WZW terms via Generalized Cohomology

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Lots of help form arXiv:2009.00033, Y. Lee, K. Ohmori, Y. Tachikawa

Contents

NOTE : Extraordinary Homology = Generalized Homology

Notation

- The ring of integers : $\mathbb{Z}$
- The prime field of order two : $\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z}$
- The field of real number : $\mathbb{R}$
- Classifying space of G : BG
- BRST transformation : δ_B
- Its generator : Q_B
- Ghost in superfield formalism : C
- Spacetime dimension : n$, d, m = 2l$ (using multiple notations)
- Chern character : ch

Contents

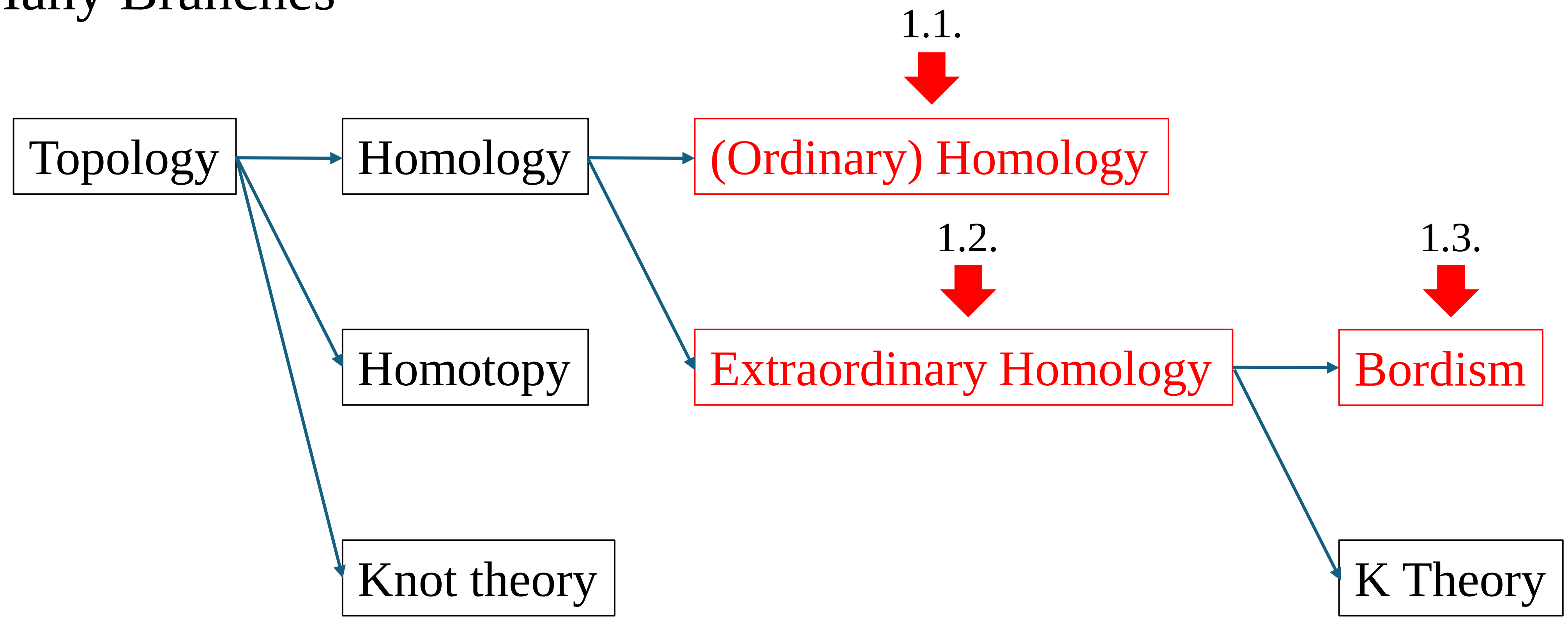
1. Topology

A geometry allowing <u>CONTINUOUS DEFORMATION</u>

$$\cong$$

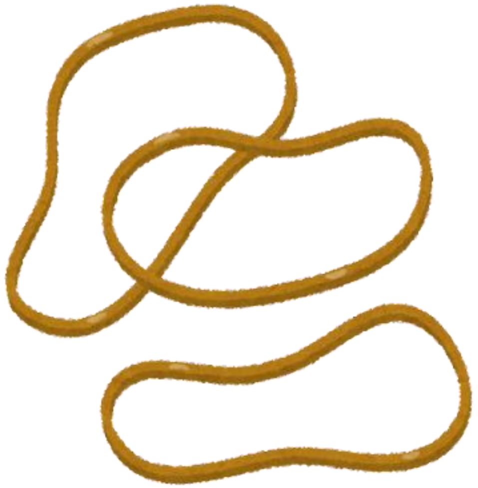
1. Topology

Many Branches



1.1. Homology

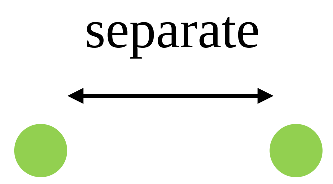
Studying <u>HOLE</u>



1-dim hole

2-dim hole

connectedness



0-dim hole

between two points!

