of alternating irreducible strings.

P_{alt} (projector onto the alternating subspace)

⇒ Two decoupled XXC chains

$$\widetilde{H}_{XXC}\big|_{W_{\mathrm{alt}}} = H^{(+)}_{XXC}\otimes\mathbf{1} - \mathbf{1}\otimes H^{(-)}_{XXC}$$



The other terms work as the (imaginary) boundary magnetic fields.

HSF-induced partially solvable Liouvillian

- Example: XXC Hamiltonian coupled to boundary dissipators [CM-Tsuji (2024)]
  - Projected effective non-Hermitian Hamiltonian

    $$\widetilde{H}_{XXC}\big|_{W_{\mathrm{alt}}} = H^{(+)}_{XXC} \otimes \mathbf{1} - \mathbf{1} \otimes H^{(-)}_{XXC}$$

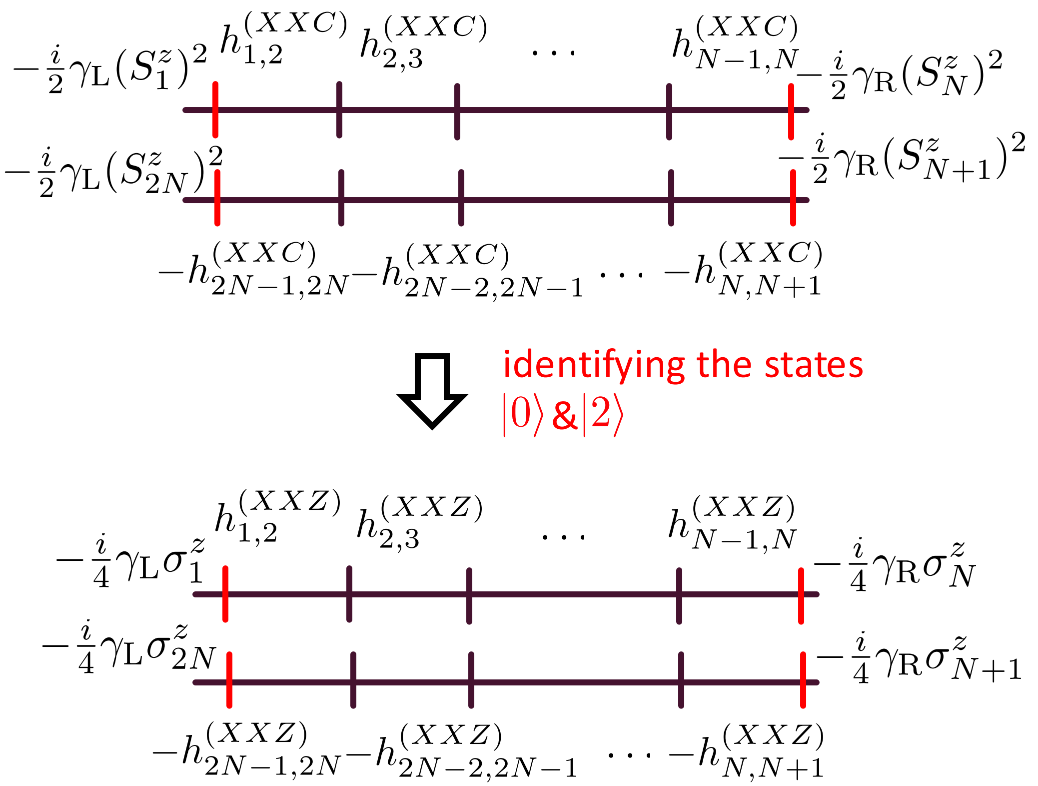
    becomes integrable when

    $$\gamma_{\mathrm{L},+} = \gamma_{\mathrm{L},-}, \quad \gamma_{\mathrm{R},+} = \gamma_{\mathrm{R},-}$$

  - Mapped to s=1/2 XXZ spin chain with diagonal boundaries

    $$H^{(\pm)}_{XXC}(\gamma_{\mathrm{L}}, \gamma_{\mathrm{R}}) \underset{\mathbb{C}^{3N}\setminus\{|0\rangle,|2\rangle\}^N}{\longmapsto} H^{(\pm)}_{XXZ}(\gamma_{\mathrm{L}}, \gamma_{\mathrm{R}})$$

    Spectrum derived by Bethe ansatz for spin-1/2 XXZ!!



