Quantum simulation of the Schwinger model with a topological term

Lento Nagano (ICEPP, The University of Tokyo)

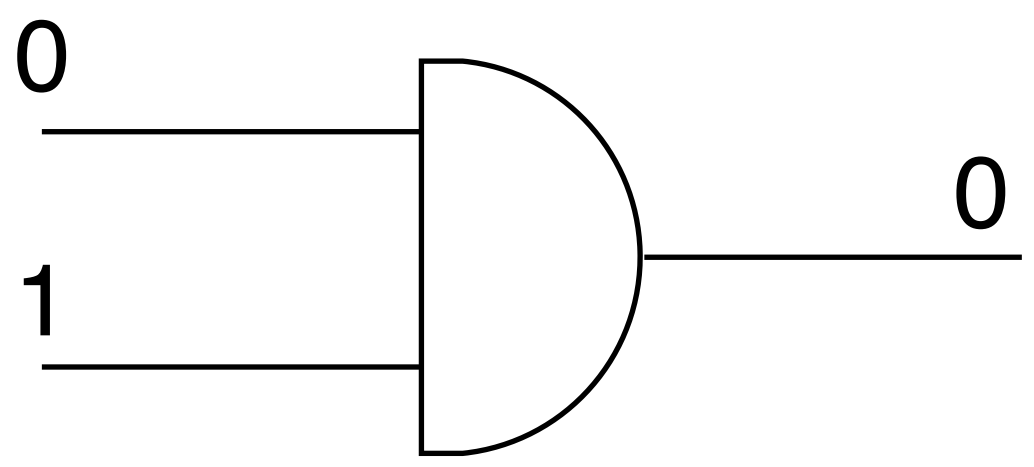
Based on:
M. Honda (Riken), E. Itou (YITP), Y. Kikuchi (Quantinuum), LN, T. Okuda (UTokyo), Phys. Rev. D 105, 014504
LN, A. Bapat and C. W. Bauer (LBNL), Phys. Rev. D 108, 034501

Quantum computing

- classical computing
  - classical bit: $s \in \{0,1\}^n$
  - classical operation: $s \mapsto f(s)$

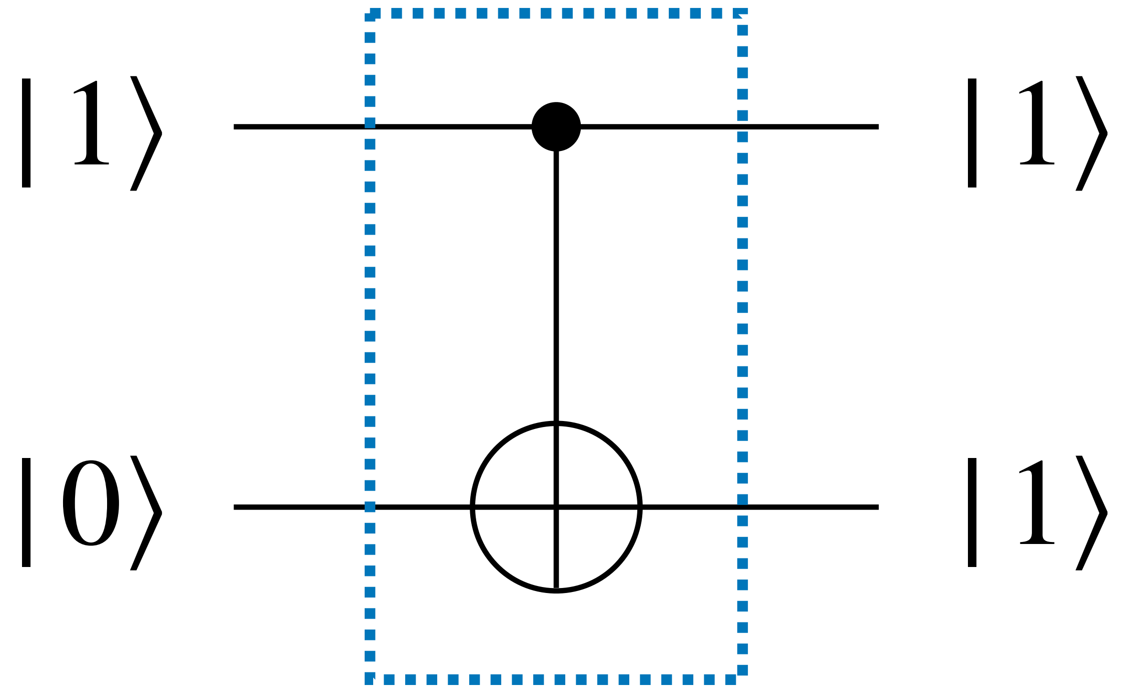
- quantum computing
  - quantum bit (**qubit**):
    $|\psi\rangle = \otimes_i (\alpha_i |0\rangle + \beta_i |1\rangle)$
    → superposition, entanglement
  - unitary operation (**quantum gate**):
    $|\psi\rangle \mapsto U|\psi\rangle$

classical (AND) gate



quantum (CNOT) gate

initial state

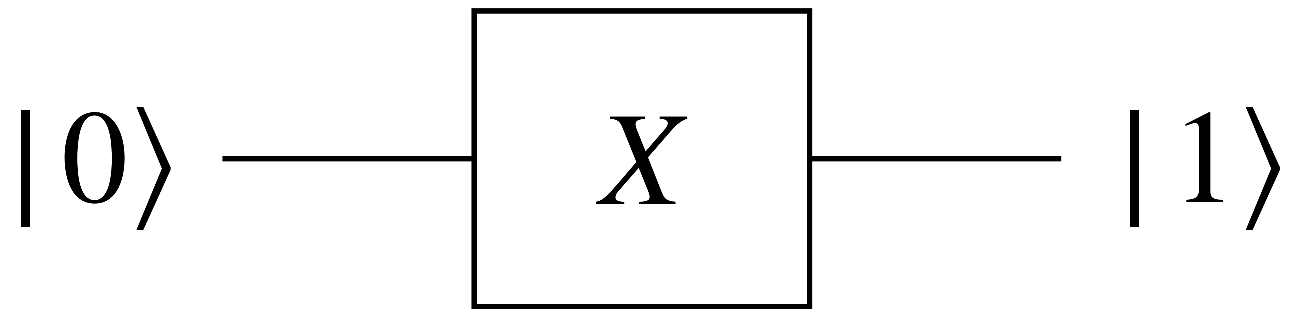


final state

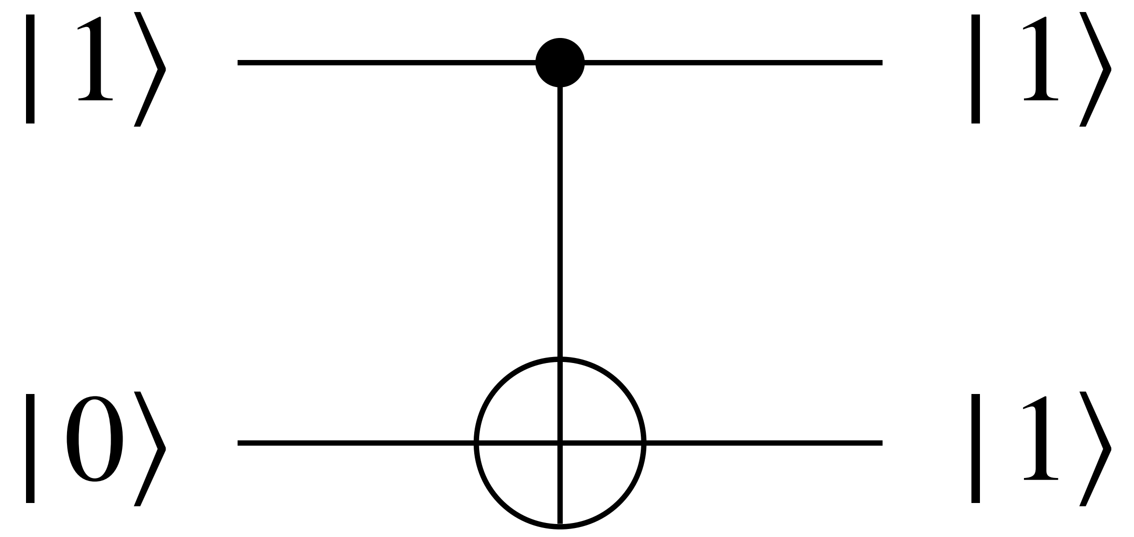
Elementary gates

- 1-qubit gate
  - Pauli gates: X, Y, Z $(\sigma_x, \sigma_y, \sigma_z)$
  - rotation gates: $R_P(\theta) = e^{-i\theta P/2}$, $(P = X, Y, Z)$
- 2-qubit gate
  - CNOT (CX) gate, etc.
- CX and Pauli rotations→universal computation

- **quantum algorithm**:
  - identify the target unitary U
  - implement unitary U by elementary gates
  - acting unitary on an initial state: $|\psi\rangle = U|0\rangle^{\otimes N}$
  - measure observable: $\langle\psi|O|\psi\rangle$
- quantum error correction: we can correct quantum errors during computation



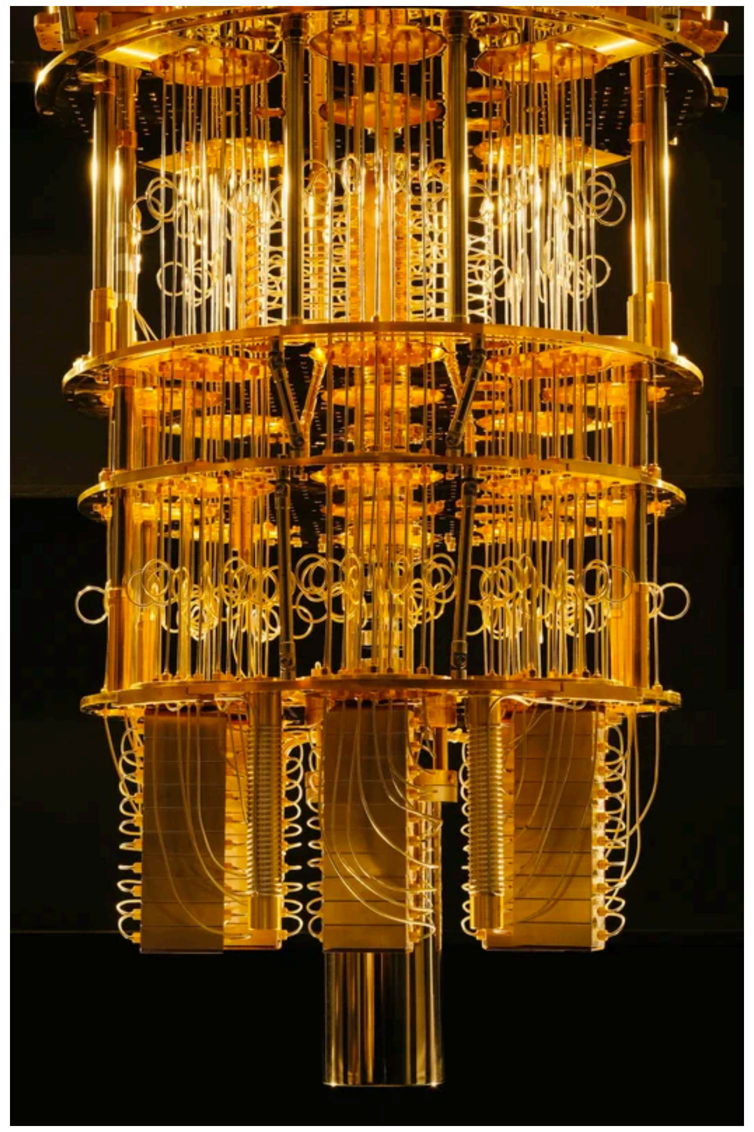
X gate



CNOT (CX) gate

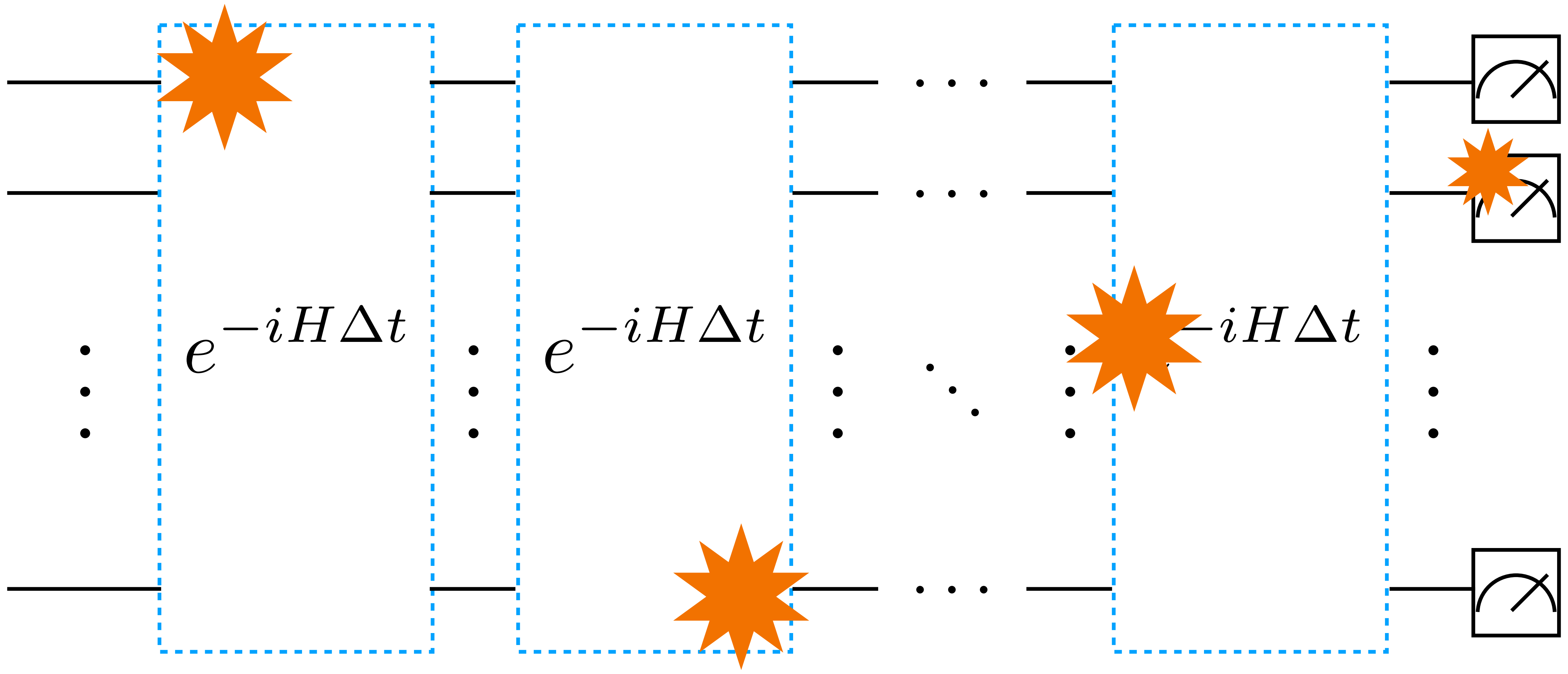
Hardware developments

- hardware realizations:
  - superconducting: IBM, google, etc
  - ion-trap: IonQ, quantinuum, etc.
  - photonic: Xanadu, etc.



[IBM research, Flickr]

- noisy intermediate-scale quantum: NISQ
  - the number of qubits $\sim \mathcal{O}(100)$
  - large quantum noise
  - no quantum error correction
    $\rightarrow$operating many gates is challenging
  - applications to physics with/without quantum error correction?

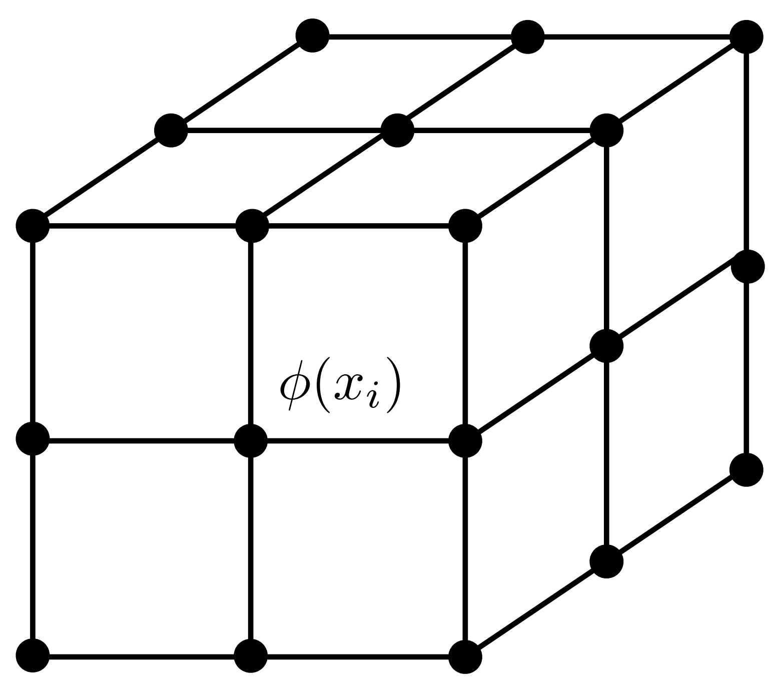


Lattice gauge theory

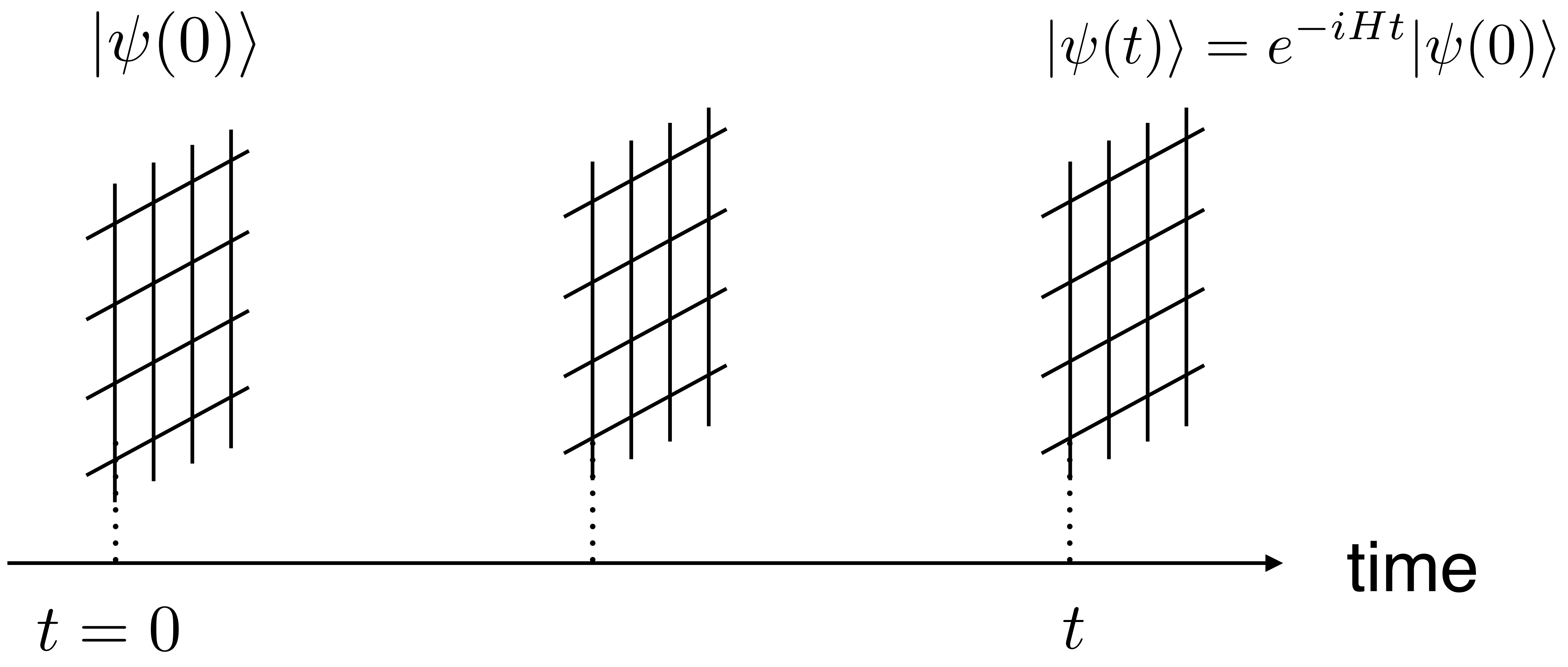
- <u>(conventional) lattice gauge theory</u>
  - discretize **spacetime**
    →using Monte Carlo method

$$Z = \int [d\phi] e^{-S[\phi]} \to \sum_{\{\phi_i\}} e^{-S(\phi_i)}$$

  - infamous sign problem
    - topological term
    - real-time dynamics, etc.



- <u>Hamiltonian simulation</u>
  - discretize **space**
  - no sign problem!
  - need exponential resources…
    - quantum computing
    - tensor network, etc.