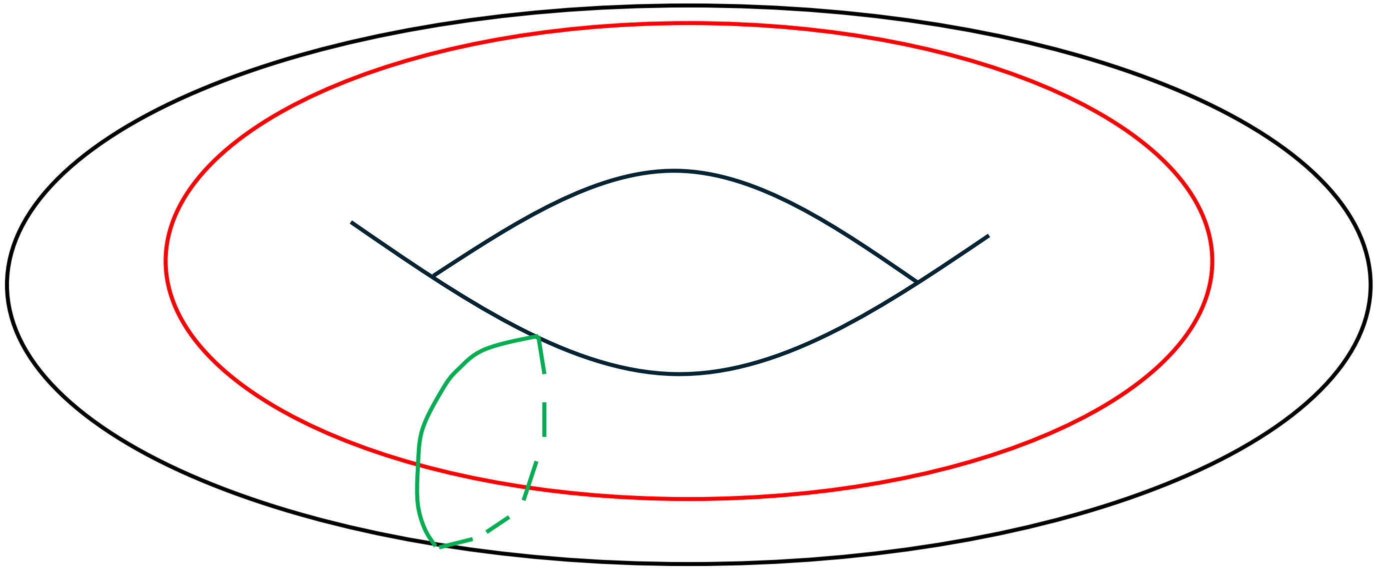
1.1. Homology

Detect the hole of torus

0-dim hole : 1

1-dim hole : 2

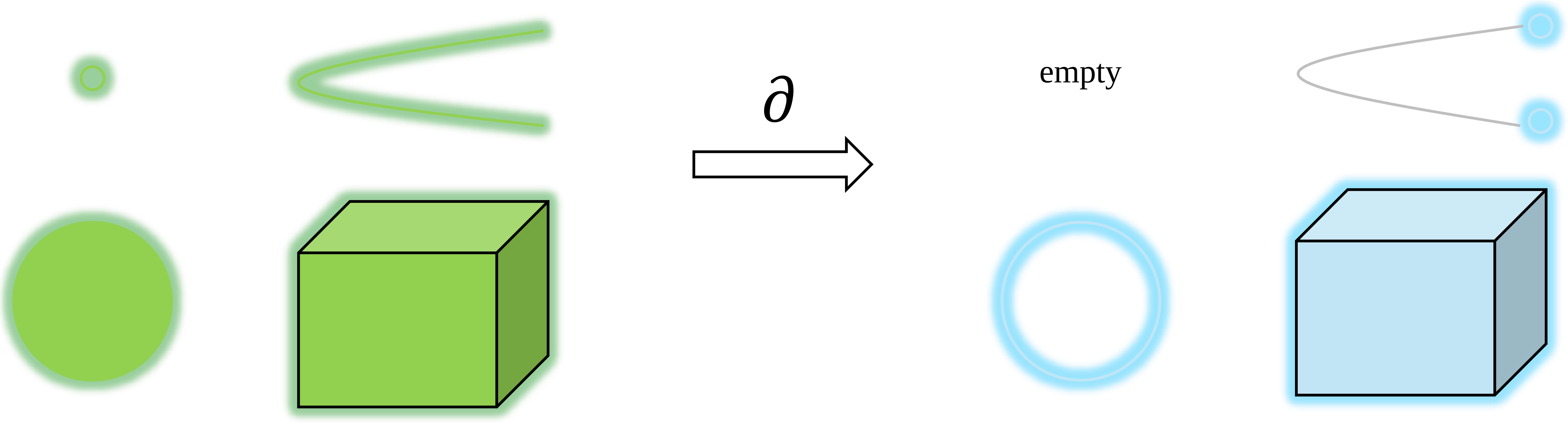
2-dim hole : 1



1.1. Homology

Let's consider <u>BOUNDARY</u>

Boundary operator : ∂



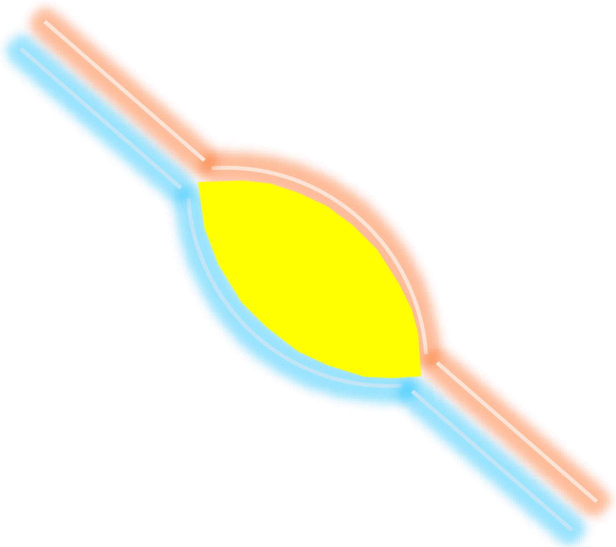
filled

Not filled

1.1. Homology

Let's study 1-dim hole

Closed loop $\in Ker\ \partial_1$



Deformation $\in Im\ \partial_2$

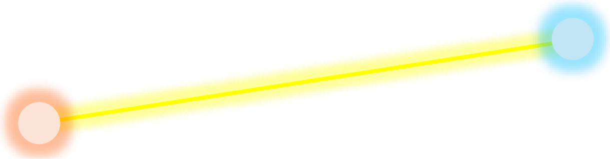
1-dim hole is determined by

$$\frac{Ker\ \partial_1}{Im\ \partial_2}$$

1.1. Homology

Let’s study 0-dim hole

All Points



Closed point $\in Ker\ \partial_0$

Deformation $\in Im\ \partial_1$

0-dim hole is determined by

$$\frac{Ker\ \partial_0}{Im\ \partial_1}$$

1.1. Homology

Let’s study 2-dim hole

2-dim hole is determined by

Closed surfaces

When deforming surface,
Passing through 3-dim ball

$$\frac{Ker\ \partial_2}{Im\ \partial_3}$$