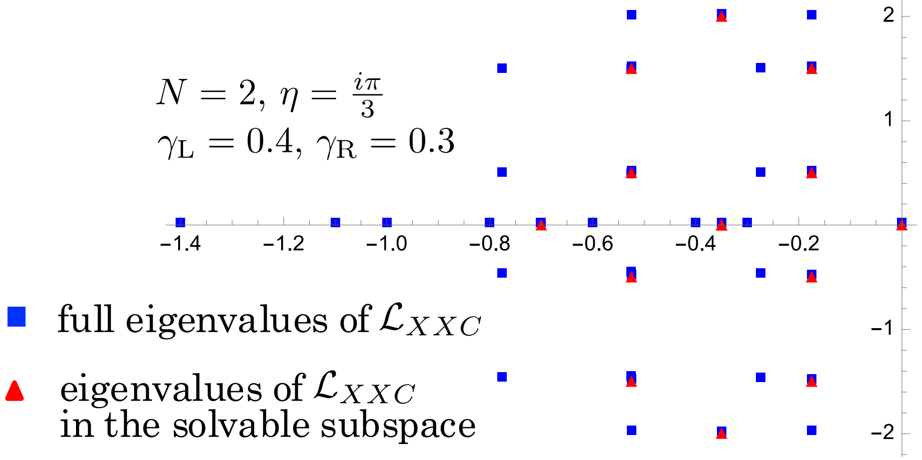
HSF-induced partially solvable Liouvillian

- Example: XXC Hamiltonian coupled to boundary dissipators [CM-Tsuji (2024)]
  - Eigenstates of effective Hamiltonian

$$\widetilde{H} \xrightarrow[P_{\text{alt}}\mathcal{H}\setminus\{|0\rangle,|2\rangle\}^N]{} H^{(+)}_{XXZ}(\gamma_{\text{L}},\gamma_{\text{R}})\otimes \mathbf{1} - \mathbf{1}\otimes H^{(-)}_{XXZ}(\gamma_{\text{R}},\gamma_{\text{L}})$$

$$H^{(+)}_{XXZ}(\gamma_{\text{L}},\gamma_{\text{R}}) = -\tfrac{i}{4}\gamma_{\text{L}}\sigma^z_1 + \sum_{j=1}^{N-1}\left(\sigma^+_j\sigma^-_{j+1} + \sigma^-_j\sigma^+_{j+1} + \tfrac{1}{2}\cosh\eta\,\sigma^z_j\sigma^z_{j+1}\right) - \tfrac{i}{4}\gamma_{\text{R}}\sigma^z_N$$

$$H^{(-)}_{XXZ}(\gamma_{\text{R}},\gamma_{\text{L}}) = -\tfrac{i}{4}\gamma_{\text{R}}\sigma^z_{N+1} - \sum_{j=N+1}^{2N-1}\left(\sigma^+_j\sigma^-_{j+1} + \sigma^-_j\sigma^+_{j+1} + \tfrac{1}{2}\cosh\eta\,\sigma^z_j\sigma^z_{j+1}\right) - \tfrac{i}{4}\gamma_{\text{L}}\sigma^z_{2N}$$



Full Liouvillian has degenerate steady states including the fully polarized state.

The steady state in the integrable subspace is given by the product state $\rho_{\text{ss}} = (|1\rangle\langle 1|)^{\otimes N}$.

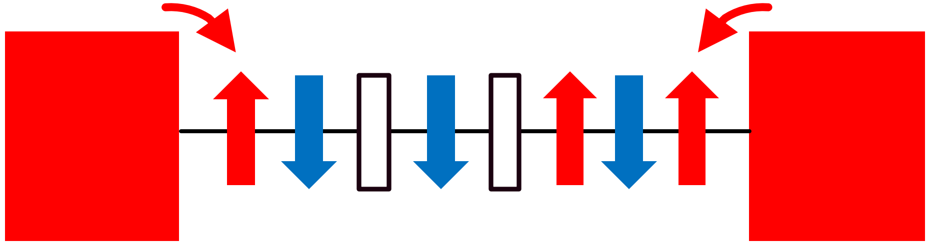
No persistent oscillation unlike the rSGA-induced solvable steady states.

rSGA-induced solvable eigenmodes

- System coupled to boundary dissipators

$$\mathcal{L}(\rho) = -i[H,\, \rho] + \sum_\mu \gamma_\mu \mathcal{D}_\mu(\rho)$$

$$\mathcal{D}_\mu(\rho) = 2A_\mu \rho A_\mu^\dagger - \{A_\mu^\dagger A_\mu,\, \rho\}$$