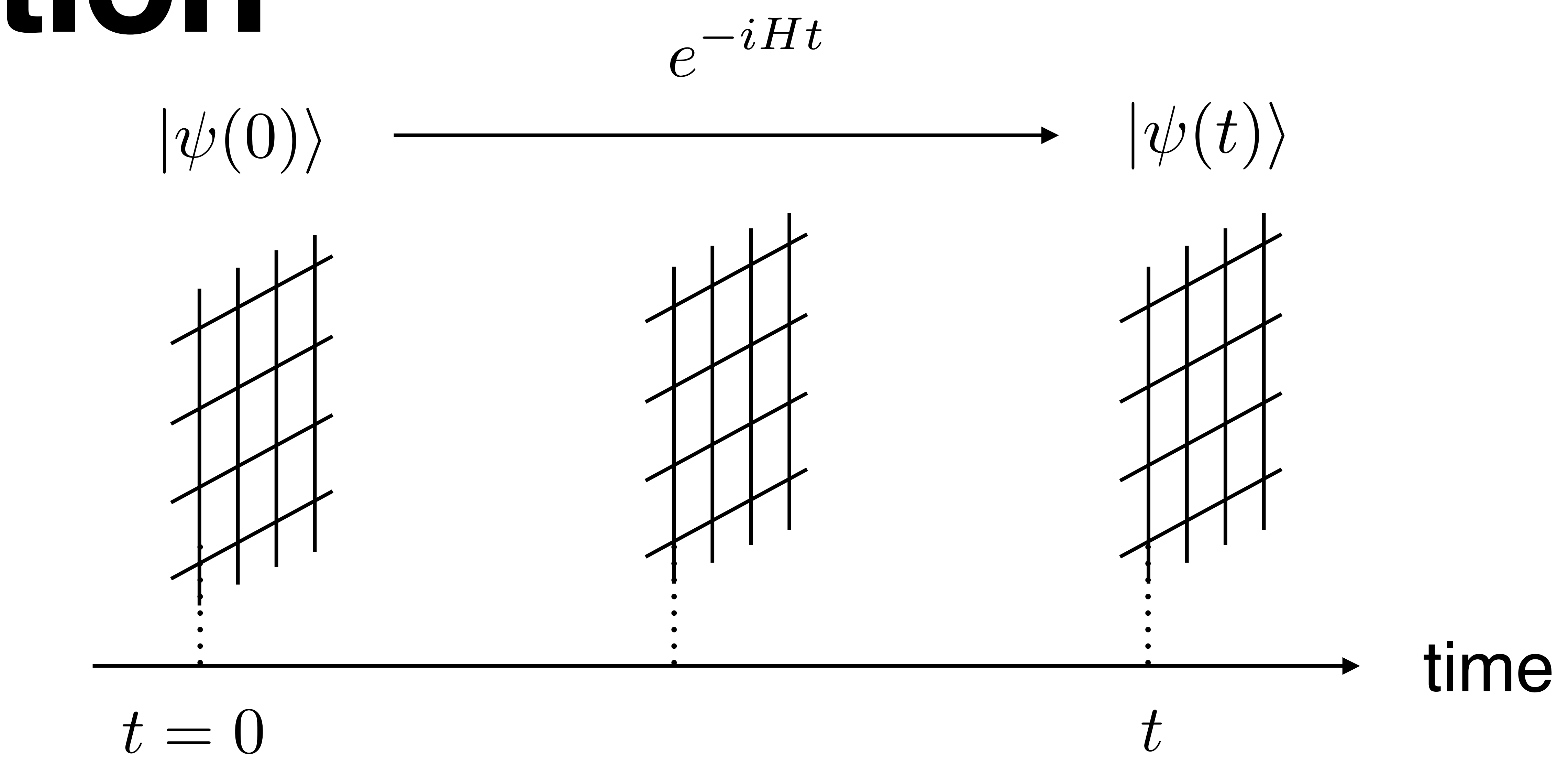


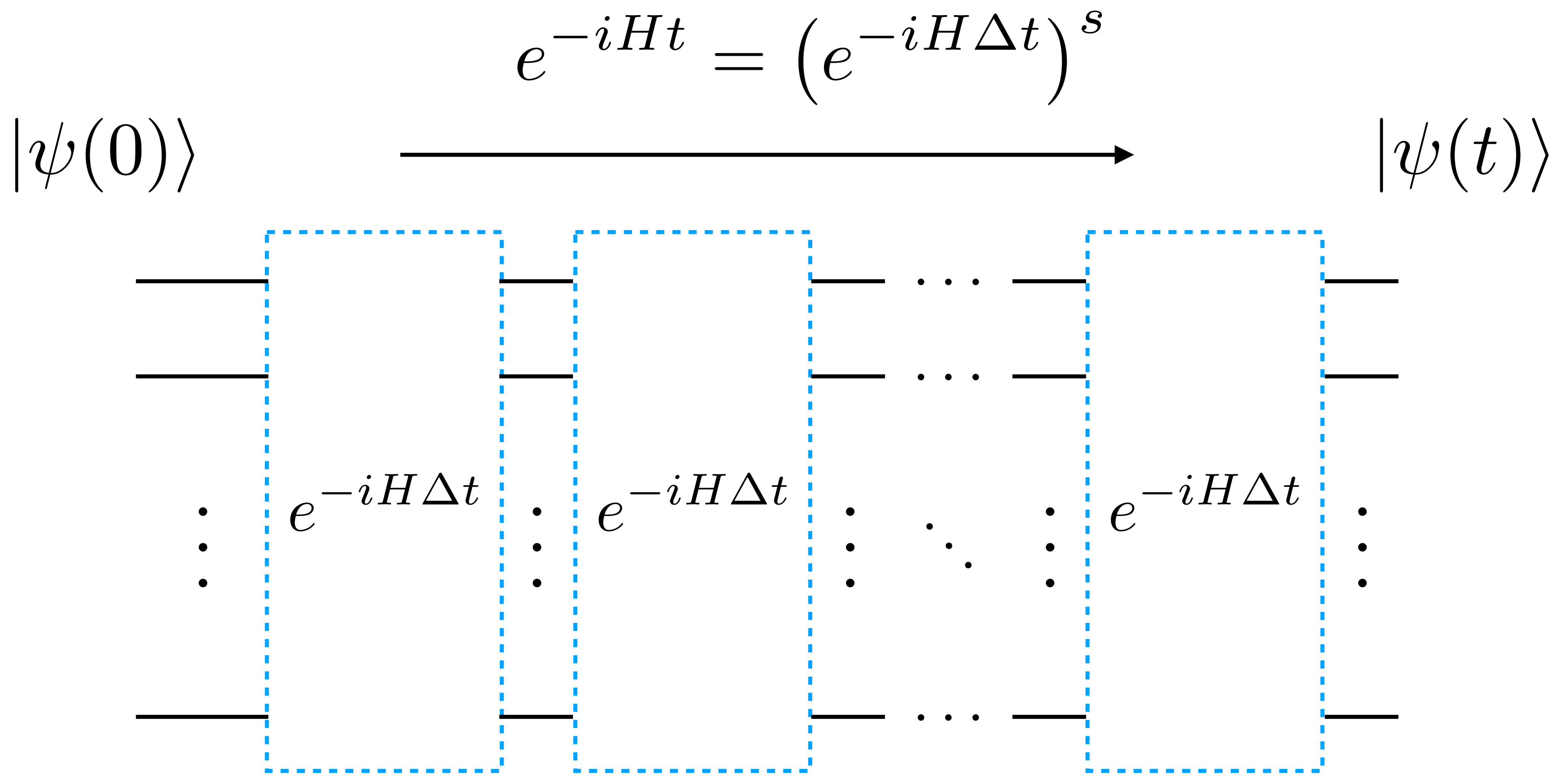
Digital quantum simulation

- quantum simulation:
  simulation using a quantum computer
  - real-time evolution $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$
  - adiabatic time evolution

$$|\psi_{\rm GS}\rangle = e^{-i\int dt H_A(t)t}|\psi^{(0)}_{\rm GS}\rangle$$

- applications to HEP: e.g. scattering problem
  [Jordan, Lee, Preskill, Science 336, 1130-1133 (2012)]
  - pros: exponential advantage
  - cons:
    - still need many resources
    - near-term (NISQ) applications?





Contents of this talk

- simple gauge theory: Schwinger model
- confinement/screening in the Schwinger model [Honda, Itou, Kikuchi, LN, Okuda, Phys. Rev. D 105, 014504]
  - obtain the ground state in the presence of probe charges
  - method: adiabatic state preparation via Suzuki-Trotter decomposition (FTQC application)
- pair-creation due to the non-perturbative effect (the Schwinger mechanism) [LN, Bapat, Bauer [arXiv:2302.10933]]
  - investigate quench dynamics in the presence of external field
  - method: real-time evolution via variational quantum algorithms (NISQ-friendly application)

Confinement in the Schwinger model

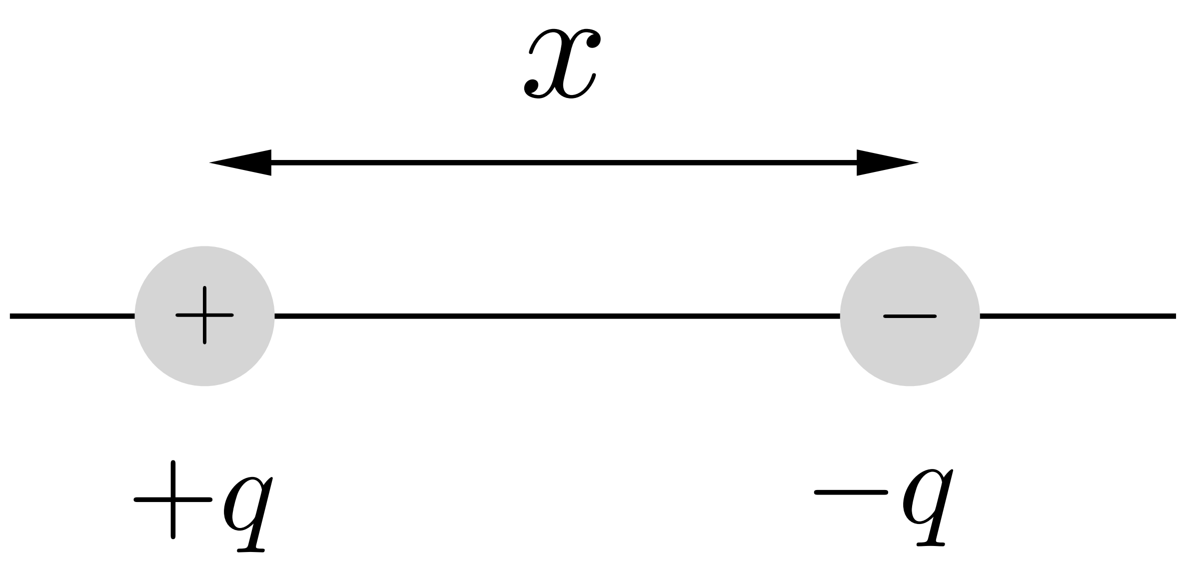
Schwinger model

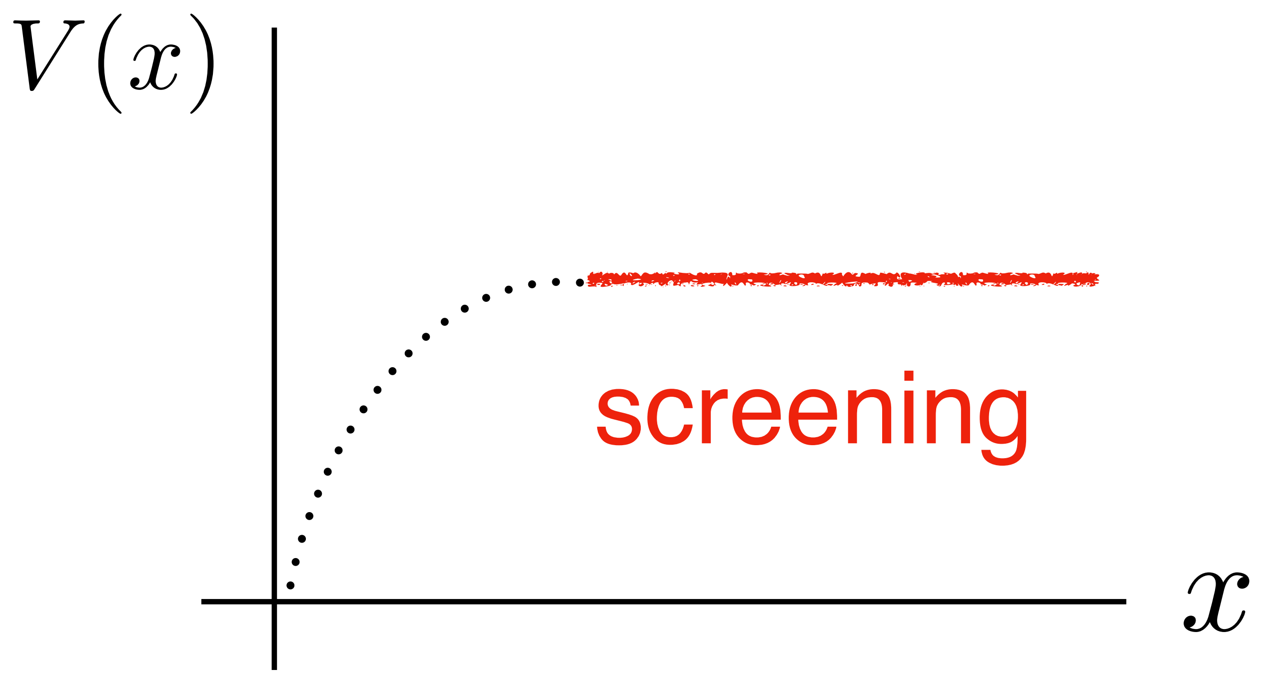
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g\theta}{4\pi} \epsilon^{\mu\nu} F_{\mu\nu} + \mathrm{i}\bar{\psi}\gamma^{\mu} \left(\partial_{\mu} + \mathrm{i}gA_{\mu}\right) \psi - m\bar{\psi}\psi$$

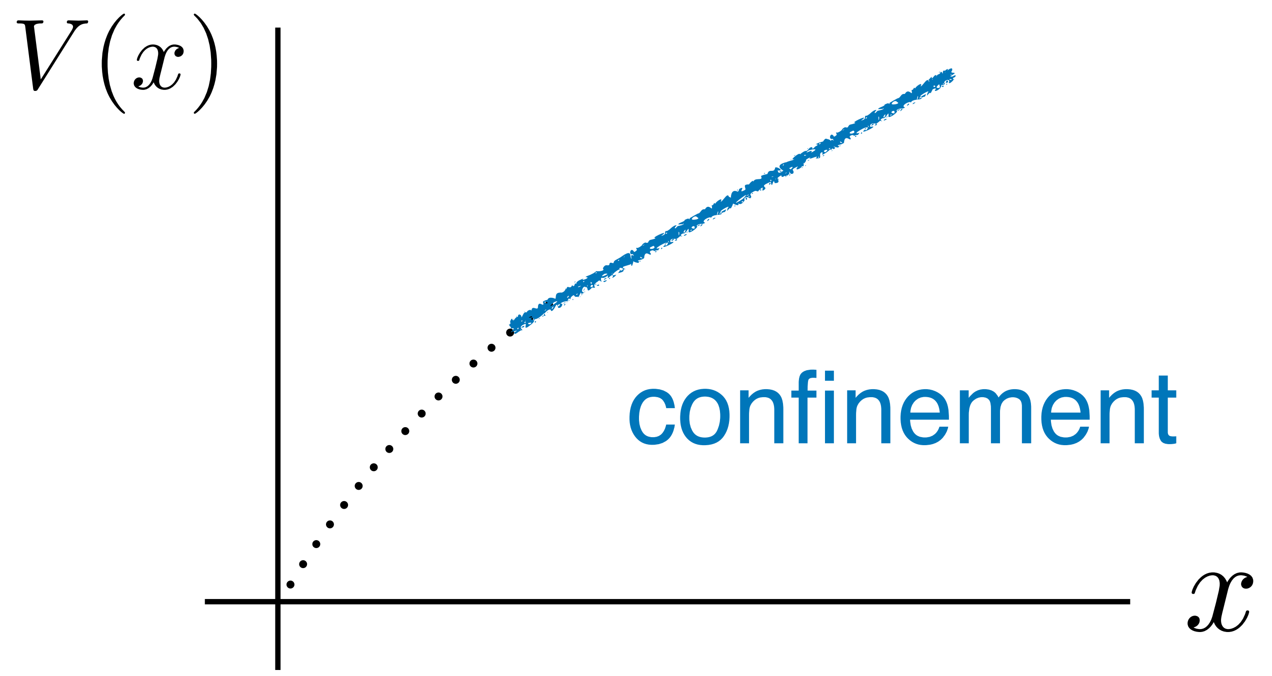
- simple toy model: 1+1d U(1) gauge theory = **Schwinger model** [Schwinger, Phys. Rev. 128, 2425]
  - exactly solvable for $m = 0$
  - mass perturbation is available for small mass regime
- simple but still non-trivial
  - screening/confinement phenomena
  - we can include the topological term (cannot be treated in the MC method)
    - the effects of the external field
    - the effects of probe charges

Screening vs Confinement in Schwinger model

[Schwinger]
[Gross-Klebanov-Matytsin-Smilga, Iso-Murayama]





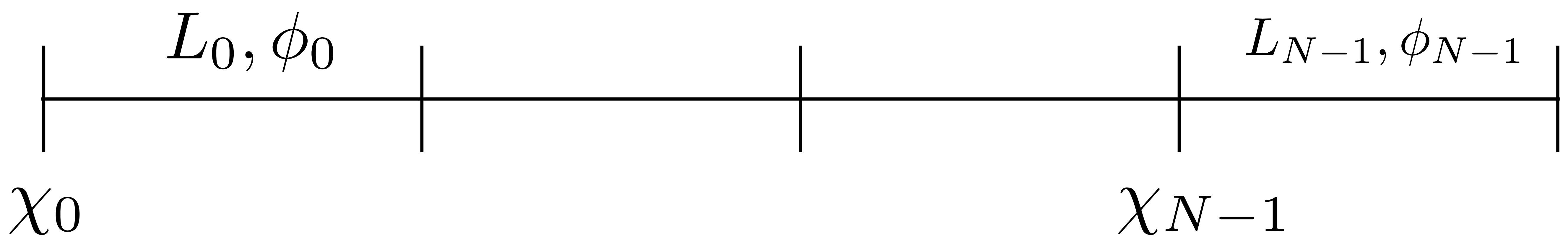


- $m = 0$ (exactly solvable): $V(x) = \dfrac{q^2 g^2}{2\mu}(1 - e^{-\mu x})$ screening

- $m \neq 0$ (mass pert.): $V(x) \sim m\Sigma\left[1 - \cos(2\pi q)\right] x$ screening, $q \in \mathbb{Z}$ / confinement, $q \notin \mathbb{Z}$

Lattice Hamiltonian of Schwinger model

- χ_n: staggered fermion [Susskind, Kogut-Susskind]
- L_n, ϕ_n: link variables (gauge field)



$$H_{\mathrm{lat}} = J \sum_{n=0}^{N-2} \left(L_n + \frac{\theta}{2\pi} \right)^2 - \mathrm{i}w \sum_{n=0}^{N-2} \left(\chi_n^\dagger e^{\mathrm{i}\phi_n} \chi_{n+1} - \mathrm{c.c.} \right) + m \sum_{n=0}^{N-1} (-)^n \chi_n^\dagger \chi_n$$

- gauge invariance: Gauss's law constraint

$$L_n - L_{n-1} = \chi_n^\dagger \chi_n - \frac{1-(-)^n}{2}$$

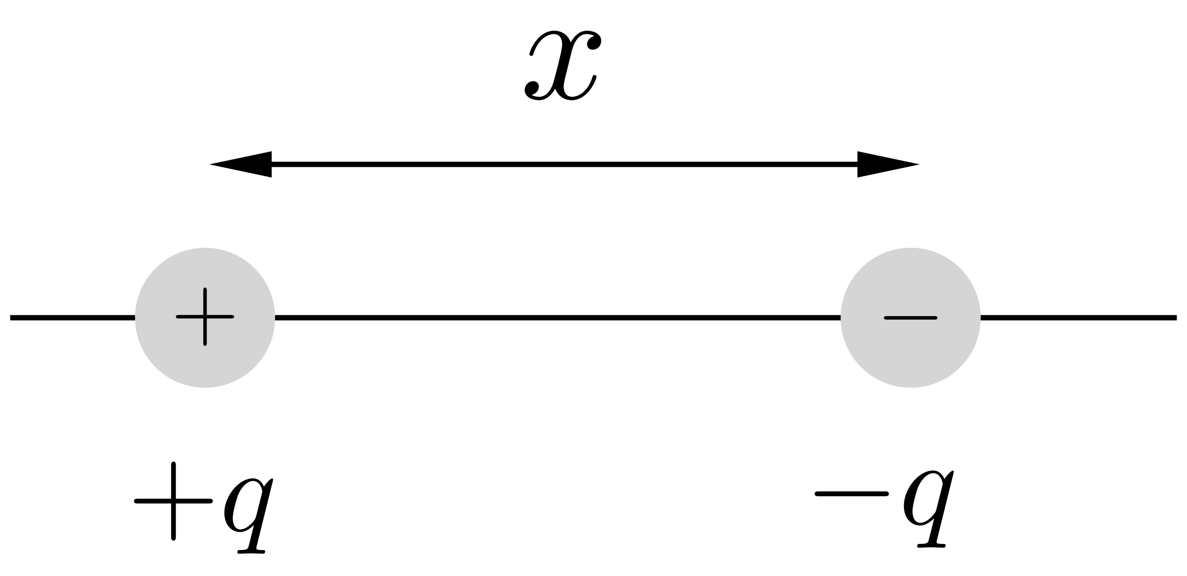
- we can **eliminate** gauge fields!
  - automatically gauge invariant, no boson fields
  - cannot be used in higher dimension