Closed surface $\in Ker\ \partial_2$

Deformation $\in Im\ \partial_3$

1.1. Homology

Mathematically,

Consider a CW complex of manifold X

$$\cdots \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

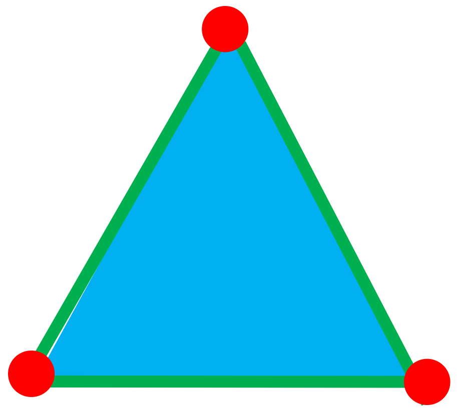
where each map is boundary operator

$$\partial_n : C_n \longrightarrow C_{n-1}$$

Homology group is defined by

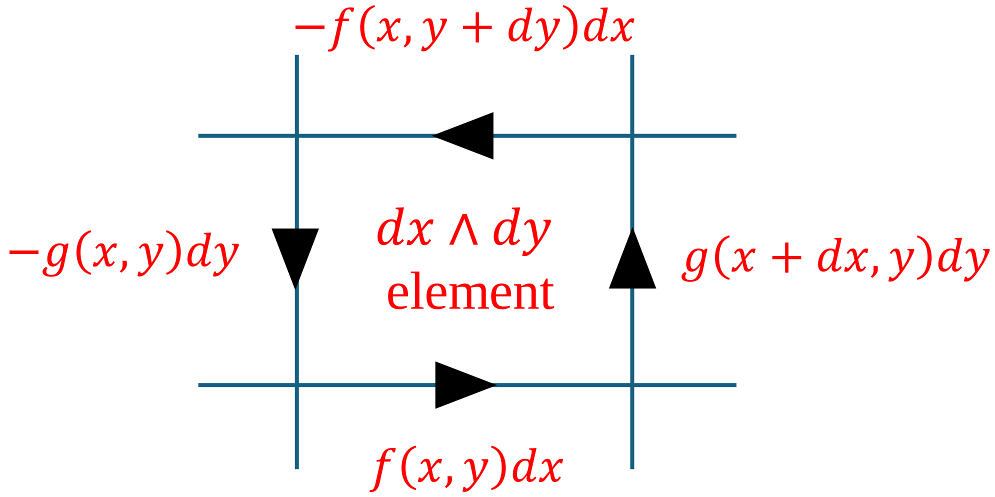
$$H_n(X) = \frac{Ker\ \partial_n}{Im\ \partial_{n+1}}$$

corresponding n-dim holes



1.1. Homology

Exterior derivative is related to boundary operator.



$$f(x, y)dx + g(x + dx, y)dy - f(x, y + dy)dx - g(x, y)dy$$
$$\rightarrow \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

$$d(fdx + gdy) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

✓ Decompose a manifold in infinitesimally, then get (co)homology

1.2. Extraordinary Homology

To know topological shapes,

Homology studies whether it has holes or not.