- System with rSGA

$$[H,\, Q^\dagger] - \mathcal{E}Q^\dagger\big|_W = 0, \quad W = \mathrm{span}\{(Q^\dagger)^n|\psi_A\rangle\}_n$$

$$Q^\dagger = \sum_x e^{i\pi x} q_x^\dagger$$

⇒ Quasiparticle excitations carrying momentum π

- Quasiparticle baths at the edges

$$A_\mathrm{L} = q_1^\dagger, \quad A_\mathrm{R} = q_N^\dagger$$

⇒ Doping quasiparticles at the boundaries

Fully occupied steady state?

rSGA-induced solvable eigenmodes

- Example: s=1 spin chains with rSGA + spin-2 magnon baths
  - s=1 spin chain with rSGA (e.g. AKLT model)

$$[H,\, Q^\dagger] - \mathcal{E}Q^\dagger\big|_{W^{(v_\mathrm{R},v_\mathrm{L})}} = 0, \quad Q^\dagger := \sum_{x=1}^{N} e^{i\pi x}(S_x^+)^2$$

$$W^{(v_\mathrm{L},v_\mathrm{R})} = \mathrm{span}\{(Q^\dagger)^n|\psi_A^{(v_\mathrm{L},v_\mathrm{R})}\rangle\}_n$$

$$|\psi_A^{(v_\mathrm{L},v_\mathrm{R})}\rangle = \langle \underline{v_\mathrm{L}}|\vec{A}\otimes_p \cdots \otimes_p \vec{A}|\underline{v_\mathrm{R}}\rangle, \qquad \vec{A} = \begin{pmatrix} a_0\sigma^+ \\ a_1\sigma^z \\ a_2\sigma^- \end{pmatrix}, \quad a_0, a_1, a_2 \in \mathbb{C}$$

Boundary vectors
$\in V_a = \mathrm{span}\{|0\rangle,\, |1\rangle\}$

⇒ Four degenerate zero-energy states.

  - Spin-2 magnon baths at the edges

$A_\mathrm{L} = (S_1^+)^2, \quad A_\mathrm{R} = (S_N^+)^2$ ⇒ Doping spin-2 magnons at the boundaries

rSGA-induced solvable eigenmodes

- Example: s=1 spin chains with rSGA + spin-2 magnon baths [CM-Tsuji (2024)]
  - The subspace $W^{(0,1)}$ consists of the dark states.

$$[H,\,(Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n]=0$$

⇒ Eigenstates of the Hamiltonian

$$\mathcal{D}_{(S_1^+)^2}\left((Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right)=0$$

$$\mathcal{D}_{(S_N^+)^2}\left((Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right)=0$$

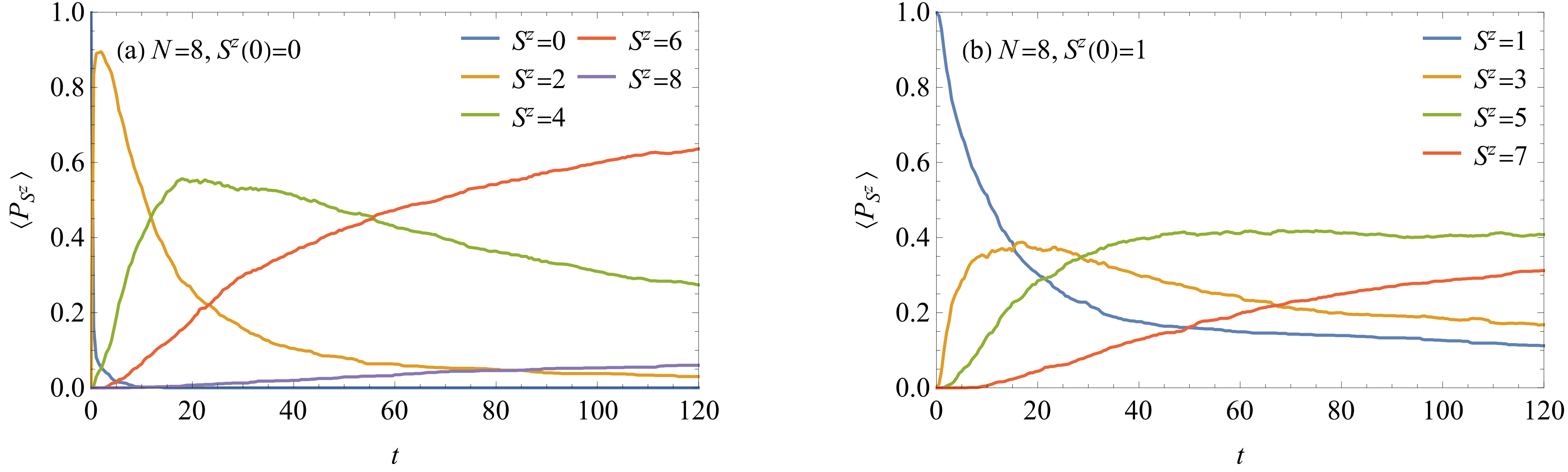
⇒ Dissipators are irrelevant.
(Robust eigenstates against boundary dissipators)