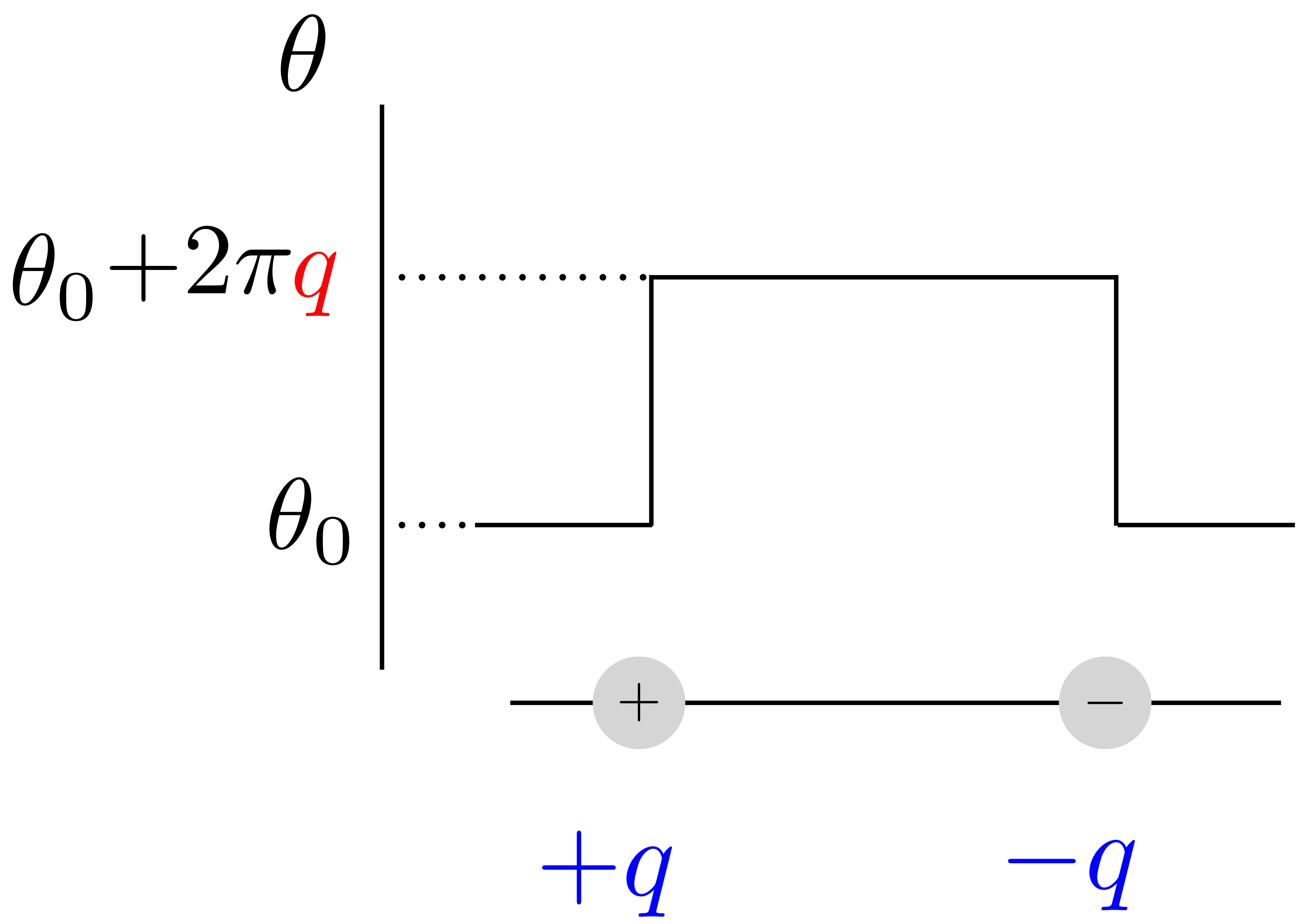
Spin Hamiltonian of the Schwinger model

- fermion formalism→spin system (Jordan-Wigner transformation)

$$H_{\mathrm{spin}} = J\sum_{n=0}^{N-2}\left(\sum_{k=0}^{n}\frac{Z_k+(-)^k}{2}+\frac{\theta_n}{2\pi}\right)^2+\frac{w}{2}\sum_{n=0}^{N-2}(X_nX_{n+1}+Y_nY_{n+1})+\frac{m}{2}\sum_{n=0}^{N-1}(-)^n Z_n$$

- constant background electric field↔ $\theta_n = \theta$ (constant)
- introducing probe charges↔position-dependent θ_n





Adiabatic state preparation

Goal: obtain ground state in the presence of probe charges (position dep. θ_n)

$$H = J \sum_{n=0}^{N-2} \left(\sum_{k=0}^{n} \frac{Z_k + (-)^k}{2} + \frac{\theta_n}{2\pi} \right)^2 + \frac{w}{2} \sum_{n=0}^{N-2} (X_n X_{n+1} + Y_n Y_{n+1}) + \frac{m}{2} \sum_{n=0}^{N-1} (-)^n Z_n$$

$H_A(0) = H_0$ $\quad$ $H_A(t)$ $\quad$ $H_A(T) = H$

t

$|\Omega_0\rangle$

$$|\Omega\rangle = \lim_{T \to \infty} \mathrm{T} \exp \left(-\mathrm{i} \int_0^T \mathrm{d}t H_A(t) \right) |\Omega_0\rangle$$

ground state of H_0

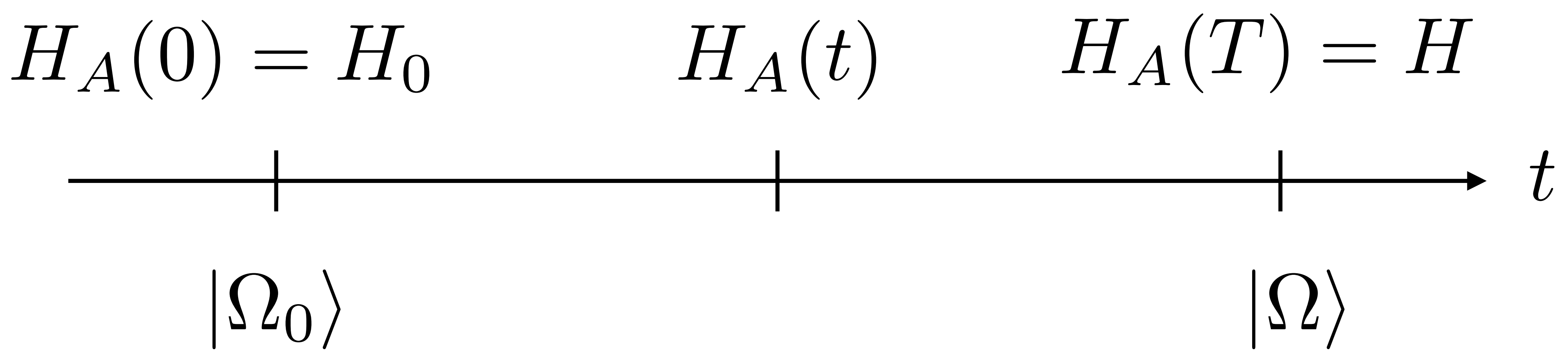
no charges, $w = 0$

ground state of H

∃probe charges

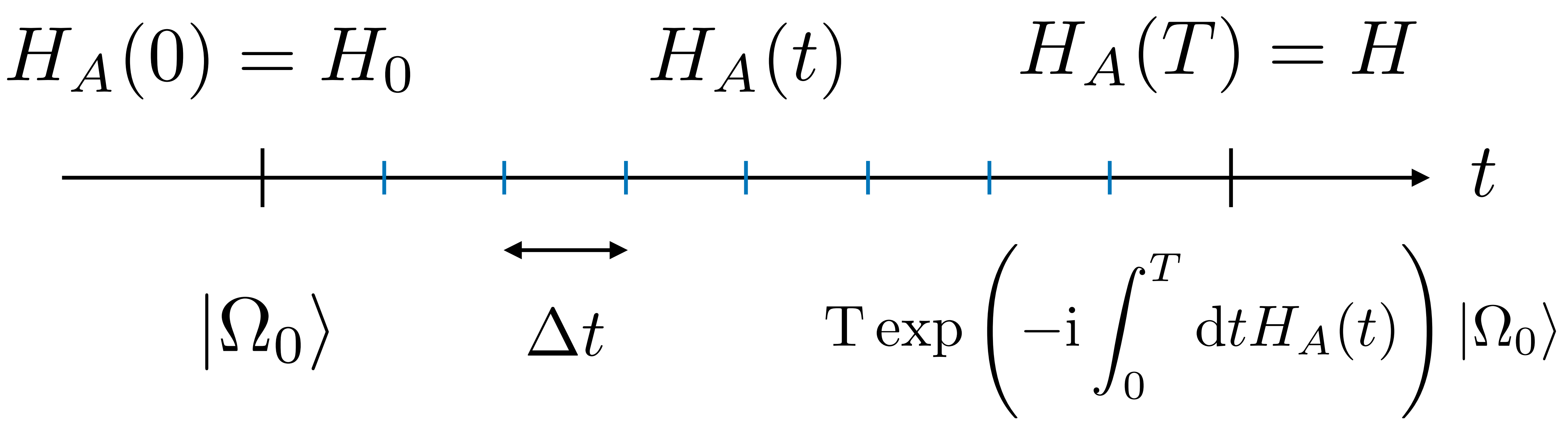
Adiabatic state preparation

- adiabatic theorem



$$|\Omega\rangle = \lim_{T\to\infty} \mathrm{T}\exp\left(-\mathrm{i}\int_0^T \mathrm{d}t H_A(t)\right)|\Omega_0\rangle$$

- Suzuki-Trotter decomposition



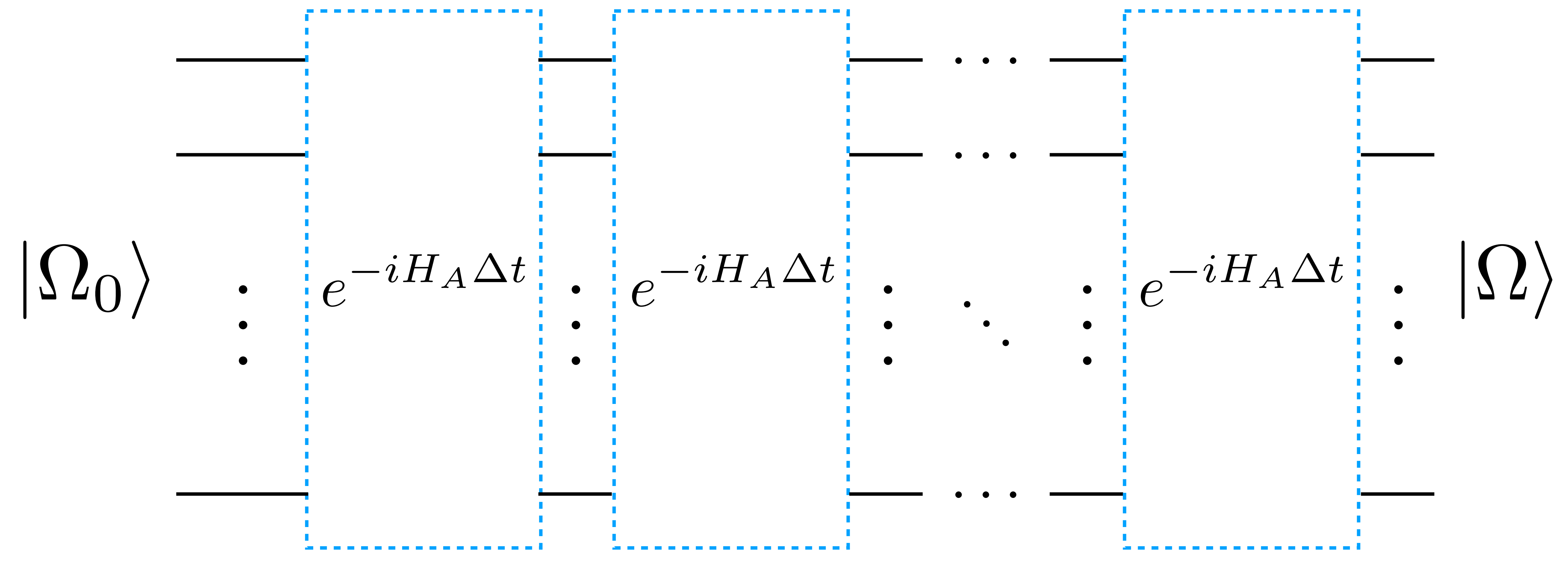
$$\mathrm{T}\exp\left(-\mathrm{i}\int_0^T dt H_A(t)\right) \simeq \prod_{s=1}^{M} \exp\left(-\mathrm{i}H_A(s\Delta t)\right)$$

$$\simeq \prod \big(\text{elementary gates}\big)$$

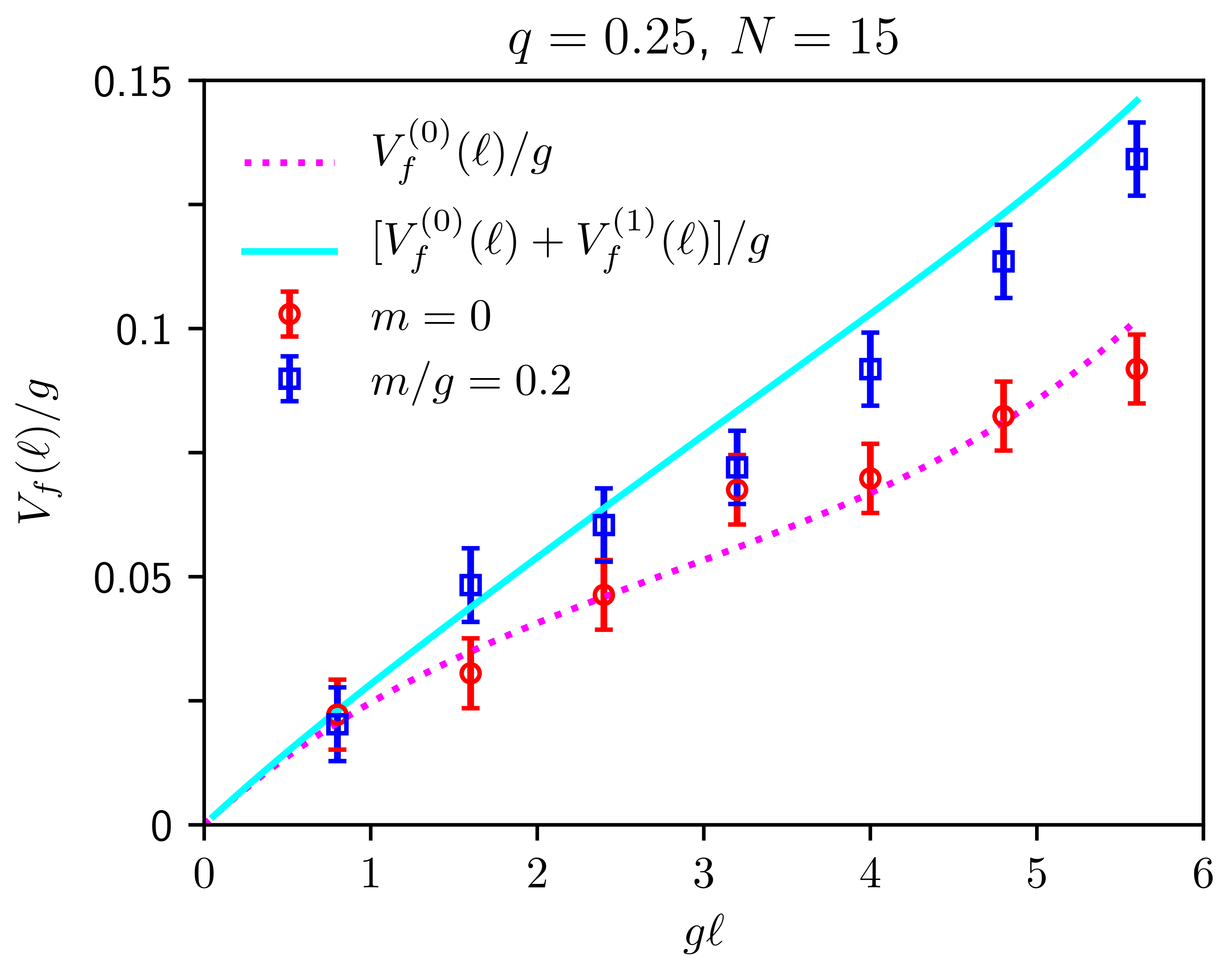
Results for $q \notin \mathbb{Z}$

[Honda, Itou, Kikuchi, LN, Okuda, Phys. Rev. D 105, 014504]

- digitized adiabatic state preparation→ compute energy $\langle \Omega | H | \Omega \rangle$
- expect confinement for massive case (in infinite volume and continuum limit)
  screening massless
- cyan/magenta curves: analytic results→no plateau due to finite volume effect
- linear behavior for massive case!



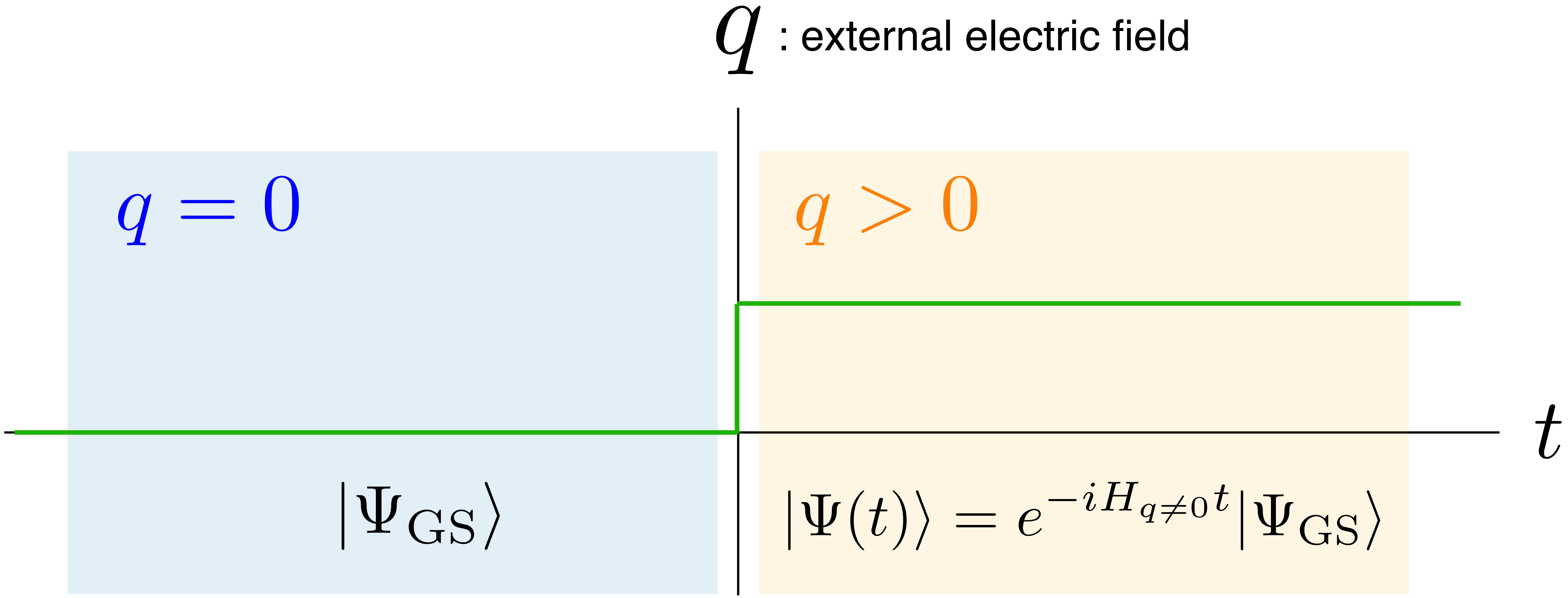
$$\mathrm{T}\exp\left(-\mathrm{i}\int_0^T \mathrm{d}t H_A(t)\right)$$