There exists other ways???

Yes, extraordinary homology.

1.2. Extraordinary Homology

To extraordinary homology is determined by information of a point.

$$G_n(X) = G_n(pt) \oplus \tilde{G}_n(X)$$

In fact, Atiyah Hirzebruch spectral sequence works on $G_n(pt)$.

From now on, we denote $G_n(pt)$ as G_n.

1.2. Extraordinary Homology

For ordinary homology,

$H_0 = \mathbb{Z},\ H_1 = 0,\ \ H_2 = 0,\ \ H_3 = 0,\ \ \cdots$

For bordism,

$\Omega_0 \ = \mathbb{Z}_2,\ \Omega_1 \ = 0,\ \Omega_2 \ = \mathbb{Z}_2,\ \Omega_3 \ = 0,\ \ \cdots$

For complex K theory,

$K_0 = \mathbb{Z},\ K_1 = 0,\ \ K_2 = \mathbb{Z},\ K_3 = 0,\ \ \cdots$

Extraordinary Homology

1.3. Bordism

Ordinary homology is

$$H_n = \frac{Ker\ \partial_n}{Im\ \partial_{n+1}}$$

where $Ker\ \partial_n$ means closed n-dim manifolds

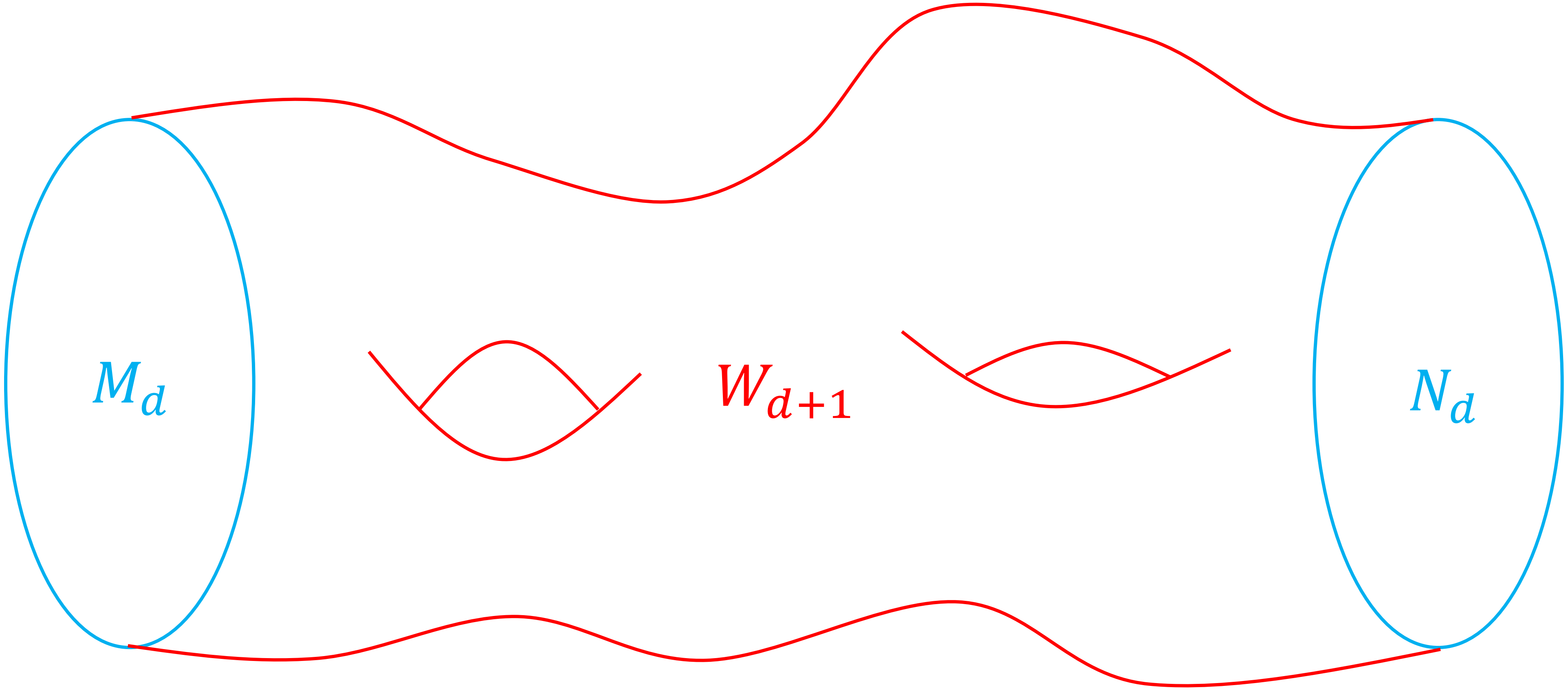
and $Im\ \partial_{n+1}$ means allowing deformations.