  - Any density matrix diagonal in $W^{(0,1)}$ becomes the steady states.

$$\mathcal{L}\left(\sum_n p_{nn}(Q^\dagger)^n|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right)=0,\quad \sum_n p_{nn}=1,\quad p_{nn}>0,\,\forall n$$

rSGA-induced solvable eigenmodes

- The ratio of the number of trajectories for each S^z.



(Left) The initial state does not overlap with solvable states ⇒ Dominated by states with large S^z.

(Right) The initial state does overlap with solvable states ⇒ States with small S^z survive!

rSGA-induced solvable eigenmodes

- Example: s=1 spin chains with rSGA + spin-2 magnon baths [CM-Tsuji (2024)]
  - Other solvable eigenmodes in $W^{(0,1)} \otimes (W^{(0,1)})^*$

$$[H, (Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n] = \sum_{n,m}(m-n)\mathcal{E}(Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n$$

⇒ Eigenstates of the Hamiltonian

$$\mathcal{D}_{(S_1^+)^2}\left((Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0$$

$$\mathcal{D}_{(S_N^+)^2}\left((Q^\dagger)^m|\psi_A^{(0,1)}\rangle\langle\psi_A^{(0,1)}|Q^n\right) = 0$$

⇒ Dissipators are irrelevant.
(Robust eigenstates against boundary dissipators)

  - Persistent oscillation emerges.

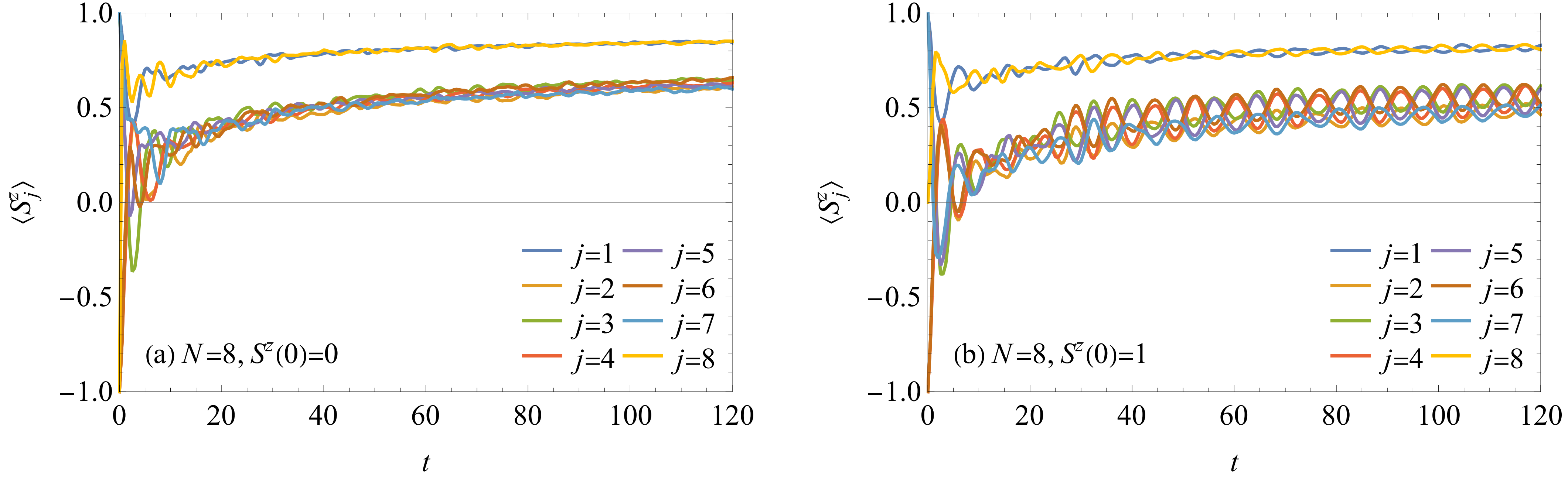
$$|\Psi(t=0)\rangle = \sum_n a_n(Q^\dagger)^n|\psi_A^{(0,1)}\rangle \in W^{(0,1)}$$