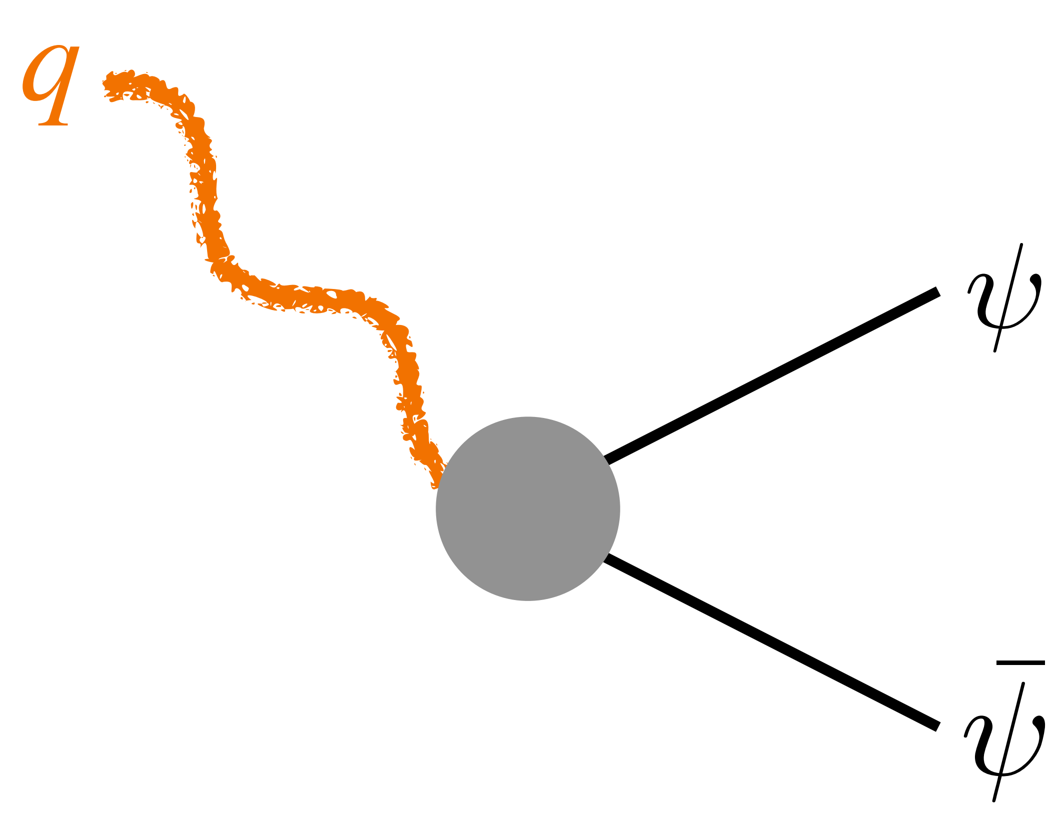


Quench dynamics in the Schwinger model

Quench dynamics in the Schwinger model

- Schwinger effect: particle pair creation due to strong external electric field [Schwinger, Phys. Rev. 82, 664, (1951)]
- Method: variational quantum algorithm (VQE+VQS)





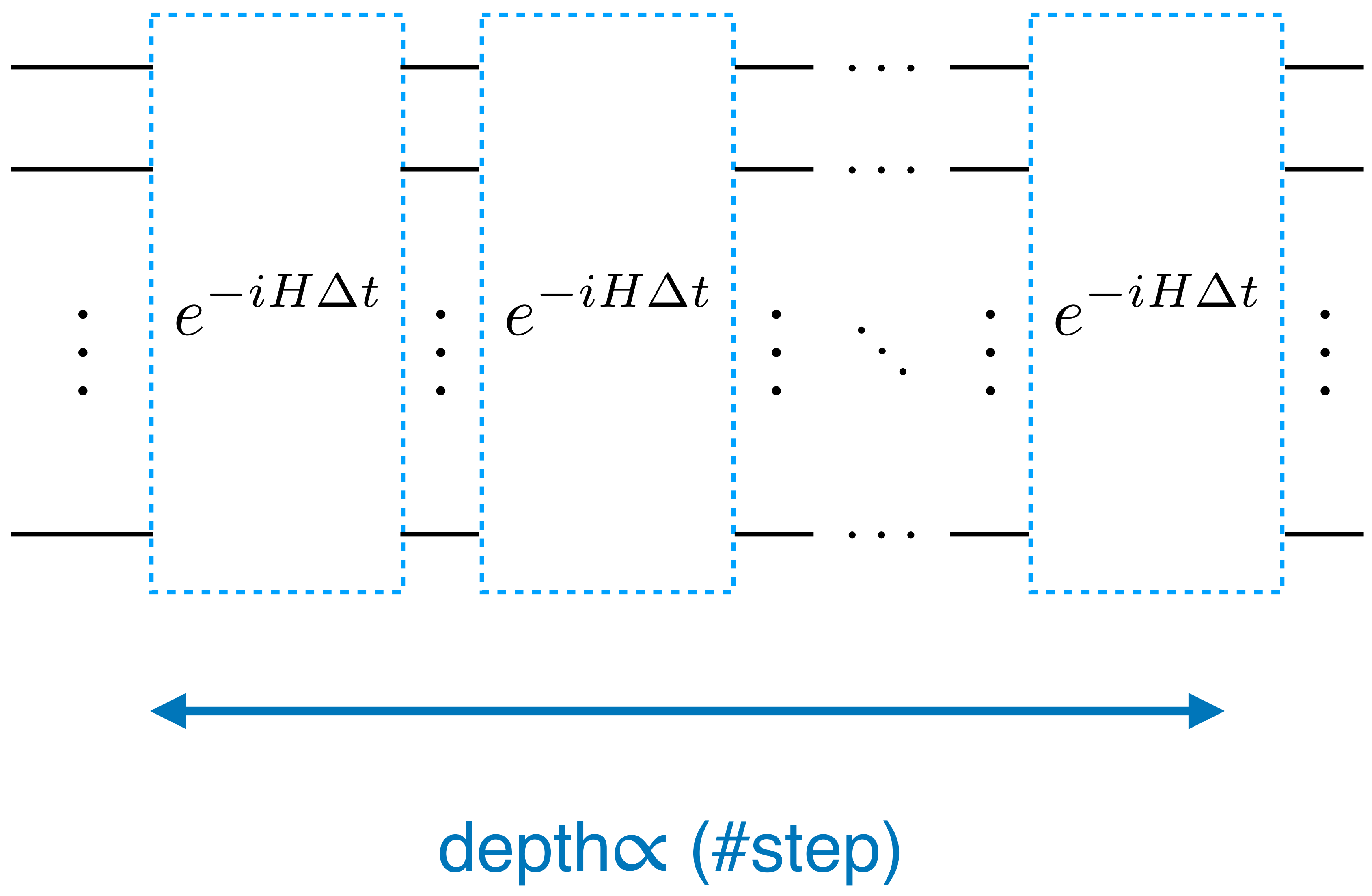
Ground state without external field

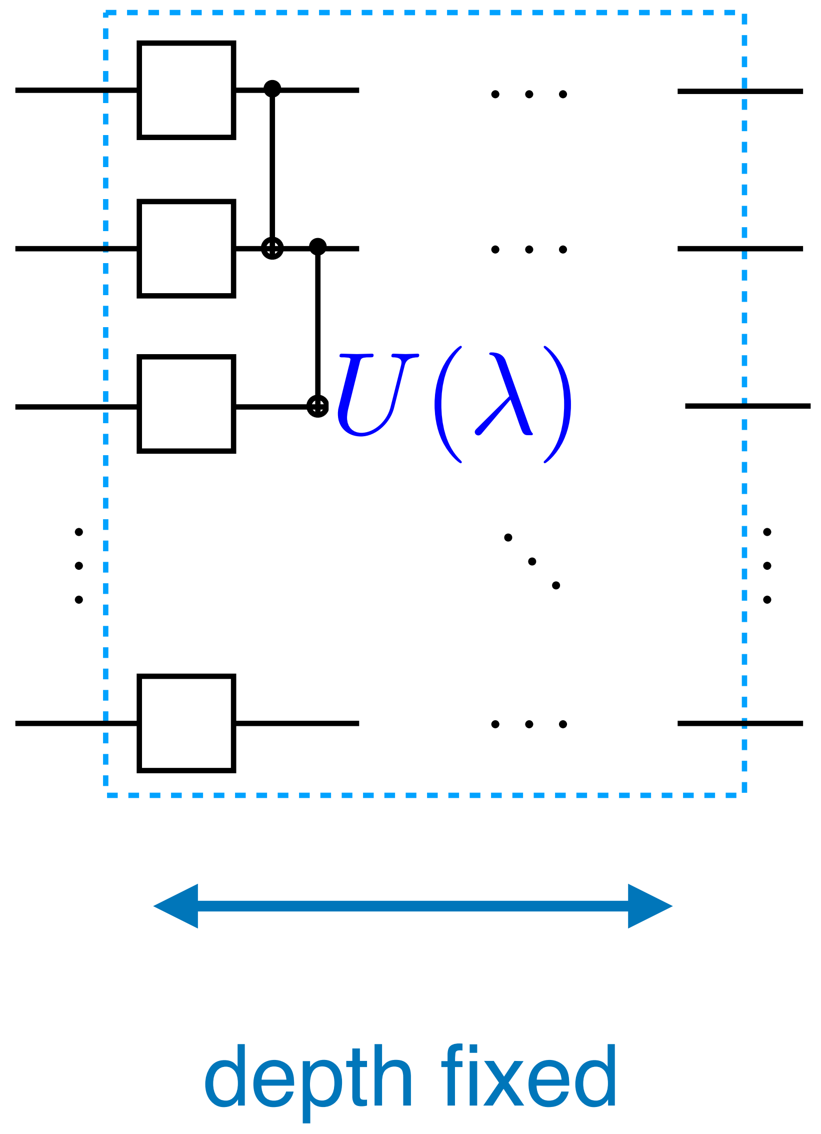
time evolution with external field
→pair-creation?

Suzuki-Trotter vs variational method

- Suzuki-Trotter method
  - #depth grows with #steps
  - decoherence problem on NISQ devices

- variational quantum algorithm (VQA)
  - approximate states by ansatz with fixed depth
  - state preparation: variational quantum eigensolver (VQE)
  - time-evolution: variational quantum simulation (VQS)



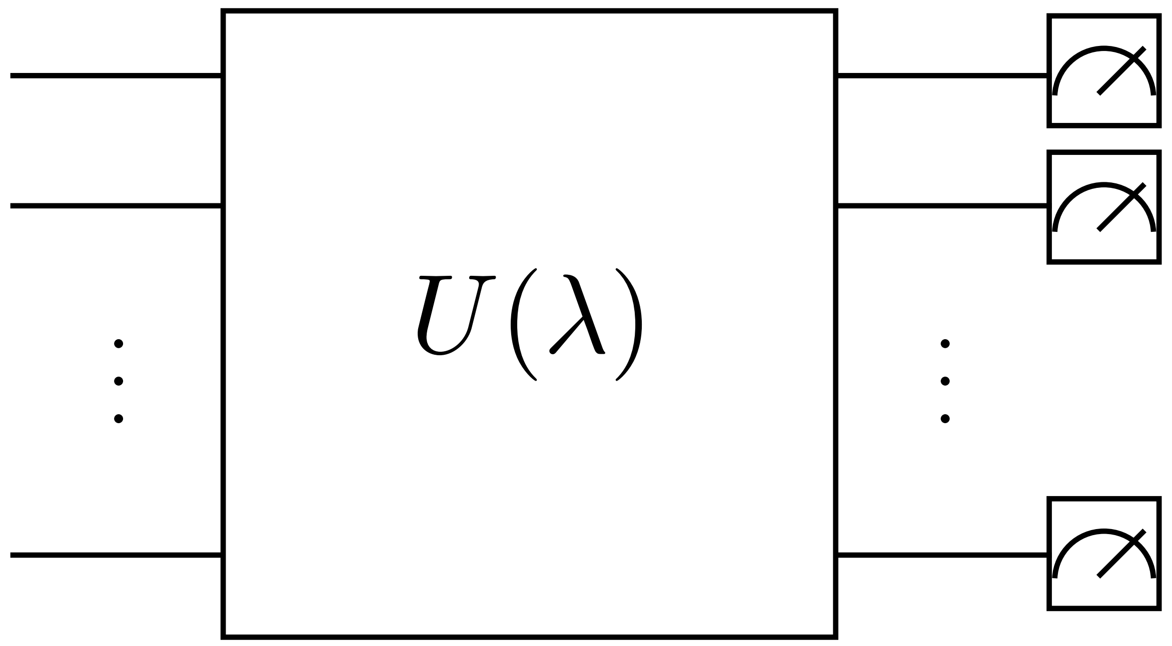


$$|\psi(\lambda)\rangle \approx \begin{cases} |\Psi_{\mathrm{GS}}\rangle & \text{(VQE)} \\ e^{-iHt}|\psi(0)\rangle & \text{(VQS)} \end{cases}$$

Variational quantum eigensolver

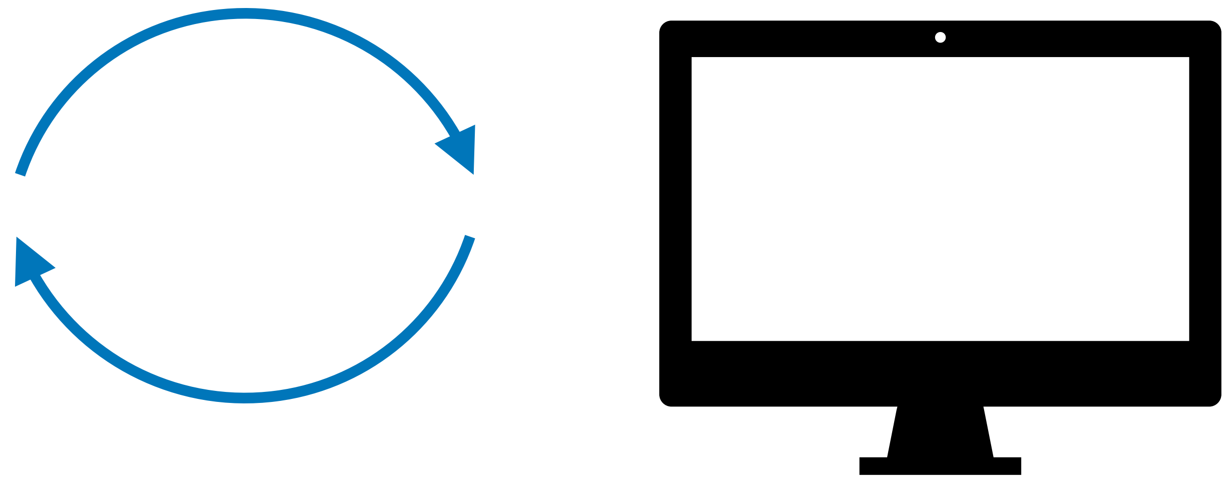
- goal: obtain the ground state
- approximate the ground state by ansatz $|\psi(\lambda)\rangle$
- optimize cost function $C(\lambda) = \langle\psi(\lambda)|H|\psi(\lambda)\rangle$ via classical computer
  $\rightarrow$ground state is given by $|\psi(\lambda_*)\rangle$

$$|\psi(\lambda)\rangle = U(\lambda)|0\rangle$$



$$C(\lambda) = \langle\psi(\lambda)|H|\psi(\lambda)\rangle$$

classical computer



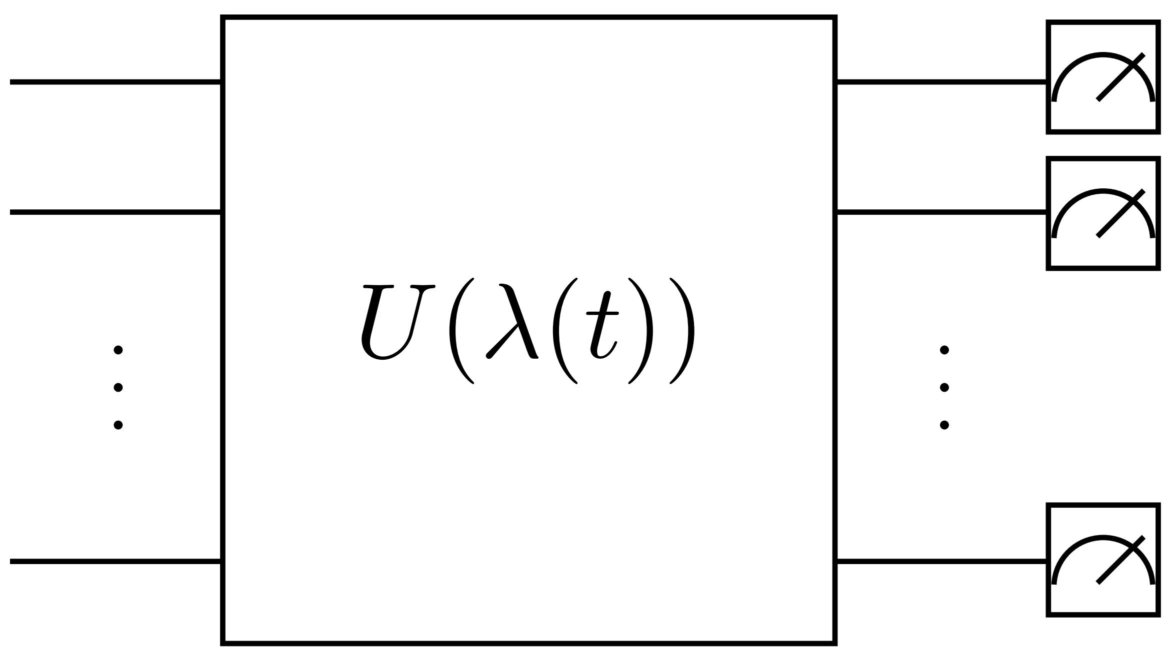
$$\lambda_* = \arg\min_{\lambda} C(\lambda)$$

Variational quantum simulation

[Li, Benjamin, Phys. Rev. X 7, 021050, (2017)]

- goal: obtain time-evolved state $|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$
- approximate $|\Psi(t)\rangle$ by ansatz $|\psi(\lambda(t))\rangle$ with time-dependent parameters
- evolution of states → evolution of parameters $\lambda(t)$ via McLacran's variational principle
- we use the same ansatz (Hamiltonian variational ansatz) for both VQE and VQS
  →quench dynamics: set $\lambda(0) = \lambda_*$ (obtained by VQE)

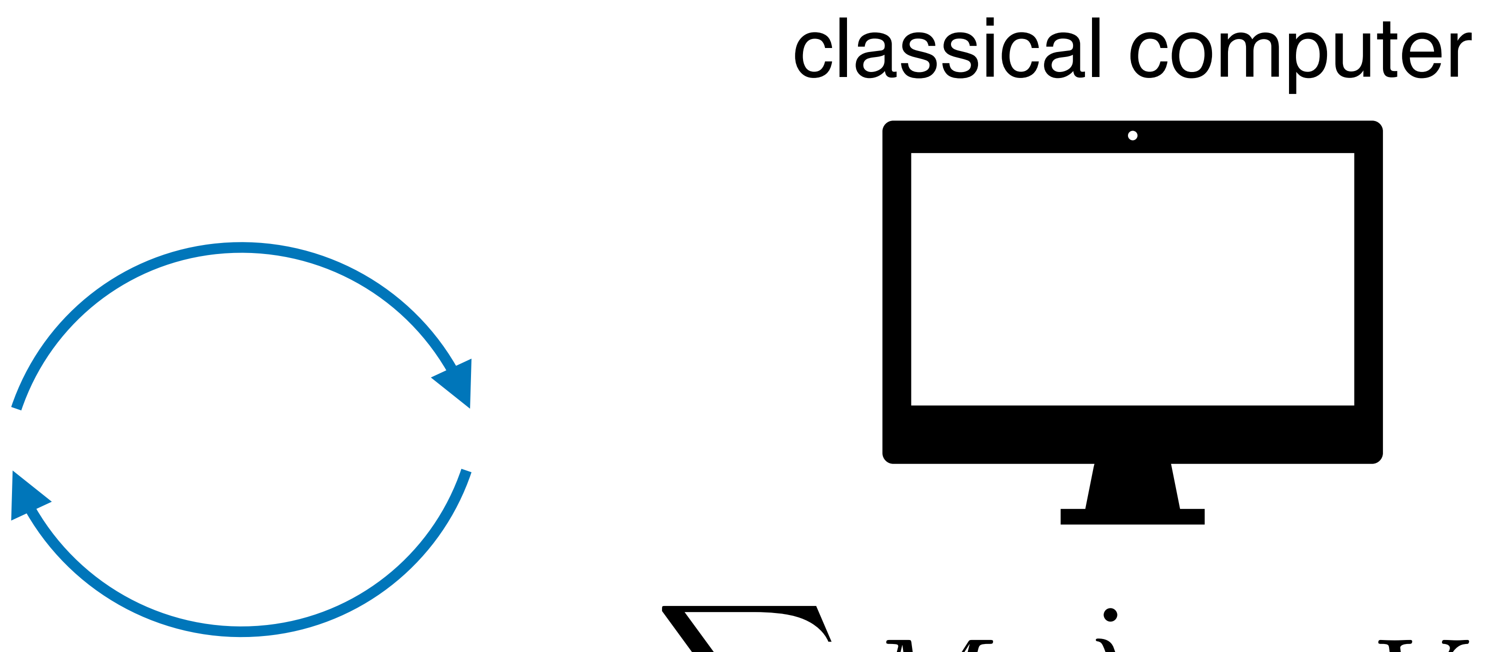
$$|\psi(\lambda(t))\rangle = U(\lambda(t))|0\rangle$$



$$M_{ij} = \mathrm{Re}\frac{\partial\langle\psi(\lambda)|}{\partial\lambda_i}\frac{\partial|\psi(\lambda)\rangle}{\partial\lambda_j}$$

$$V_i = \mathrm{Im}\frac{\partial\langle\psi(\lambda)|}{\partial\lambda_i}H|\psi(\lambda)\rangle$$

(+correction terms)