Bordism is defined as

$$\Omega_n = \frac{closed\ n - dim\ manifolds}{bordant}$$

1.3. Bordism

M_d and N_d are bordant if …



1.3. Bordism

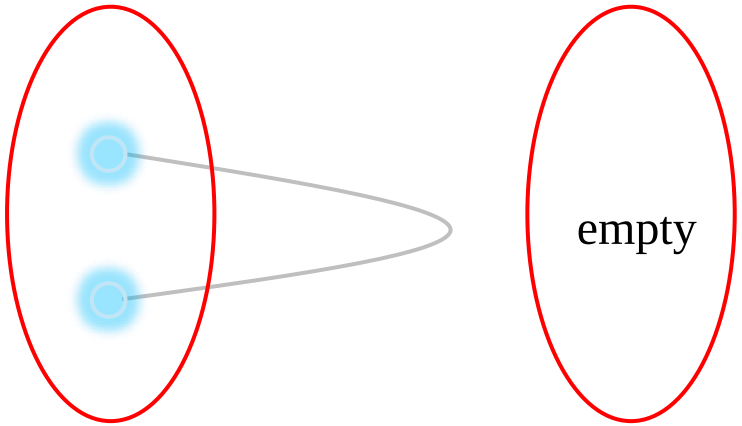
If M_d, N_d and W_{d+1} have an additional structure S,

Another bordism theory Ω_n^S is defined.

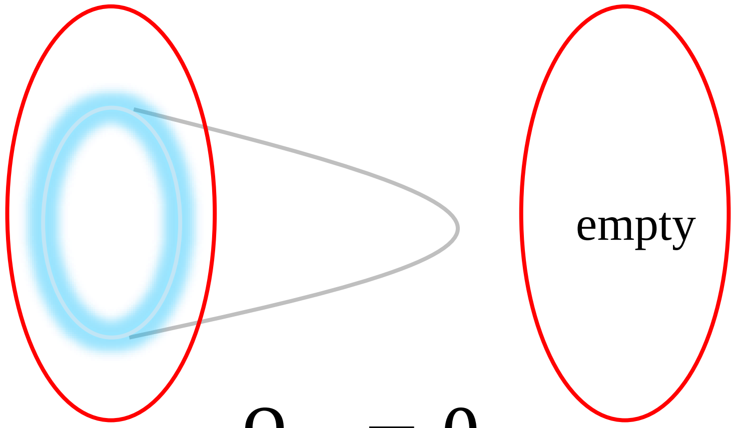
For example, when manifolds are equipped with a spin structure,
spin bordism is defined

$\Omega_0^{spin} = \mathbb{Z},\ \Omega_1^{spin} = \mathbb{Z}_2,\ \Omega_2^{spin} = \mathbb{Z}_2,\ \Omega_3^{spin} = 0,\ \cdots$

1.3. Bordism



$$\Omega_0 = \mathbb{Z}_2$$



$$\Omega_1 = 0$$

2-dim closed manifolds $\begin{cases} 0 \in \mathbb{Z}_2 : orientable. \\ 1 \in \mathbb{Z}_2 : not\ orientable \end{cases}$