$$\Rightarrow \lim_{t\to\infty}\rho(t) = e^{\mathcal{L}t}\rho(0) = \sum_{n,m} e^{-i(m-n)\mathcal{E}t}(Q^\dagger)^m|\psi_A^{(0,1}\rangle\langle\psi_A^{(0,1)}|Q^n$$

$$\lim_{t\to\infty}\langle O(t)\rangle = \sum_{n\leq m} 2\cos((m-n)\mathcal{E}t)a_m a_n \,\mathrm{Re}\, O_{nm}$$

rSGA-induced solvable eigenmodes

- Time evolution of the local magnetization starting from a nearly Neel state at $\gamma_{\mathrm{L}} = \gamma_{\mathrm{R}} = 1.$



(Left) The initial state does not overlap with solvable states ⇒ No oscillations

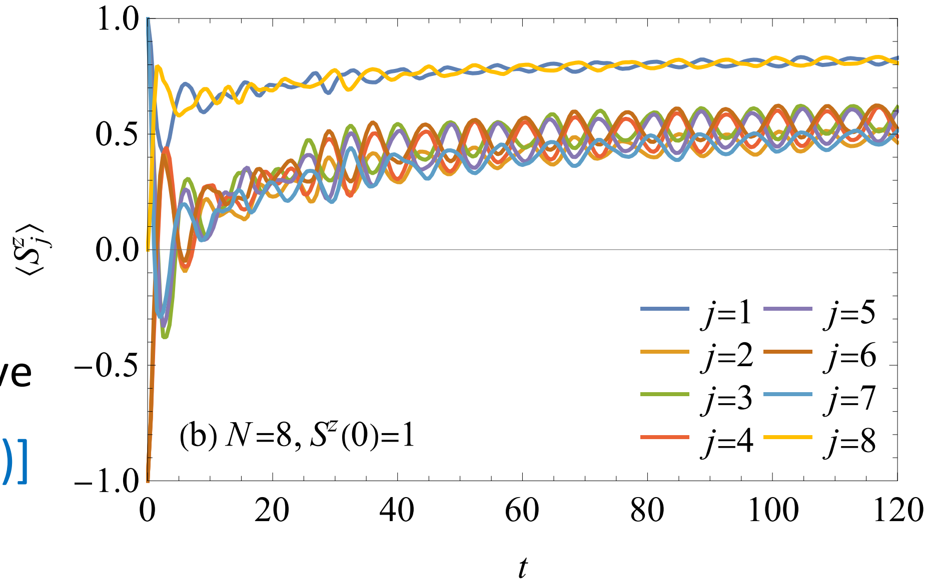
(Right) The initial state does overlap with solvable states ⇒ Long-lived oscillations emerge!

Outline

- What is "partial solvability"?
  - Definition of partial solvability
  - Thermalization & quantum many-body scars (QMBS)
- Closed partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Open partially solvable models
  - Restricted spectrum generating algebra (rSGA)
  - Hilbert space fragmentation (HSF)
- Concluding remarks

Concluding remarks

- Boundary dissipative spin chains
  - can have solvable eigenmodes inherited from the partially solvable system Hamiltonians.
  - reach a non-trivial steady state or never reach a steady state by exhibiting long-lived oscillations.



Time evolution of $\langle S^z_j \rangle$ for the boundary dissipative AKLT model at $\gamma_\mathrm{R} = \gamma_\mathrm{L} = 1$.

[CM-Tsuji (2024)]

Future works

- From the phenomenological viewpoints,
  - Can we observe the KPZ universality class in the integrable subspace?
    $\Rightarrow$ Robustness of the KPZ universality class against boundary conditions.
  - Overlap between the initial state & solvable eigenmodes?
    $\Rightarrow$ Needs determinant formula for boundary cases.
- From the mathphys aspects,
  - What is the algebraic structure behind the XXC & related models? [de Leeuw et al. (2023)]
  - What are the conserved quantities for partially solvable models?
  - Is it possible to extend the notion of partial integrability to QFT models?

Thank you for listening!