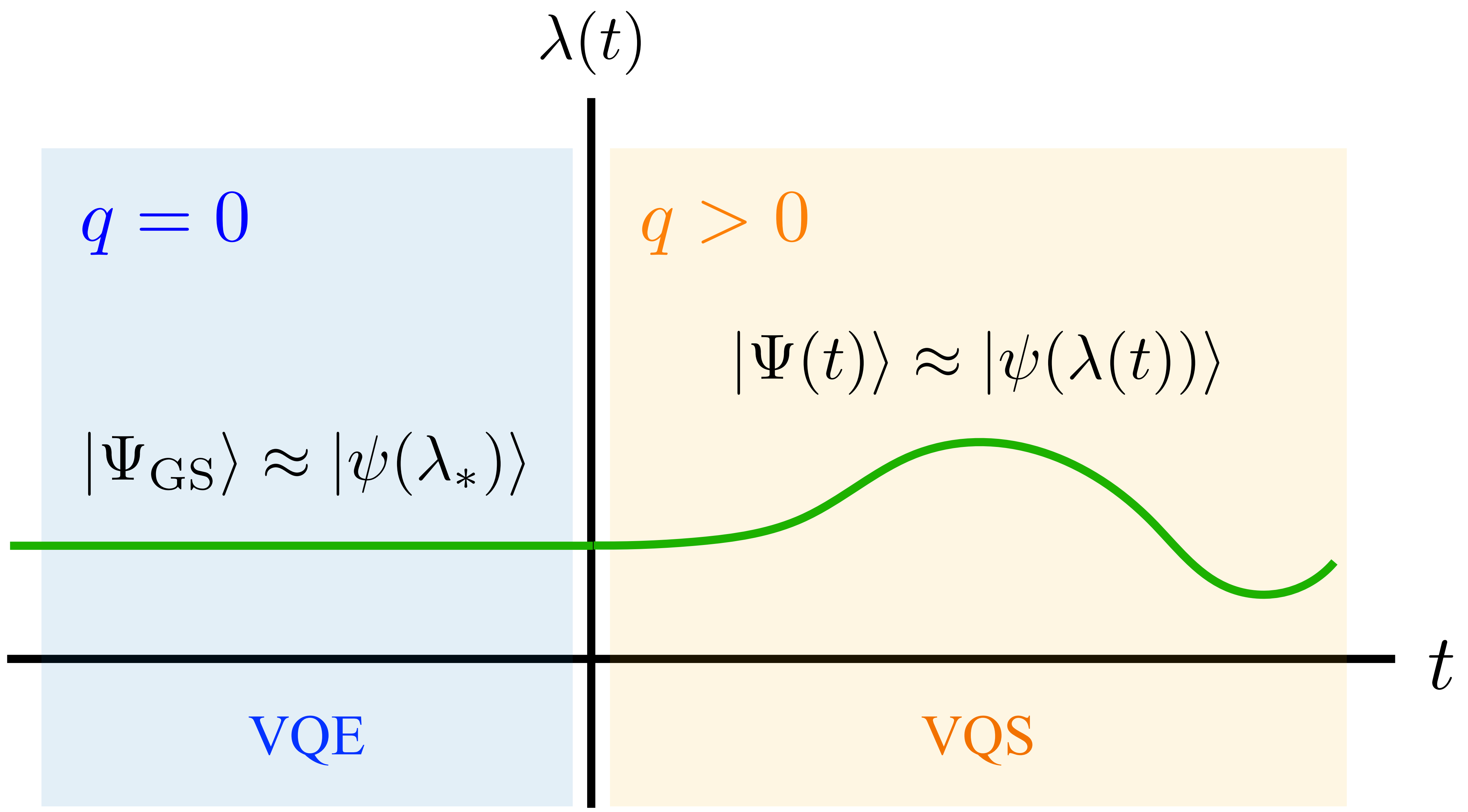
$$\sum_j M_{ij}\dot{\lambda}_j = V_i$$

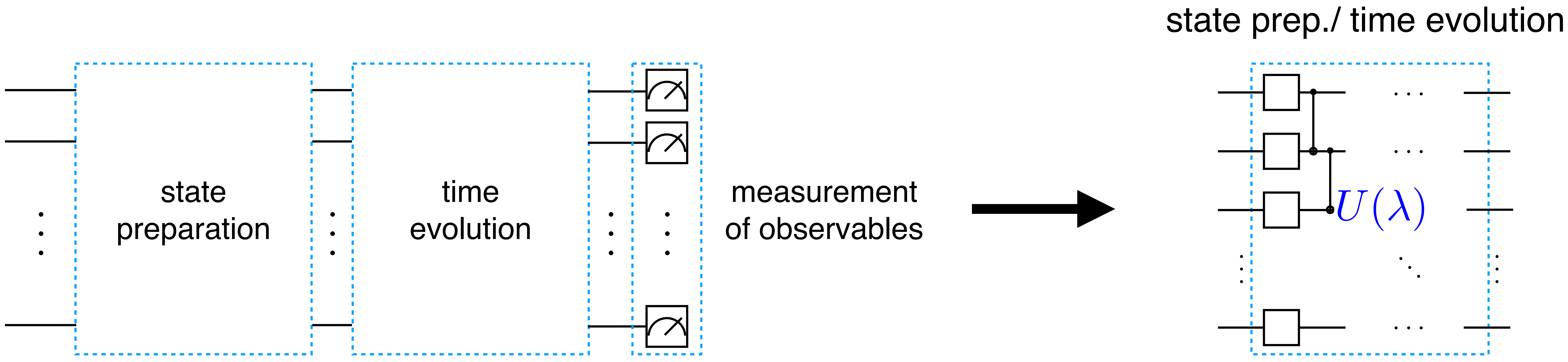
Summary of our protocol

- Quench dynamics in the Schwinger model
  - ground state without external field q: $|\Psi_{\rm GS}\rangle$
  - time evolution via Hamiltonian with external field q:

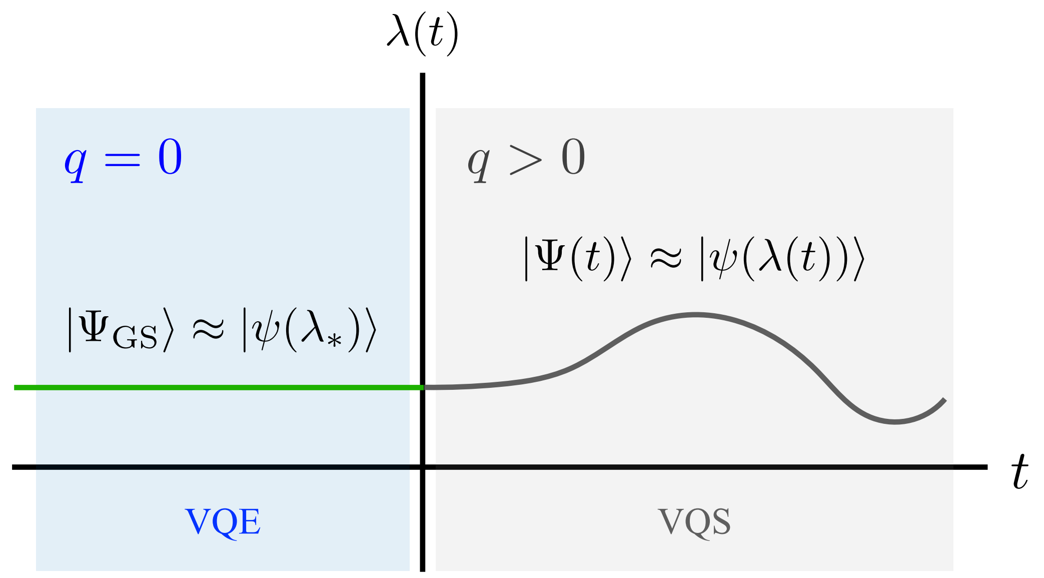
$$|\Psi(t)\rangle = e^{-iH_{q\neq 0}t}|\Psi_{\rm GS}\rangle$$

- perform VQE and VQS using the same ansatz $|\psi(\lambda)\rangle$
  - simulation with fixed depth
  - reduce overall circuit depth

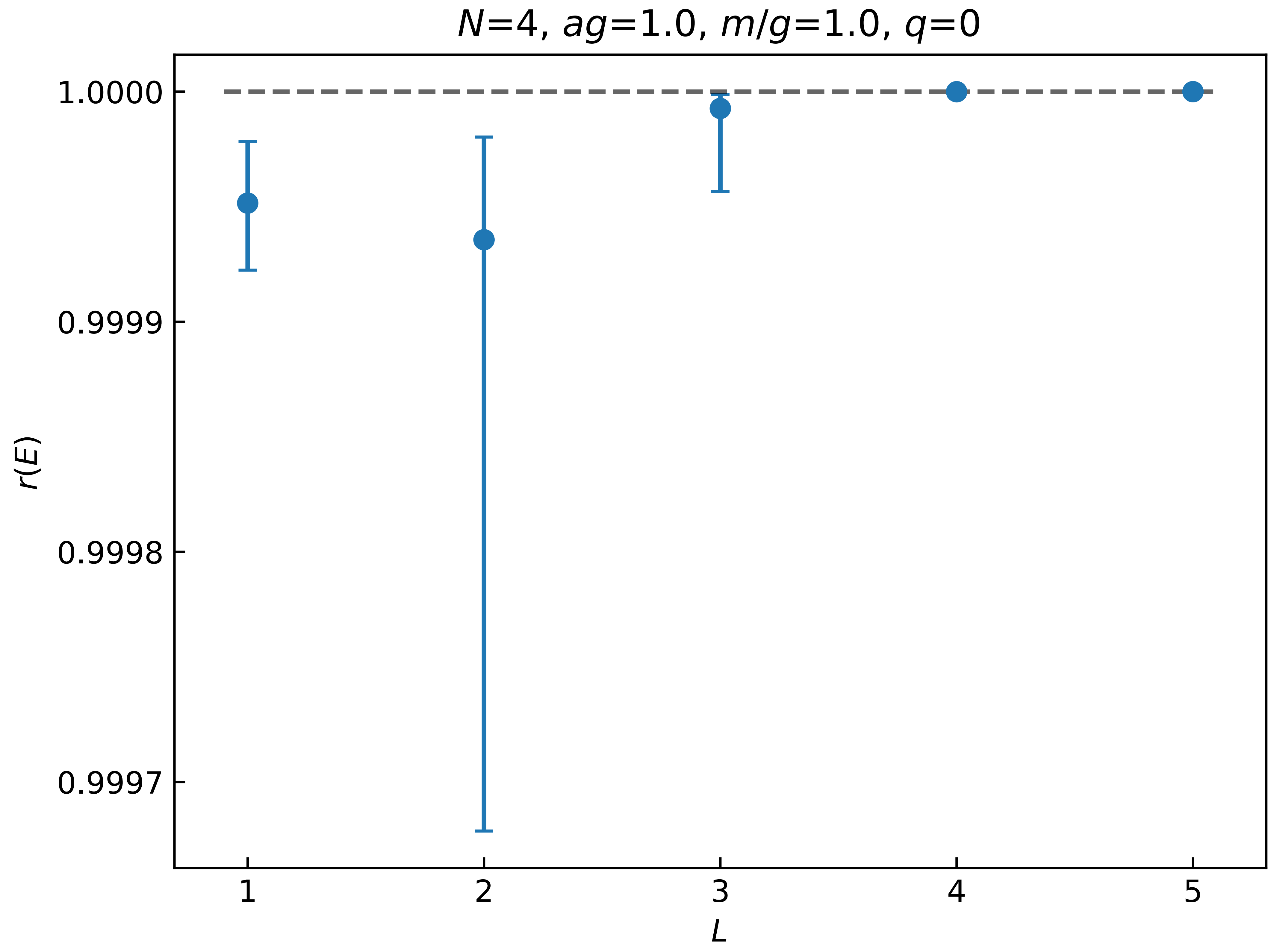




Ground state preparation via VQE

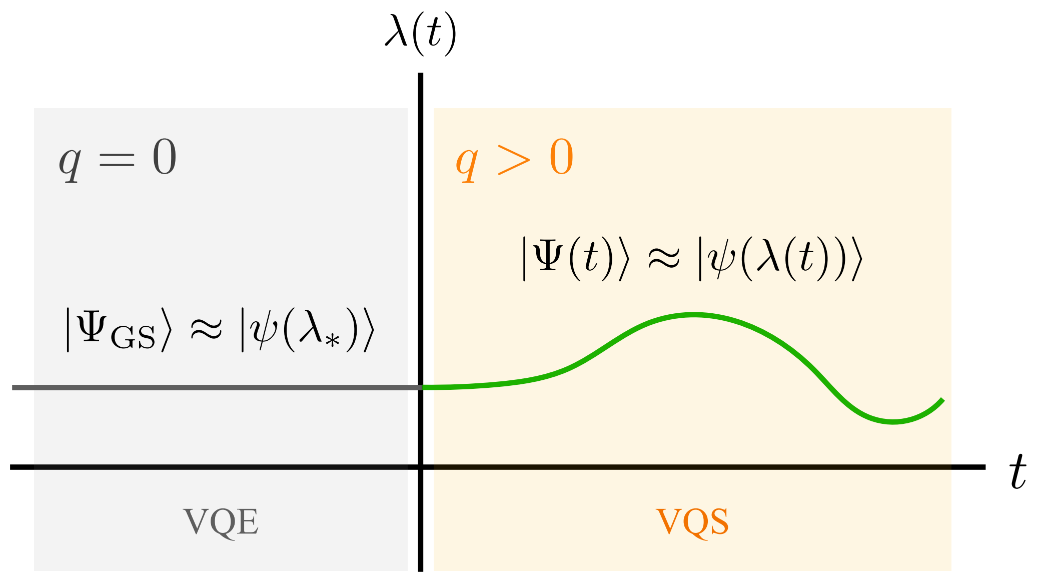


- compare VQE results with exact diagonalization (ED)
- a metric of accuracy: $r(E) = \dfrac{E_{\rm max} - E_{\rm VQE}}{E_{\rm max} - E_{\rm min}}$
  - $E_{\rm max}, E_{\rm min}$: max/min energy obtained by ED
  - $r(E) = 1$ for the best case
  - $r(E) = 0$ for the worst case
- L : depth of ansatz
- quality drastically improves for $L \geq 4$



- 20 samples with different initialization
- dots/bars represent medians and 25-75 percentiles

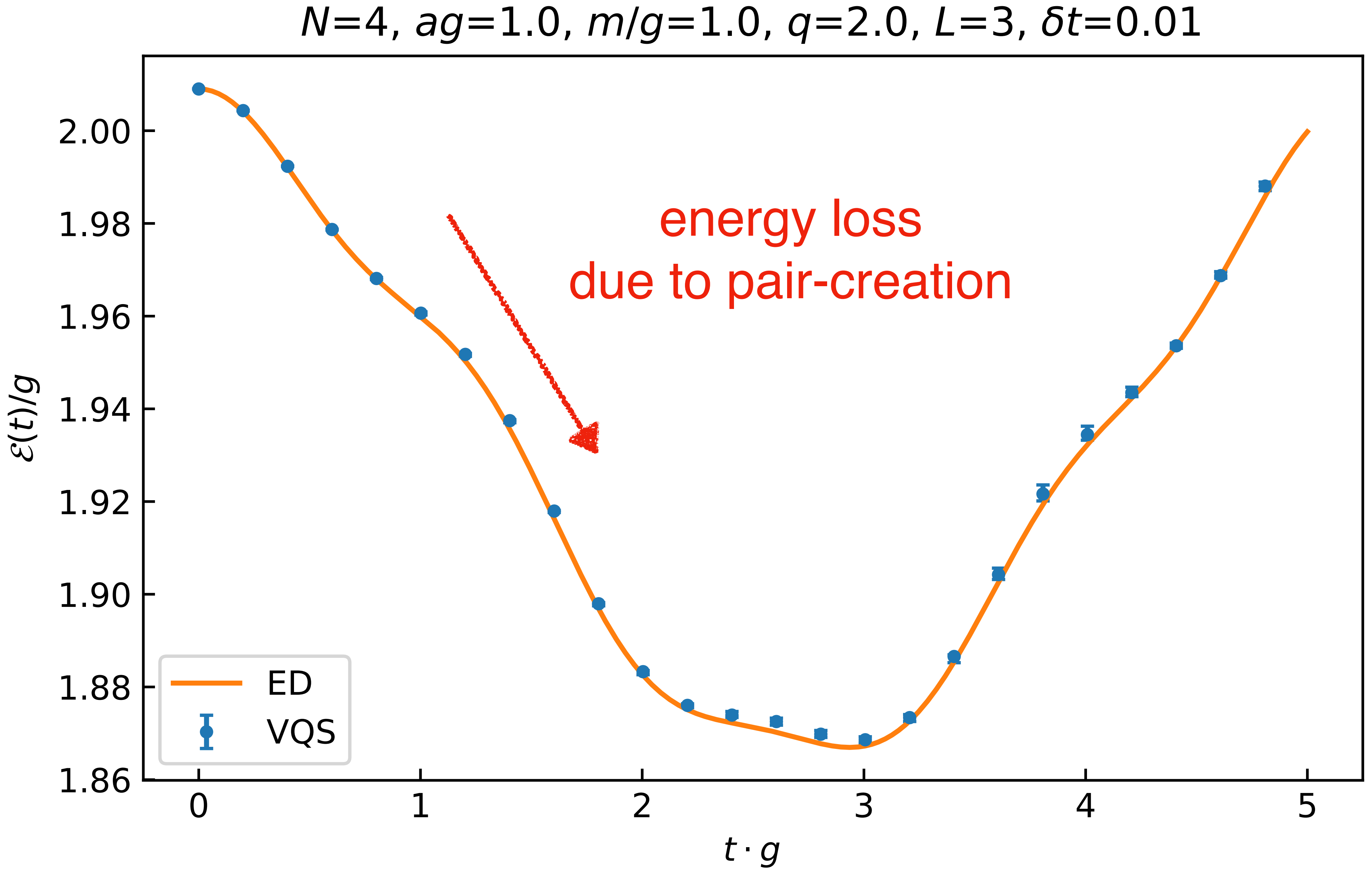
Real-time evolution via VQS



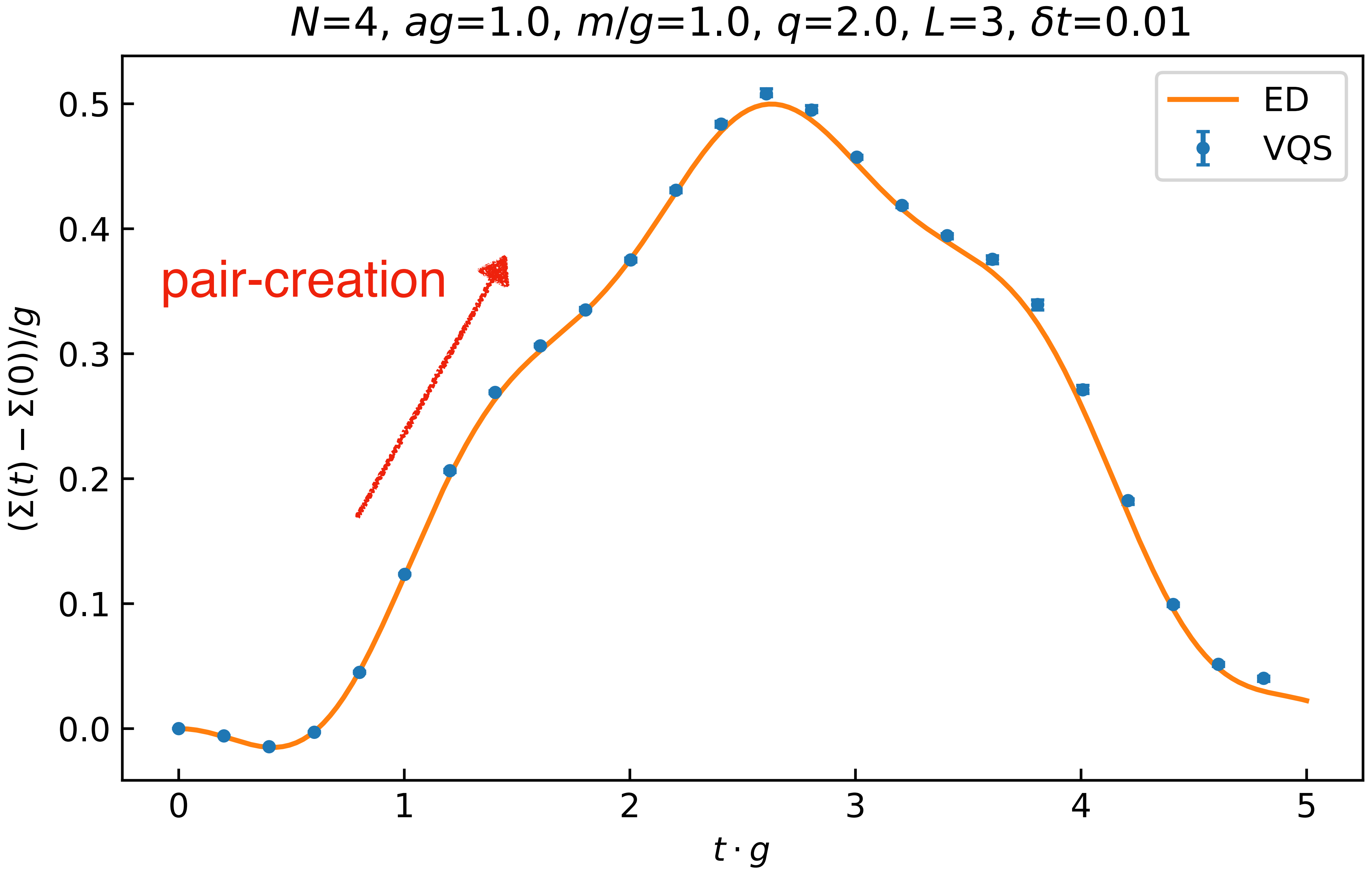
- two observables:
  - total electric field $\mathscr{E}$
  - chiral condensation $\langle \bar{\psi}\psi \rangle$ (~particle number density)
- observing energy loss and pair-creation!