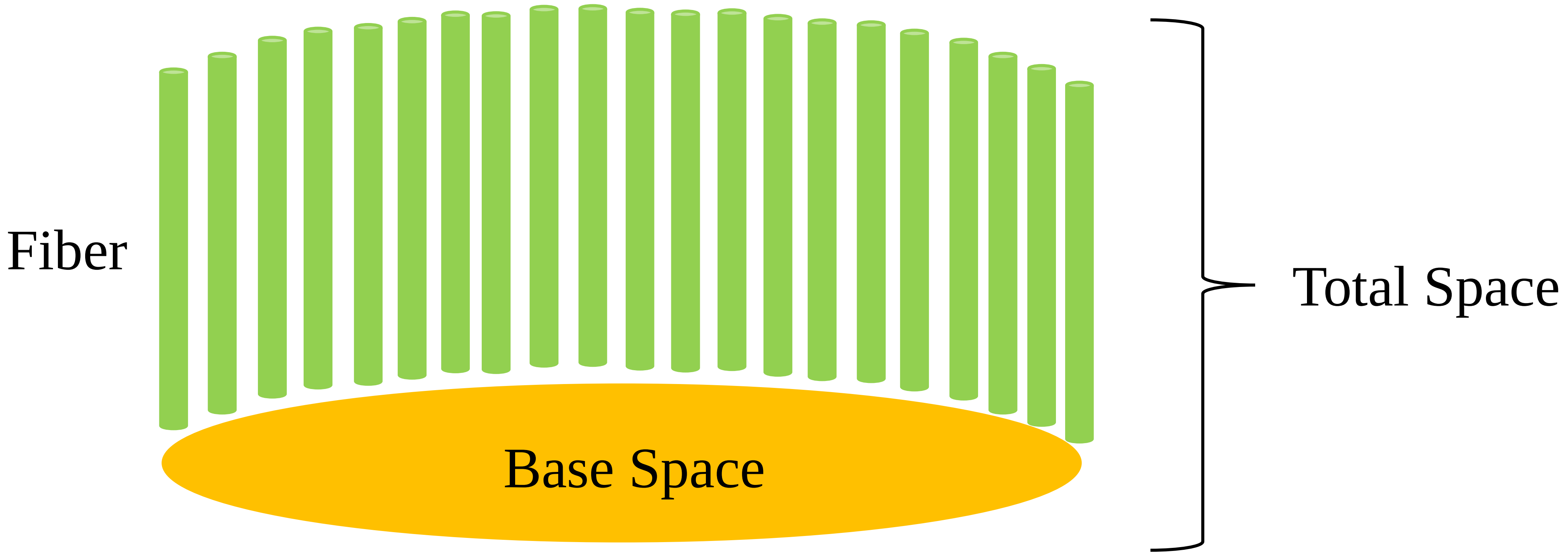
$$\Omega_2 = \mathbb{Z}_2$$

Dehn Surgery

$$\Omega_3 = 0$$

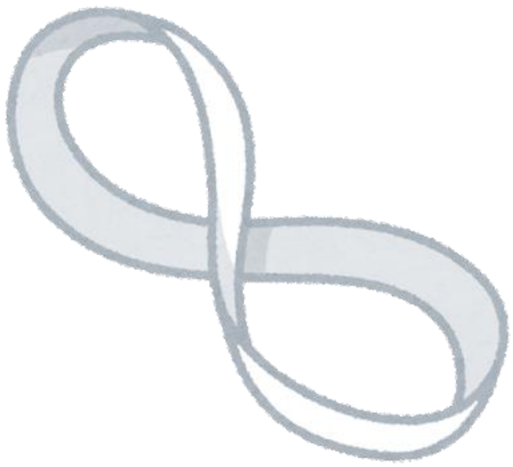
1.4. Spectral Sequence

Fibration : $F \longrightarrow E \longrightarrow B$: fiber, total space, base space respectively



1.4. Spectral Sequence

Example of $F \longrightarrow E \longrightarrow B$,



Möbius band : $B = S^1$ and $F = \mathbb{R}$

Hopf fibration : $S^1 \to S^3 \to S^2$

Gauge Theory : B is spacetime, F is gauge group

1.4. Spectral Sequence

How to compute ordinary/extraordinary homology?

Consider fibration : $F \longrightarrow E \longrightarrow B$

Spectral Sequence : from $H_*(B)$ and $H_*(F)$, compute $H_*(E)$

$$H_p(B; H_q(F)) \Rightarrow H_{p+q}(E)$$

There are maps, <u>transgression</u> $\tau : H_n(B) \longrightarrow H_{n-1}(F)$

Contents

2. Non-Abelian Gauge Theory

Let's consider QCD

$$S_{QCD} = S_{gauge} + S_{matter} + (GF \,\&\, FP)$$

where

$$S_{gauge} = -\frac{1}{2g^2}\int d^4x \, trF^{\mu\nu}F_{\mu\nu}$$

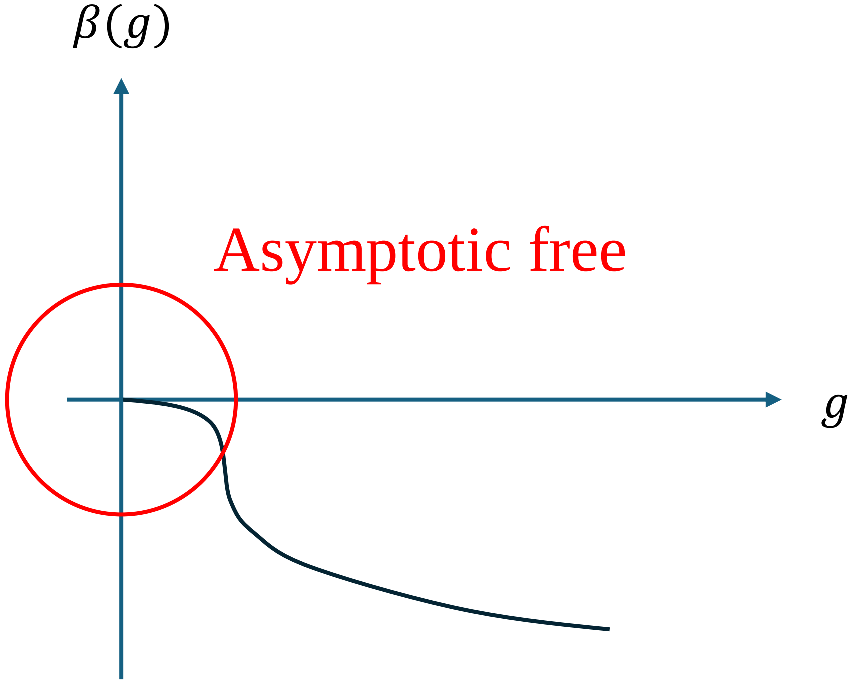
$$S_{matter} = \int d^4x \, i\bar{\psi}\gamma^{\mu}D_{\mu}\psi - m\bar{\psi}\psi$$