- 20 samples with different initialization
- dots/bars represent medians and 25-75 percentiles

electric field



chiral condensation



Summary and outlooks

- confinement/screening in the Schwinger model via adiabatic state preparation
  - obtain ground state in the presence of probe charges
  - implement adiabatic evolution via Suzuki-Trotter decomposition
- quench dynamics in the Schwinger model via VQAs
  - ground state w/o external field q via VQE
  - time evolution via Hamiltonian w/ external field q via VQS
  - we can reduce circuit depth
- results from quantum algorithms agree well with analytic/exact results
- **future directions**:
  - (screening): NISQ-friendly algorithm?
  - (VQA): reducing measurement costs [in progress], error analysis
  - extension to higher dimensional and/or non-Abelian theories

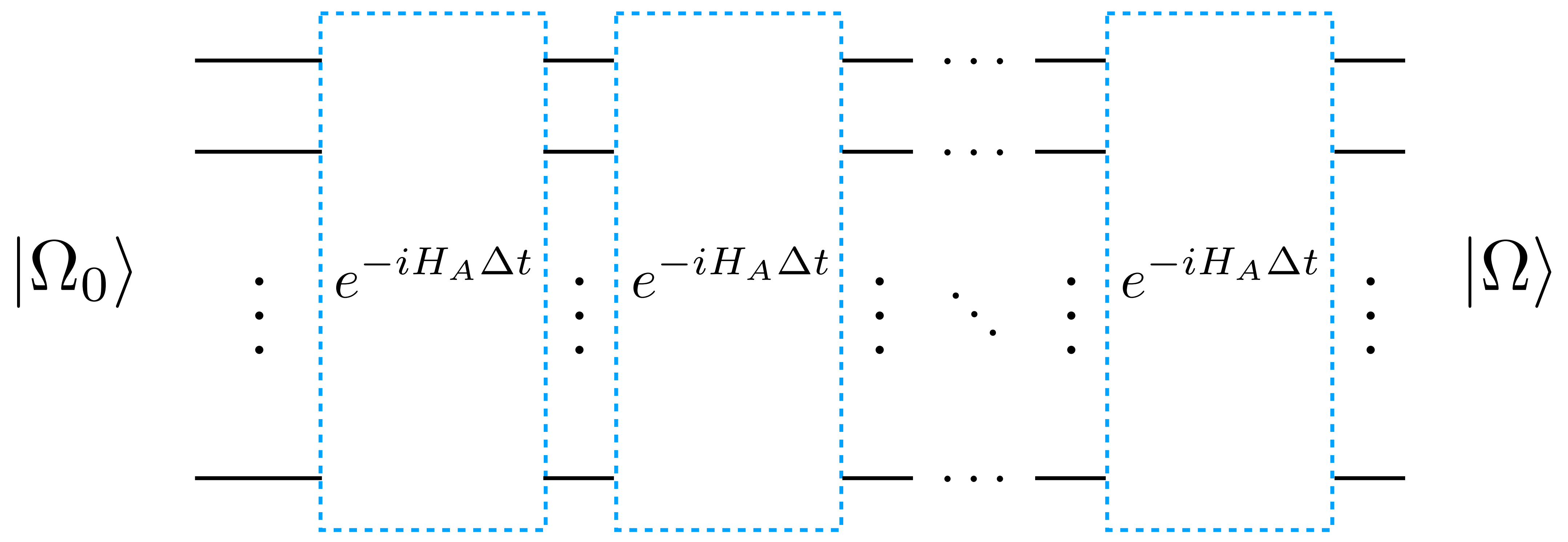
Backups

Suzuki-Trotter decomposition

- adiabatic state preparation

$$|\Omega\rangle = \lim_{T\to\infty} \mathrm{T}\exp\left(-\mathrm{i}\int_0^T \mathrm{d}t H_A(t)\right)|\Omega_0\rangle$$

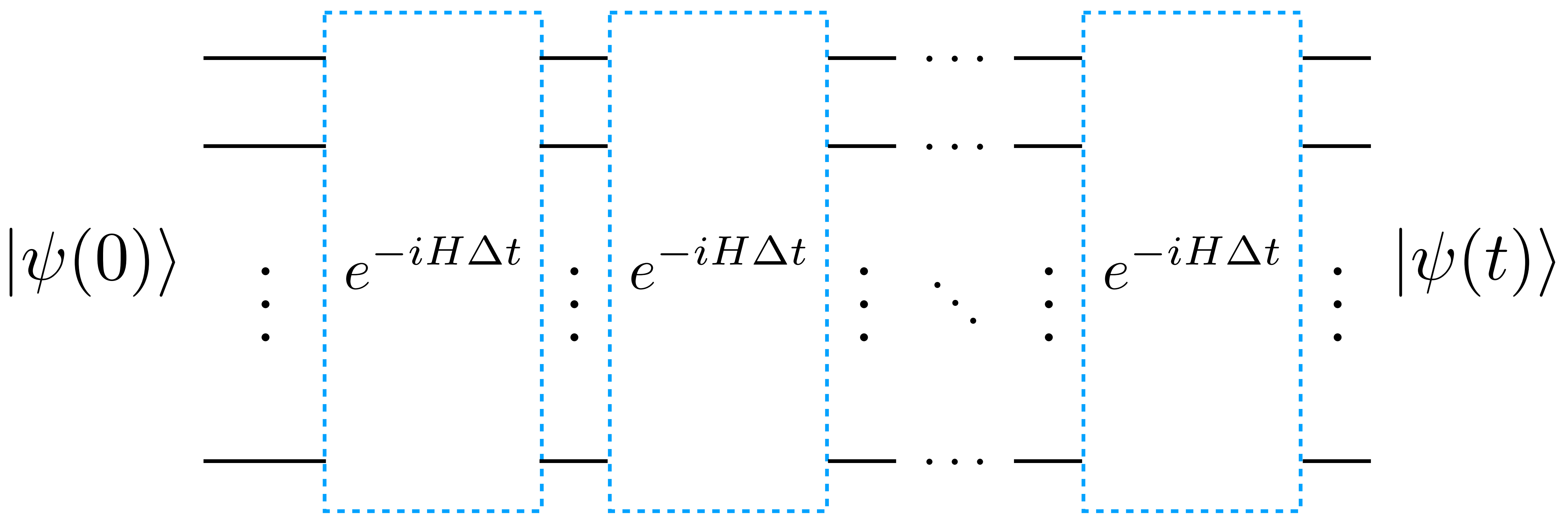
$$\simeq \prod_s e^{-\mathrm{i}H_A(s\Delta t)\Delta t}|\Omega_0\rangle$$



- real-time evolution

$$|\psi(t)\rangle = e^{-\mathrm{i}Ht}|\psi(0)\rangle = \left(e^{-\mathrm{i}H\Delta t}\right)^s|\psi(0)\rangle$$

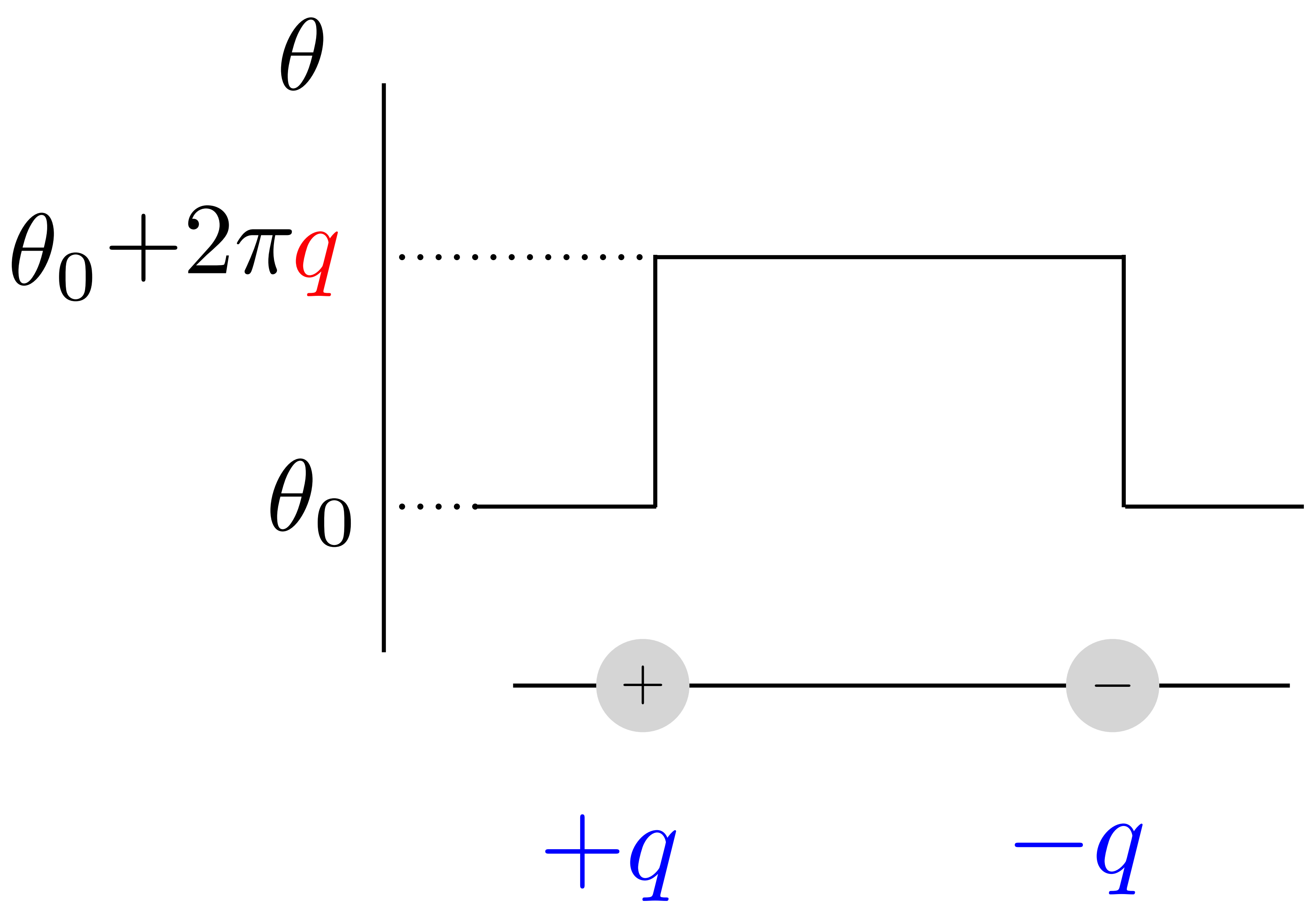
- drawback: #depth grows with #steps



Simulation setup

[Honda-Itou-Kikuchi-LN-Okuda]

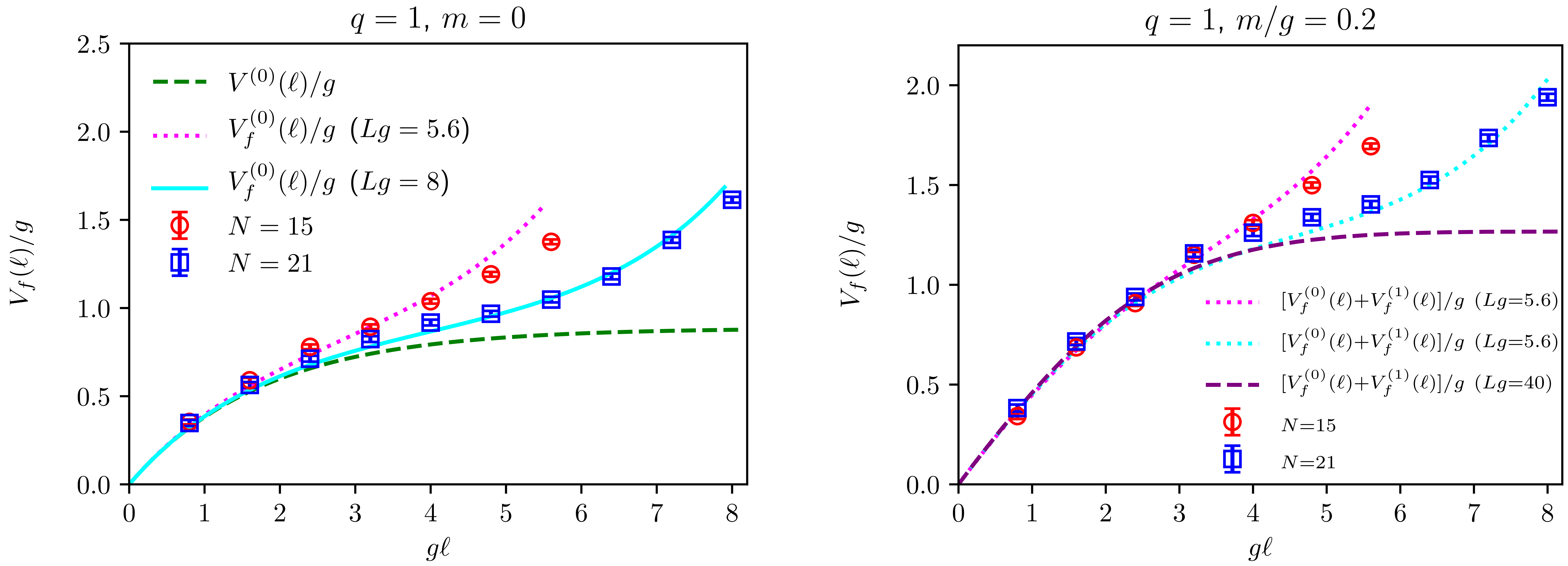
- We test our method by classical simulator of ideal quantum device (IBM Qiskit)
- We compare simulation results with analytic results for finite/infinite volume
- lattice spacing $ag = 0.4$ (fixed), # of sites $N = 15,21$
- adiabatic time $T = 99$, # of steps $M = 330$, # measurements = $\mathcal{O}(10^5)$
- $\theta_0 = 0$, $q \in \mathbb{Z}$ and $q \notin \mathbb{Z}$
- $\theta_0 \neq 0$



Results for $\theta_0 = 0, \quad q \in \mathbb{Z}$

[Honda-Itou-Kikuchi-LN-Okuda]

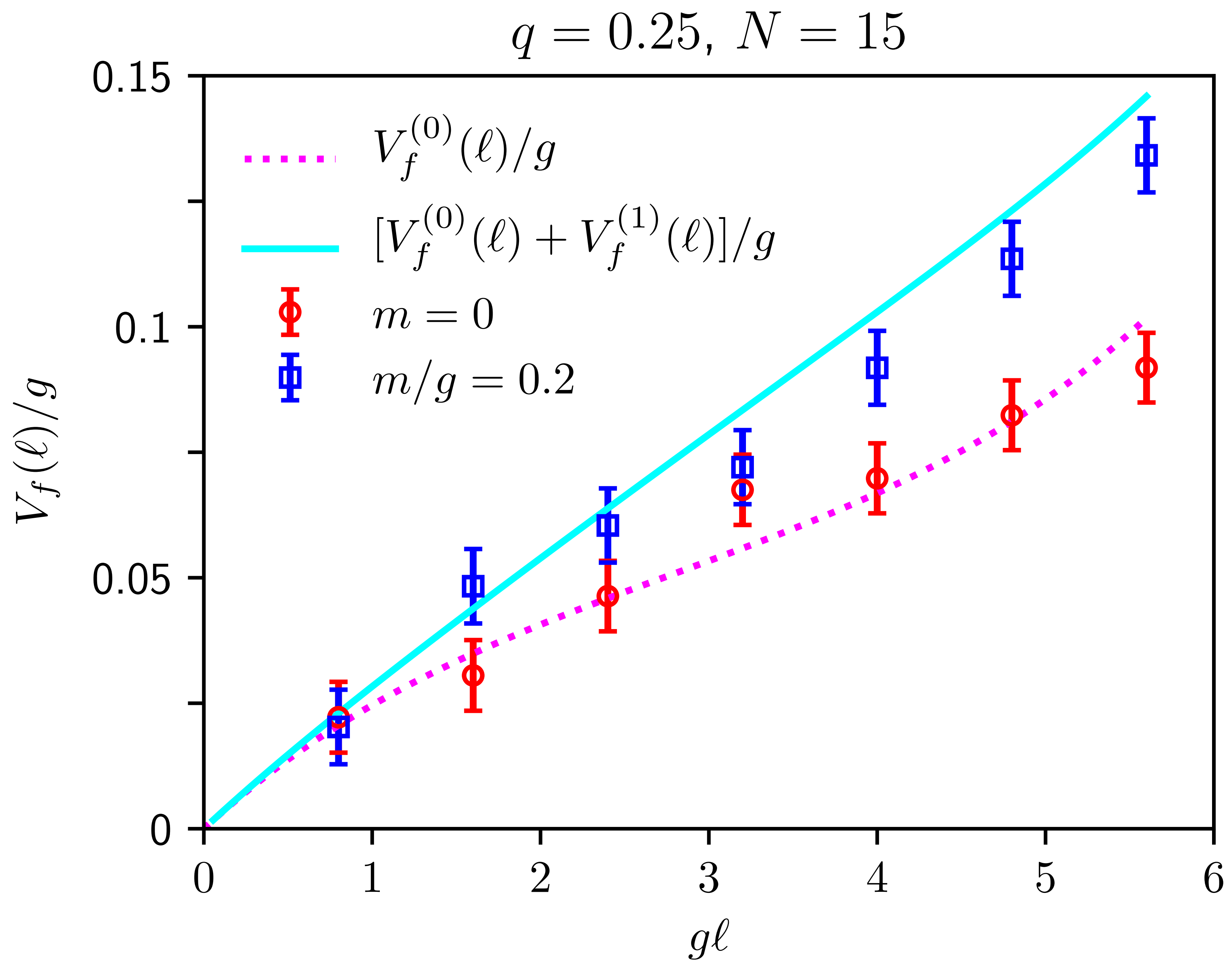
- expect screening both for massless/massive case in infinite volume/continuum limit
- green/purple curves: analytic (infinite volume)
- magenta/cyan curves: analytic (finite volume)

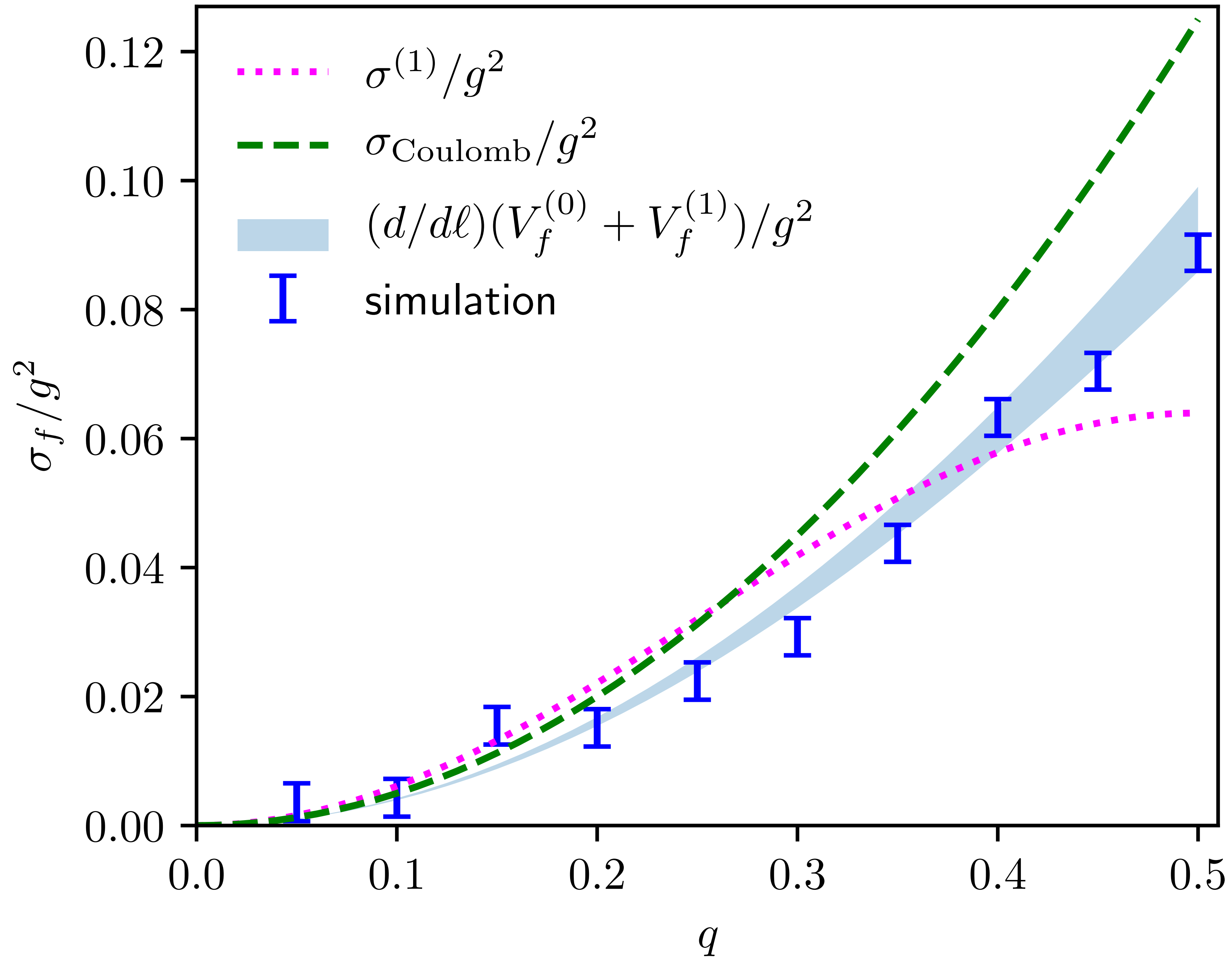


Results for $\theta_0 = 0, \quad q \notin \mathbb{Z}$

[Honda-Itou-Kikuchi-LN-Okuda]

- expect confinement for massive case in infinite volume and continuum limit
  screening massless
- linear behavior for massive case (left) → plot slopes for various q (right)





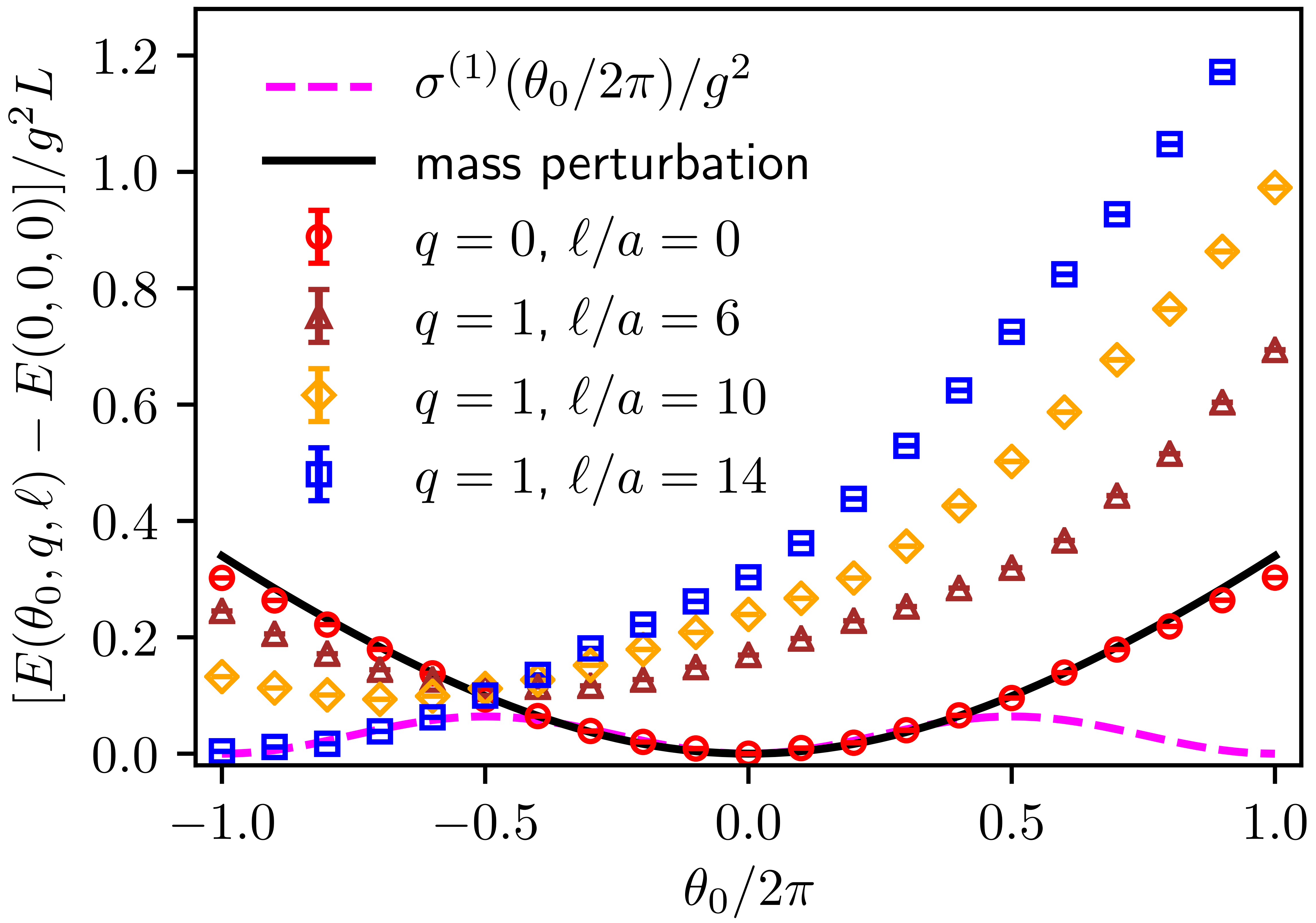
green curve: classical coulomb

blue band: analytic results from mass pert.

Results for $\theta_0 \neq 0$

[Honda-Itou-Kikuchi-LN-Okuda]

- cannot be obtained by Monte Carlo method
- plot of energy vs θ_0 with probe distance fixed



black curve: analytic results for finite volume

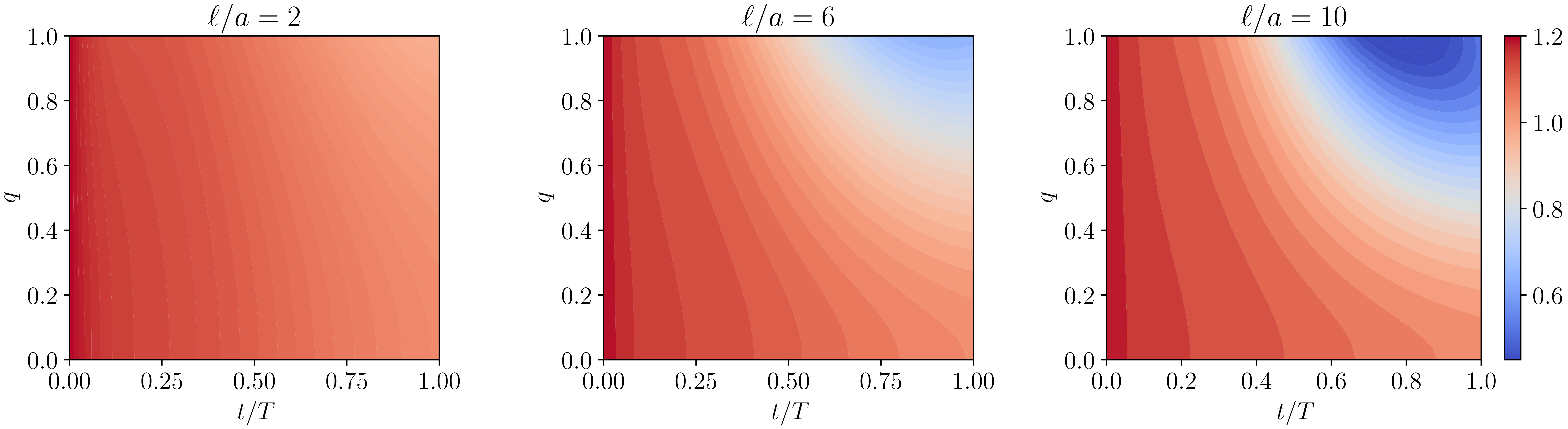
magenta curve: analytic results for infinite volume

Adiabatic errors

- energy for $N = 15, m/g = 0.15, ag = 0.4$
- blue region → small gap when distance is large!

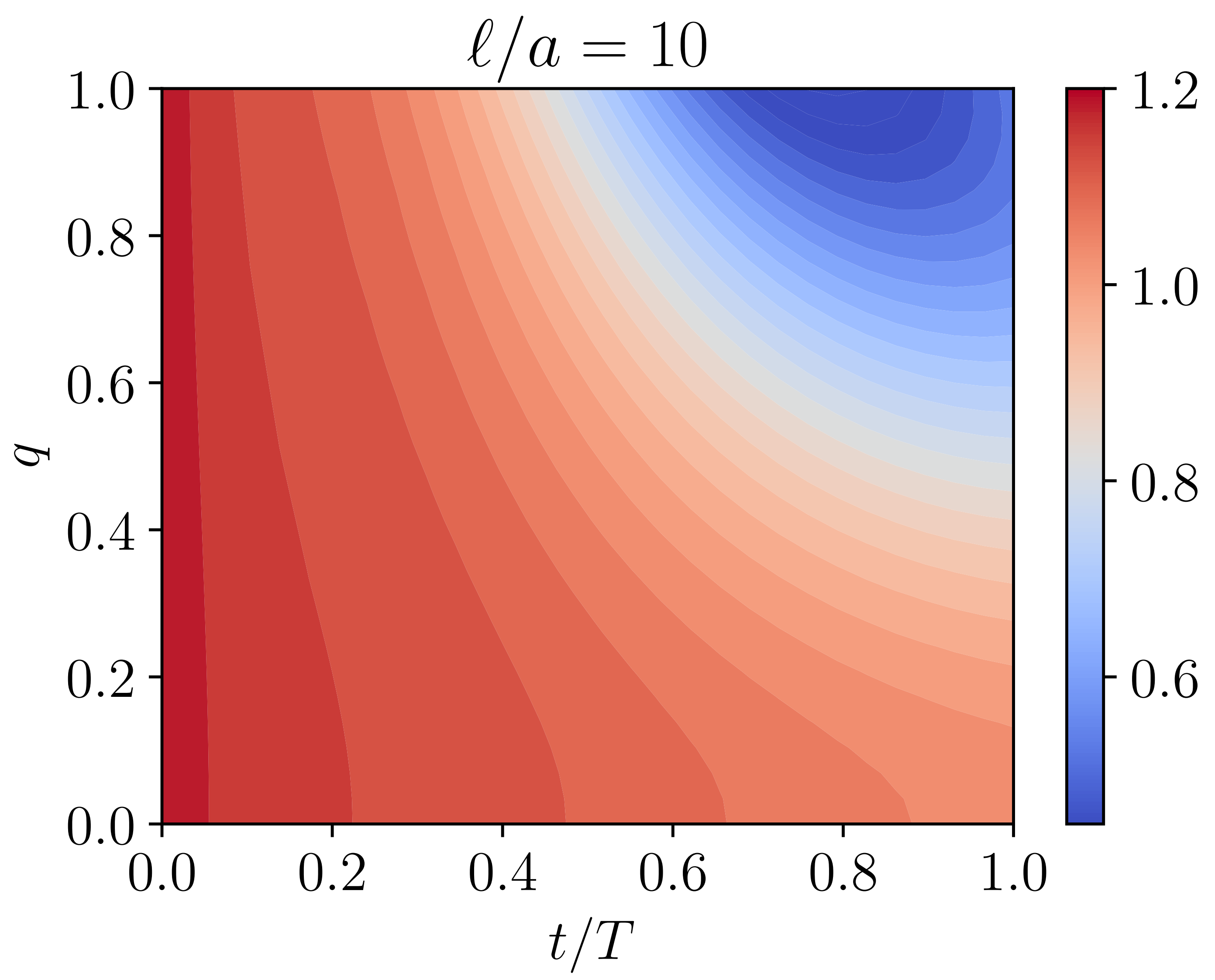
$$\epsilon_{\rm ad} \lesssim \frac{1}{T} \max \frac{\|dH_A/dt\|}{\Delta^2}$$

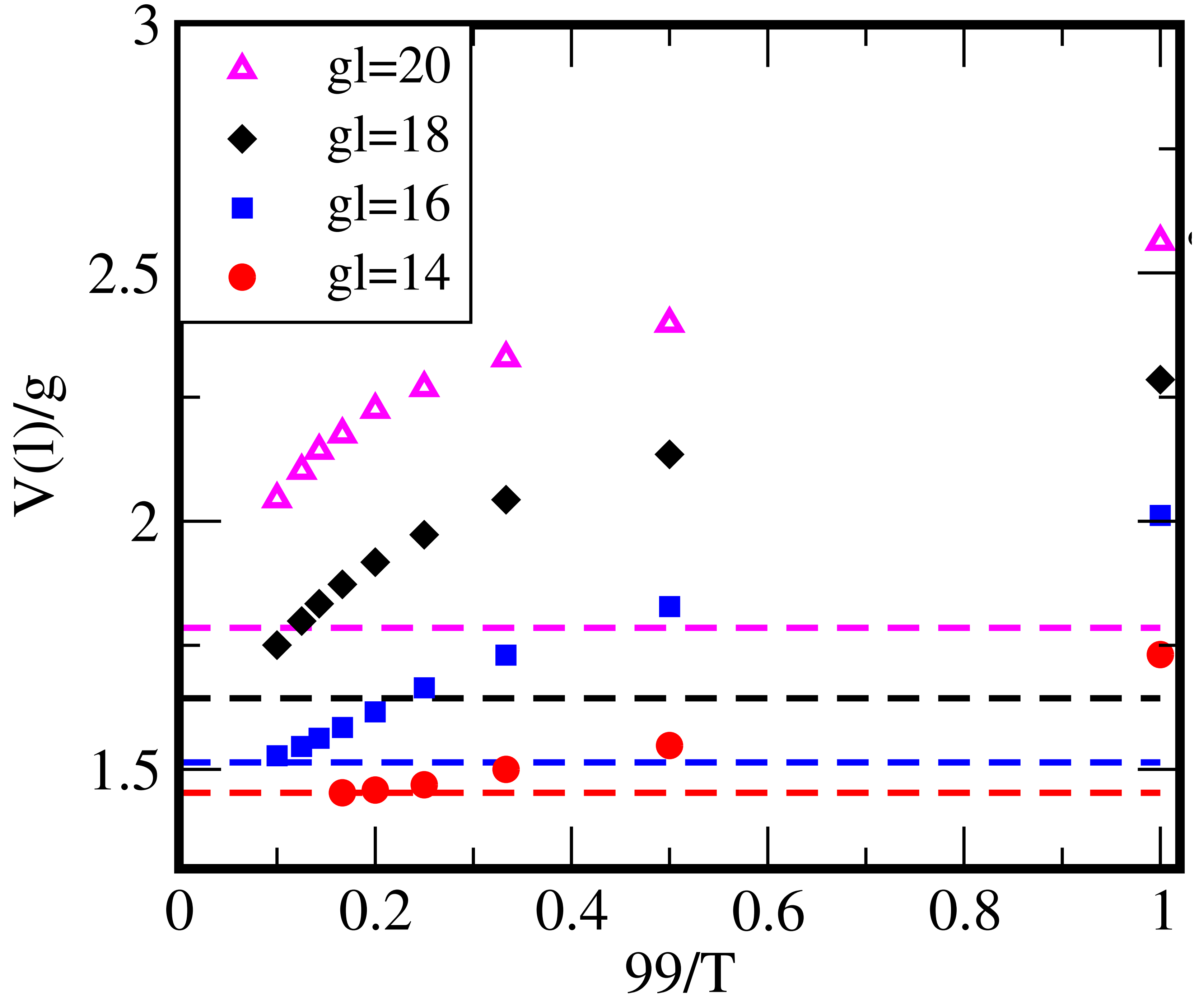
(Δ: energy gap)



Adiabatic errors

- blue region → small gap when distance is large!
- energy vs adiabatic time for $N = 21, m/g = 0.25, q = 1$
  →we need more adiabatic time for small gap region

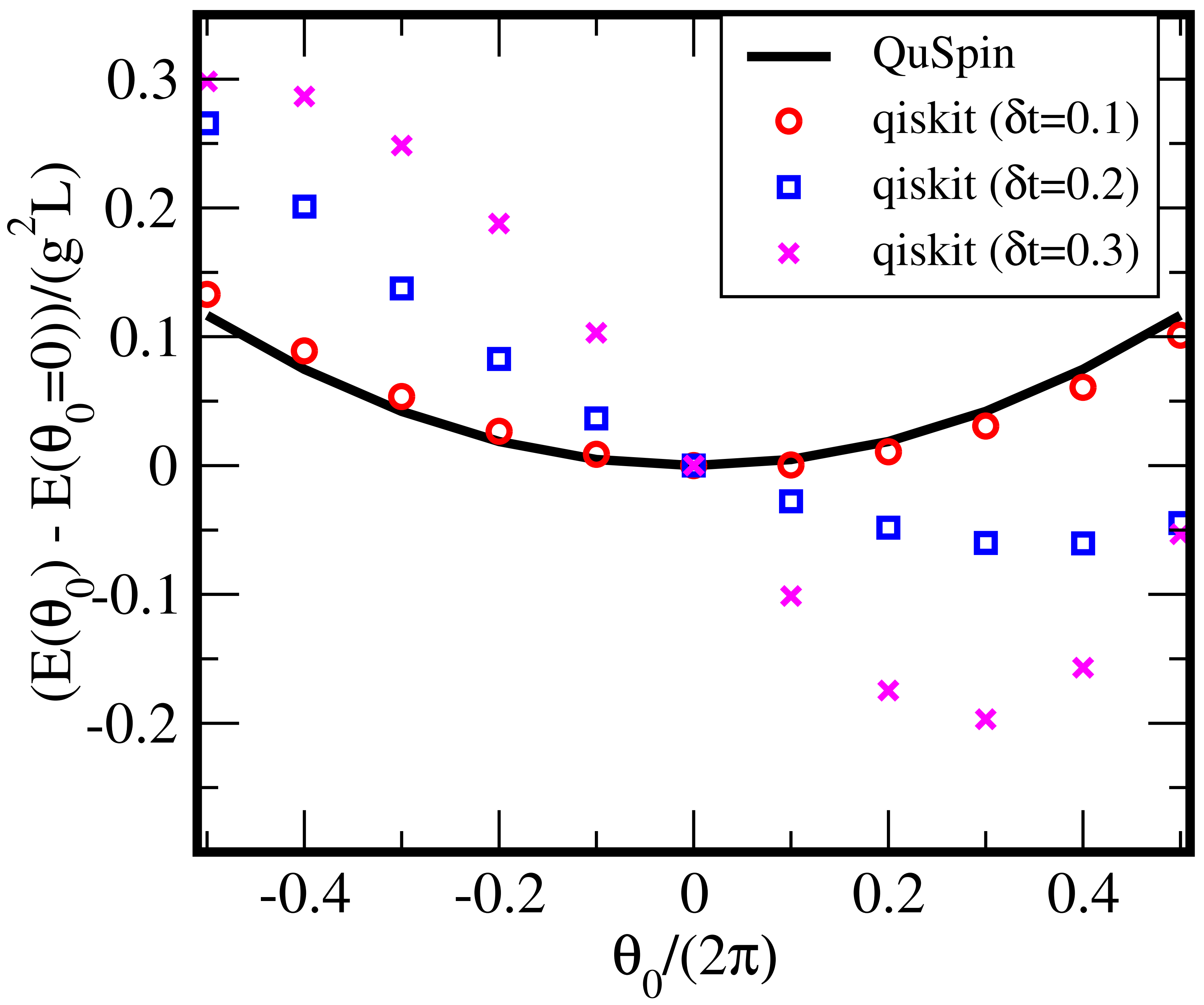




$$\epsilon_{\mathrm{ad}} \lesssim \frac{1}{T} \max \frac{\|dH_A/dt\|}{\Delta^2}$$

Suzuki-Trotter error

- plot of energy with varying background theta θ_0
- violating symmetry under $\theta \to -\theta$ due to the decomposition
  →restored when taking $\delta t \to 0$



- $N = 21, ag = 0.05\ m/g = 0.05, q = 0$
- change M with T fixed

$$\delta t = T/M$$

$$\epsilon_{\rm ST} \sim \mathcal{O}(\delta t^2)$$

Decomposing Hamiltonian

$$H = H_{XY}^{(0)} + H_{XY}^{(1)} + H_Z$$

$$H_{XY}^{(0)} = \frac{w}{2} \sum_{m=0}^{\frac{N-3}{2}} \left(X_{2m} X_{2m+1} + Y_{2m} Y_{2m+1} \right)$$