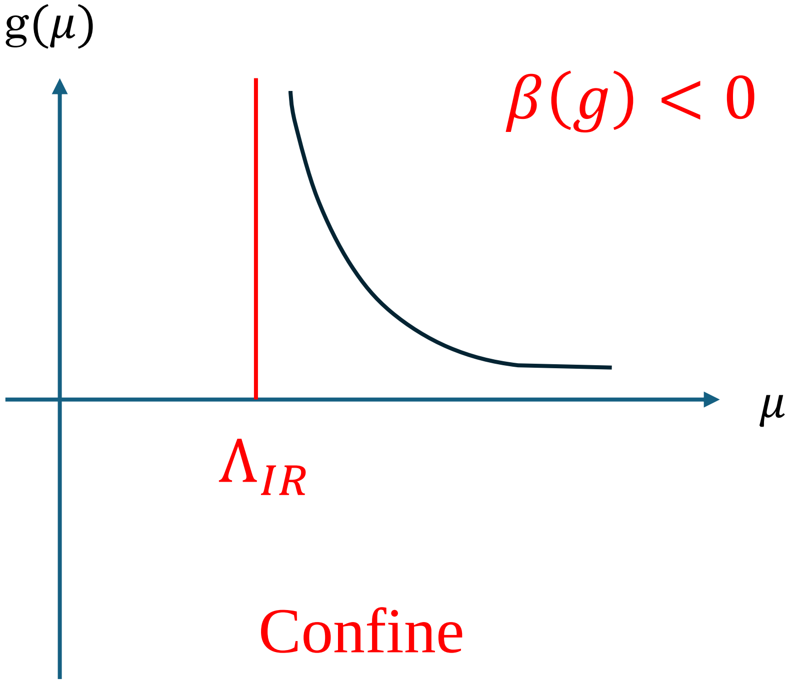
Fermions are charged under

$$G = G_{gauge} \times G_{flavor}$$

2.1. Asymptotic Freedom and Confinement

Beta function : $\mu \frac{dg}{d\mu} = \beta(g) = \beta_0 g^3 + \beta_1 g^5 + \cdots$





2.2. Non-linear Sigma Model

In QCD, quarks condensate : $\langle \bar{\psi}_i \psi_j \rangle = v^3 \sigma_{ij}$

Flavor symmetry breaks : $G_{flavor} \longrightarrow H_{flavor}$

NG field : $\xi(\pi) = e^{i\pi^a(x)T^a} \in G_{flavor}/H_{flavor}$ (coordinate of coset space)

Lagrangian : $\mathcal{L} = f_\pi^2 tr(\alpha_\mu(\pi)|_{G_{flavor}/H_{flavor}})^2 + \cdots$

where Maurer-Cartan 1 form : $\alpha_\mu(\pi) = i^{-1}\xi^{-1}(\pi)\partial_\mu \xi(\pi)$

2.3. Anomalies

Contents of this section

2.3.1. Introduction to Anomalies

1. Gauge Anomalies as obstructions of renormalization

- To renormalize a gauge theory,

  we require a gauge invariant (GI) regularization.
- Chiral fermions let the gauge transformation contain γ_5,

  and we give up the (GI) dimensional regularization.

- If anomalies exist, WT identities of a gauge theory break as
  $$\langle 0|[iQ_B, Texp\{i(external\ source)\}]|0\rangle = 2\Delta\hbar^n + \mathcal{O}(\hbar^{n+1})$$
- $\delta_B\Delta = 0$: WZ consistency condition.

2.3.1. Introduction to Anomalies

2. Anomalies as symmetry breaking by the quantum effect

- Classical theory : the invariance of Lagrangian
- Quantum theory : the invariance of the partition function

- By the quantum effect, <u>a conservation law sometimes breaks</u>

$$D^\mu \langle J_\mu^a \rangle = -\mathcal{a}'^a$$

where $\delta_B \Gamma[A] = \Delta = \int d^m x\, \mathcal{a} = \int d^m x\, c^a \mathcal{a}'^a \neq 0.$

- $\delta_B \Delta = 0$ or $\delta_B \mathcal{a} = d\mathcal{b}$: WZ consistency condition.

2.3.2. Computation of Anomalies

1. First method : DESCENT EQUATION

use BRST and superfield formalism

$\tilde{d} = d + \delta$, $\tilde{A} = A + C$, with the horizontality condition $\tilde{F} = F$

$$tr\tilde{F}^{l+1} = \mathrm{tr}F^{l+1}, m = 2l$$

Descent equation is given by

$$\tilde{d}\omega_{2l+1}\left(\tilde{A}, \tilde{F}\right) = d\omega_{2l+1}(A, F)$$

Taking terms of the ghost number two,

$$\delta_B\omega_{2l}^{(1)} + d\omega_{2l-1}^{(2)} = 0$$

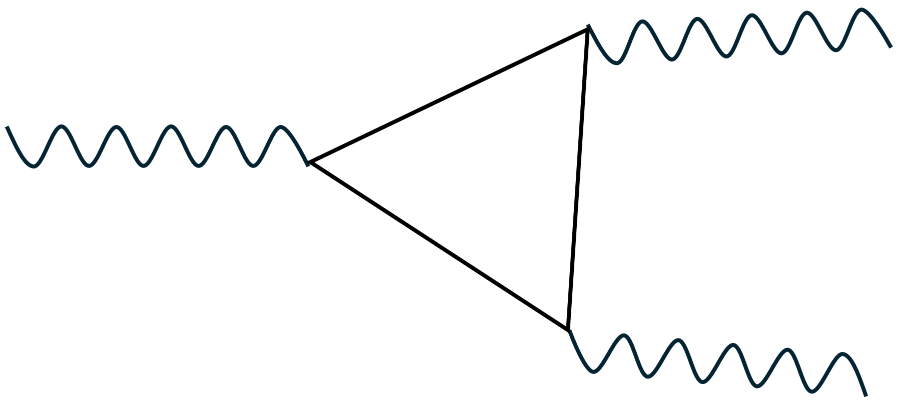
We get anomaly : $a = \omega_{2l}^{(1)}$