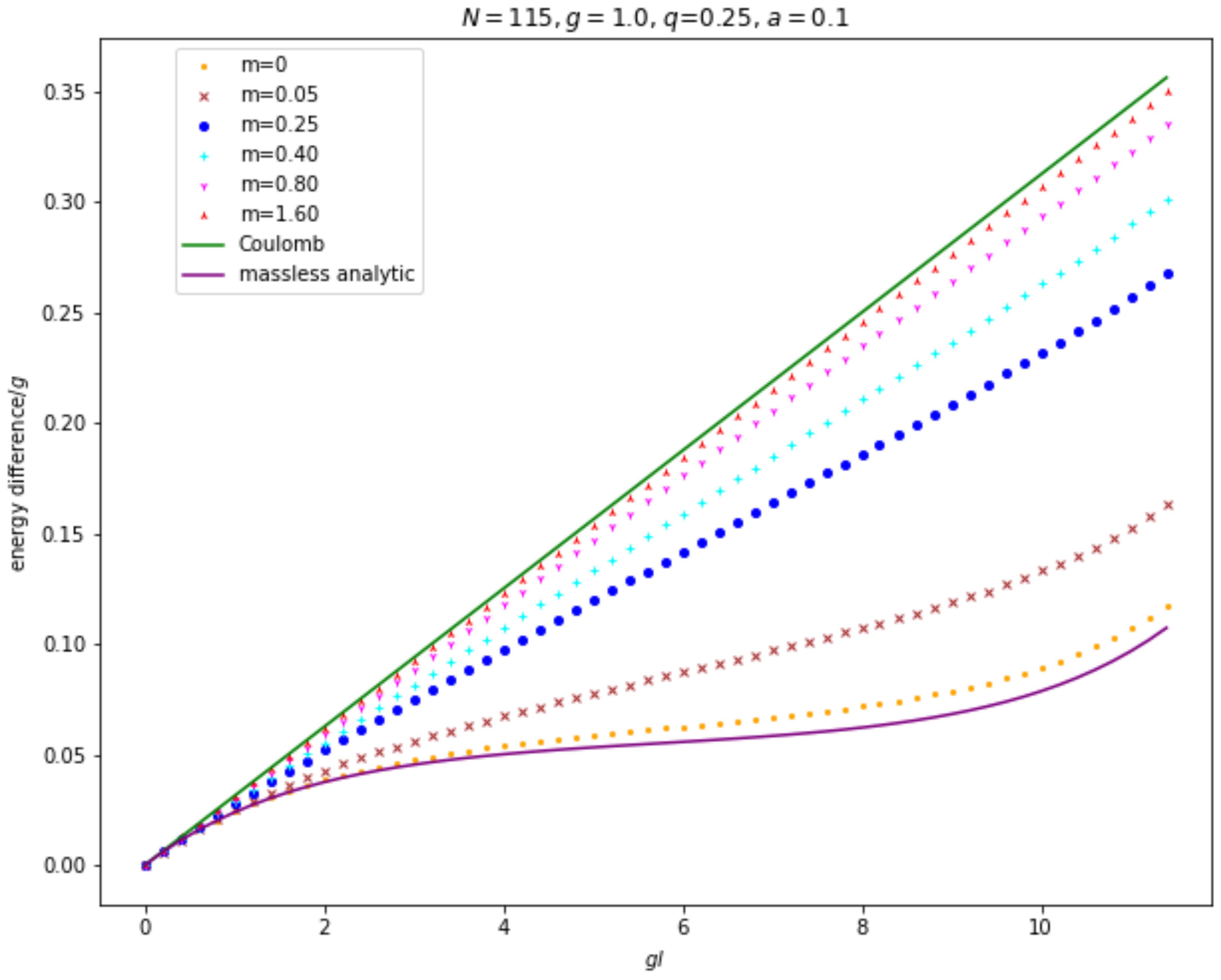
$$H_{XY}^{(1)} = \frac{w}{2} \sum_{m=1}^{\frac{N-1}{2}} \left(X_{2m-1} X_{2m} + Y_{2m-1} Y_{2m} \right)$$

$$H_Z = \frac{J}{2} \sum_{n=0}^{N-3} \sum_{m=n+1}^{N-2} (N - m - 1) Z_n Z_m + \sum_{n=0}^{N-1} c_n^{(Z)} Z_n$$

- preserves particle number
- violates parity: $\theta \to -\theta$

Analytic results on finite interval

- computed by DMRG

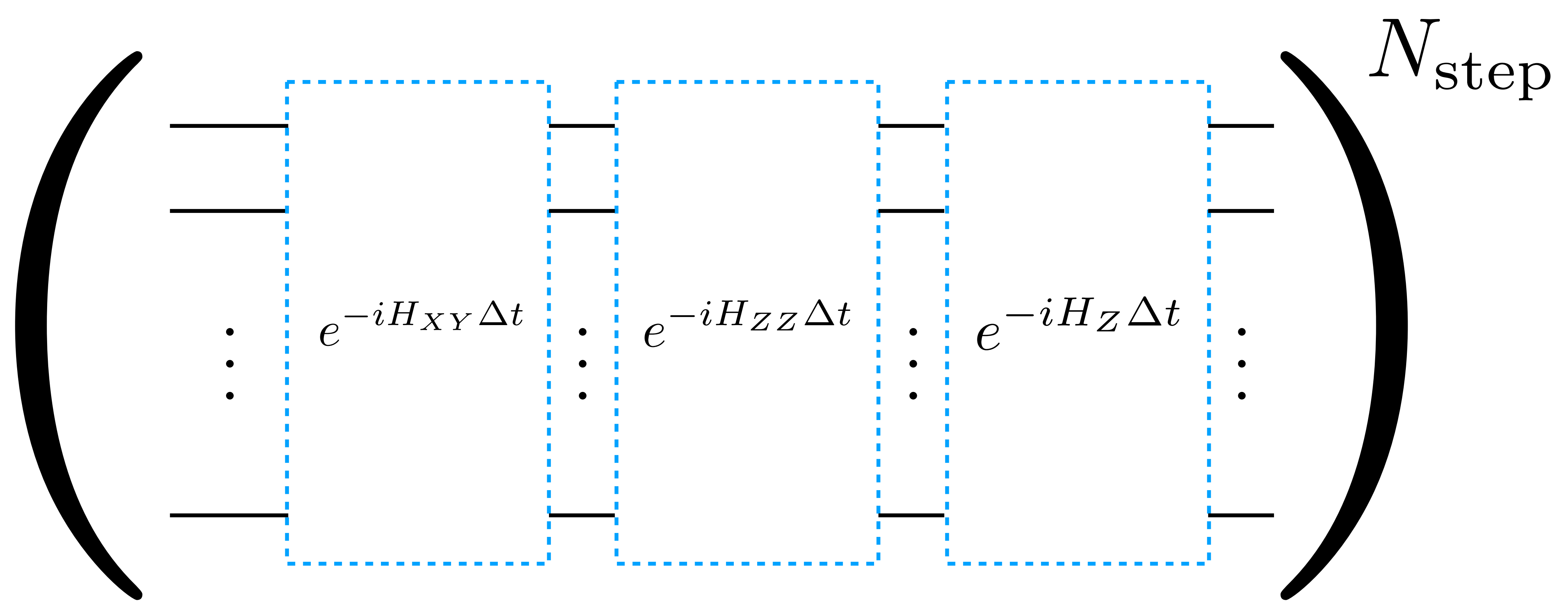


Hamiltonian variational ansatz (HVA)

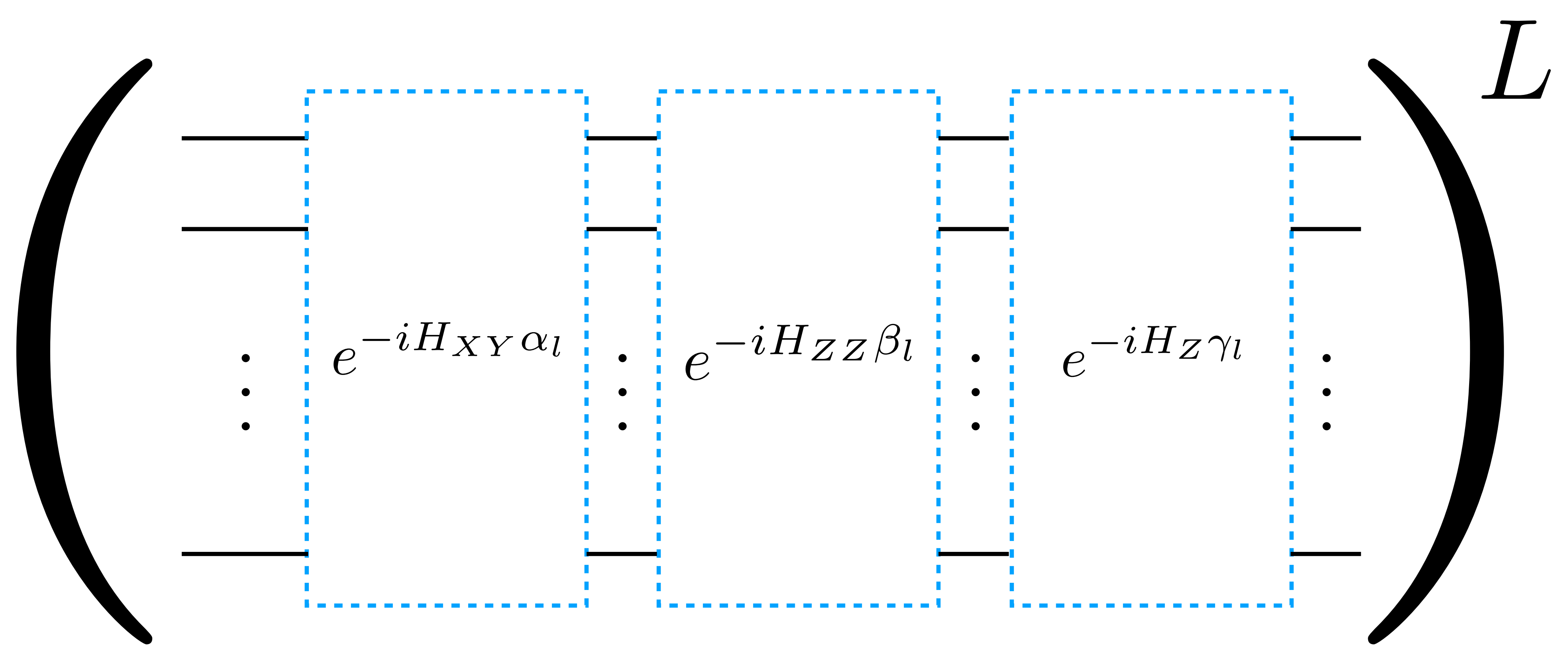
- motivation: mimic Suzuki-Trotter decomposition of adiabatic or real-time evolution
- parameters (α, β, γ) can depend on sites
- we use U(1) preserving decomposition

$$H = H_{XY} + H_{ZZ} + H_Z$$

Suzuki-Trotter evolution

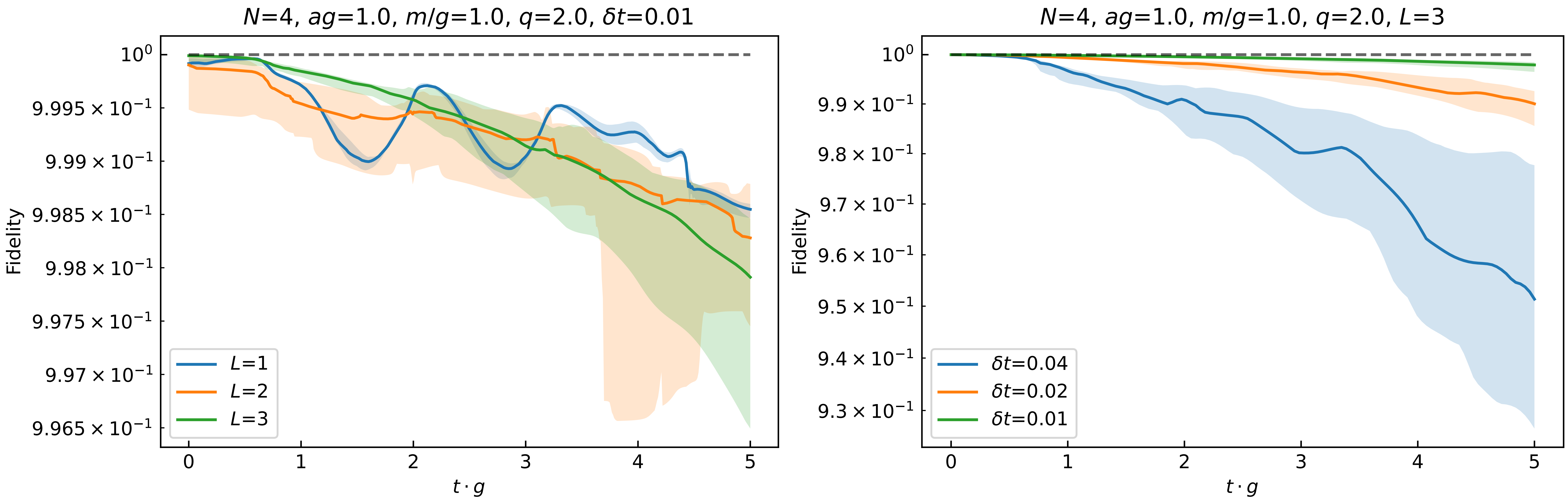


HVA

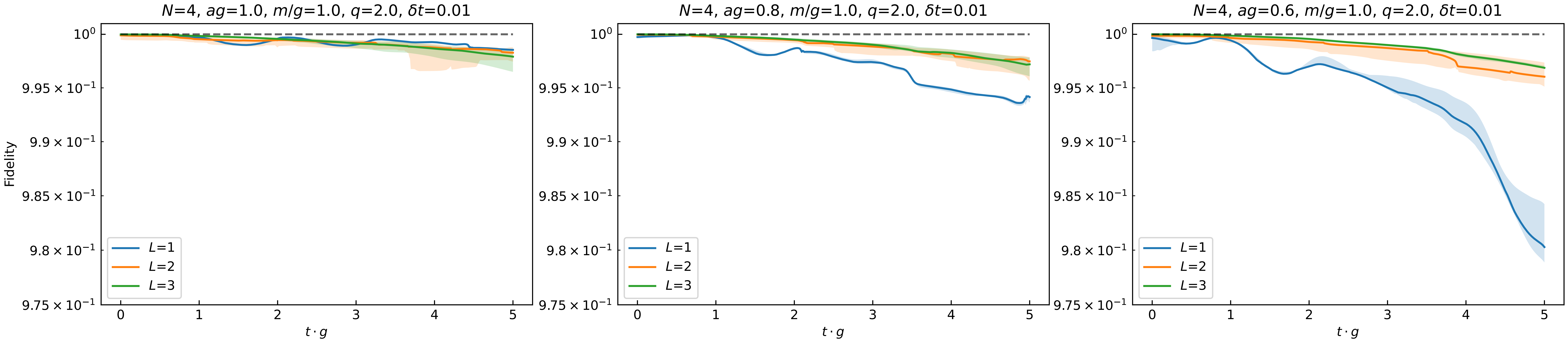


Fidelity and algorithmic errors

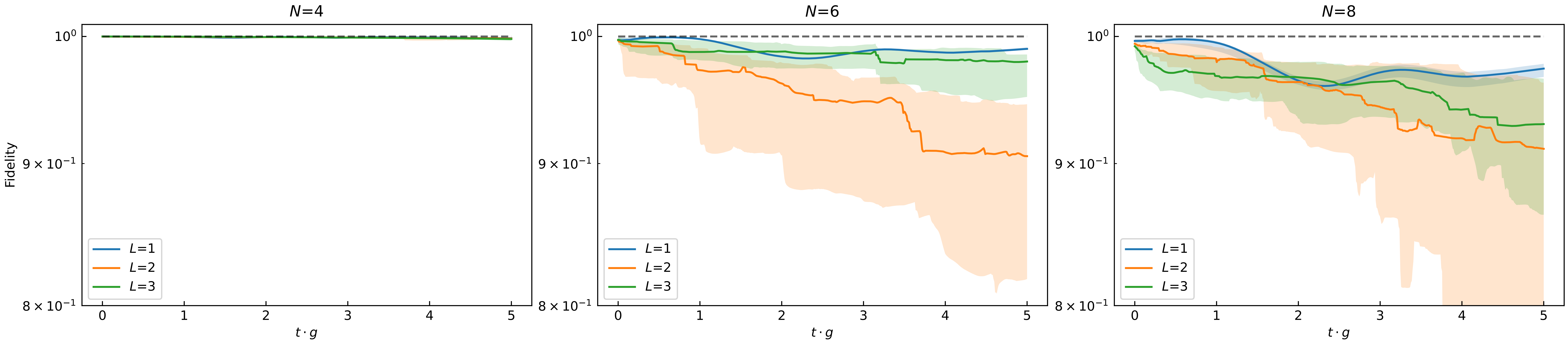
- (avaraged) fidelity improves as increasing L and/or decreasing $\delta t = T_{\mathrm{max}}/N_{\mathrm{step}}$
- effects from δt is significant



Lattice spacing dependence



System size dependence



McLachlan's variational principle

$$\delta \left\| \left(\frac{d}{dt} + iH \right) |\psi(\lambda)\rangle \right\| = 0$$

$$\Rightarrow \sum_j M_{ij} \dot{\lambda}_j = V_i$$

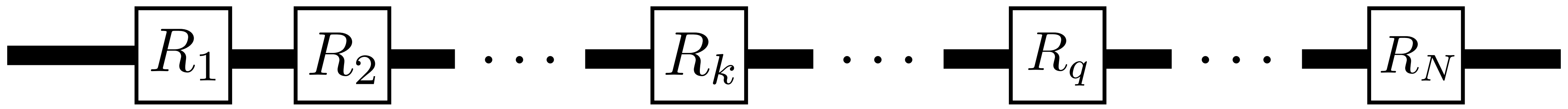
$$M_{ij} = \mathrm{Re} \frac{\partial \langle \psi(\lambda)|}{\partial \lambda_i} \frac{\partial |\psi(\lambda)\rangle}{\partial \lambda_j}$$

$$V_i = \mathrm{Im} \frac{\partial \langle \psi(\lambda)|}{\partial \lambda_i} H |\psi(\lambda)\rangle$$

Quantum circuit for VQS

[Li, Benjamin, Phys. Rev. X 7, 021050 (2017)]
[Yuan et al., Quantum 3, 191 (2019)]

$$U(\lambda) = R_N(\lambda_N) \cdots R_1(\lambda_N)$$



$$|\psi(\lambda)\rangle = U(\lambda)|\psi_0\rangle$$

- evaluation of matrix elements $M_{kq} = \mathrm{Re}\frac{\partial\langle\psi(\lambda)|}{\partial\lambda_k}\frac{\partial|\psi(\lambda)\rangle}{\partial\lambda_q}$
- derivative of each component w.r.t. parameters $\frac{\partial}{\partial\lambda_k}R_k(\lambda) = U_k R_k(\lambda)$
- quantum circuit:

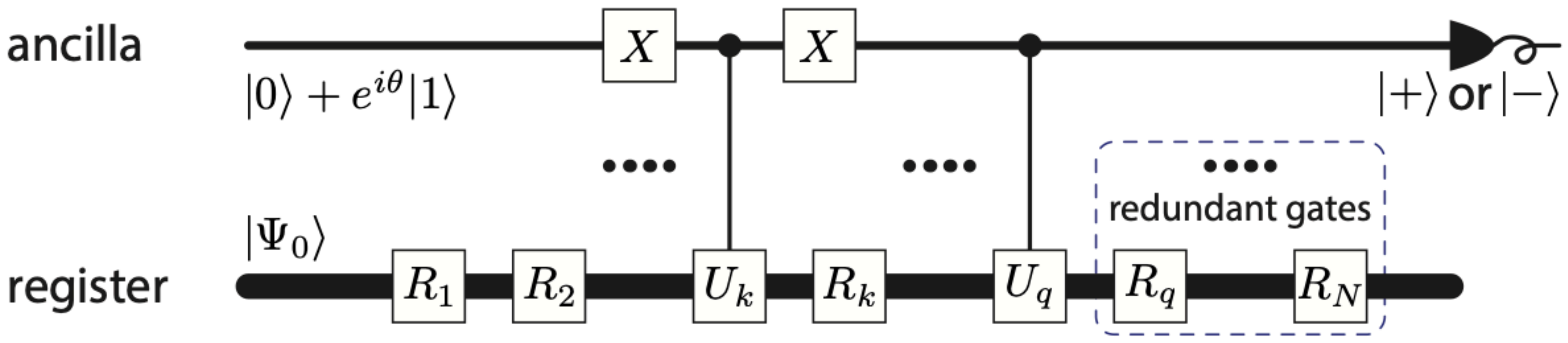


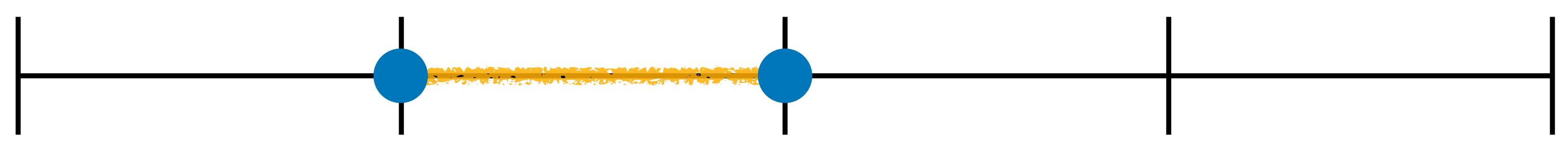
Figure from Yuan et al.

Schwinger model

[Schwinger, Phys. Rev. 128, 2425, (1962)]

$$H_{\text{spin}} = J \sum_{n=0}^{N-2} \left(\sum_{k=0}^{n} \frac{Z_k + (-)^k}{2} + q \right)^2 + \frac{w}{2} \sum_{n=0}^{N-2} \left(X_n X_{n+1} + Y_n Y_{n+1} \right) + \frac{m}{2} \sum_{n=0}^{N-1} (-)^n Z_n$$

- simple toy model: 1+1d U(1) gauge theory (=quantum electrodynamics): Schwinger model
- still nontrivial: confinement, phase transition
- discretized Hamiltonian → spin Hamiltonian (gauge field elimination, Jordan-Wigner transformation)
- phase transition at $q = 1/2$ and $m = m_c$



● : fermion (electron)

— : gauge (electric) field