2.3.2. Computation of Anomalies

2. Second method : FEYNMAN DIAGRAMS

Known to be required triangular diagrams only :

$$\delta_B \Gamma[A] = \delta_B\left[i^{-1} lnDet\left(i\gamma^\mu\left(\partial_\mu + A_\mu\right)P_\pm\right) + \cdots\right]$$



2.3.2. Computation of Anomalies

By triangular diagrams, you can classify the following anomalies.

Three vortexes connect with

- three gauge currents : <u>gauge anomaly</u>
- two gauge and one global currents : <u>chiral anomaly</u>
- three global currents : <u>‘t Hooft anomaly</u>
  1. This is symmetry in quantum theory.
  2. Conservation law breaks when couple to background gauge field.
  3. Theory dies when couple to dynamical gauge field.

  Still useful for anomaly matching condition (matching of UV and IR anomalies)

✓ Perturbative ways are blind to global gauge anomalies.

2.3.2. Computation of Anomalies

3. Third method : PATH INTEGRAL (PI)

In QFT, we always have to consider the variation of the measure of PI.

It corresponds to the breaking of the conservation law.

Under the axial rotation, the measure changes as

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \longrightarrow \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left\{\int d^4x\,\alpha(x)\,\frac{1}{16\pi^2}tr\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\right\}$$

Thus, the conservation law becomes

$$\partial_\mu j_5^\mu = \frac{1}{16\pi^2}tr\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$$

2.3.2. Computation of Anomalies

The integration of last formula is Atiyah-Singer index theorem.

$$\nu_+ - \nu_- = \int_M d^m x\, \partial_\mu j^\mu_{m+1} = \int_M ch_l(F)$$

With spacetime curvature,

$$\nu_+ - \nu_- = \int_M \left[\hat{A}(R) ch_l(F)\right]_m$$

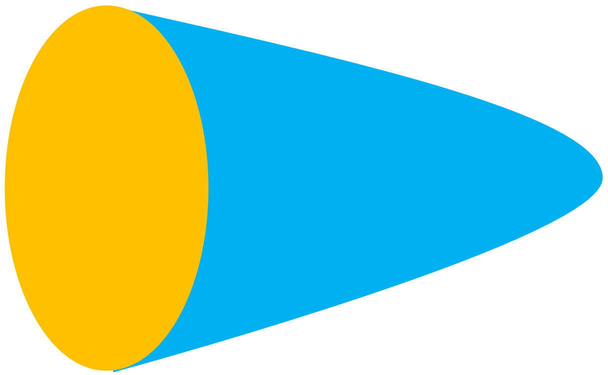
where

$$\hat{A}(R) \in H^*(BSpin), ch_l(F) \in H^*(BG_{gauge})$$

A-hat genus

✓ Inside the integral is called <u>anomaly polynomial</u>.

2.3.3. Generality



<u>ANOMALY INFLOW</u> :

Anomalous d-dim QFT with symmetry G is realized by the boundary of Invertible $(d+1)$-dim QFT with symmetry G. When combined, GI.

$$Z = \exp\{i \cdot (phases)\}$$

Anomalies are given by invertible phases (the center of below sequence).

$$0 \to Ext_{\mathbb{Z}}(\Omega_{d+1}^{spin}(BG), \mathbb{Z}) \to Inv_{spin}^{d+1}(BG) \to Hom_{\mathbb{Z}}(\Omega_{d+2}^{spin}(BG), \mathbb{Z}) \to 0$$

Free part : perturbative gauge anomaly (e.g., triangular diagrams)

Torsion part : non-perturbative gauge anomaly (e.g., SU(2) anomaly)

2.3.3. Generality

In general, anomaly polynomial is given by

$$\left[\hat{A}(R)ch(F)\right]_{d+2} \in H^{d+2}(BSpin \times BG; \mathbb{Q})$$

where

R is the spacetime curvature

F is the gauge curvature

$\hat{A}$ is the A-hat genus

ch is the Chern character

$H^{d+2}(BSpin \times BG; \mathbb{Q}) \cong Hom_{\mathbb{Z}}\left(\Omega_{d+2}^{spin}(BG), \mathbb{Z}\right) \otimes \mathbb{Q}$

✓ Anomaly polynomial contains perturbative information of anomalies.

2.4. WZW terms

Pions are described by

$$\mathcal{L} = \frac{f_\pi}{4} tr\left(\partial_\mu U^\dagger \partial^\mu U\right), U = \xi^2(\pi), \xi(\pi) \in G/H$$

Symmetries are

- Charge conjugation, $C : U \mapsto U^*$
- Naive parity, $P_0 : x \mapsto -x$
- Another, $(-1)^{N_\pi}: U \mapsto U^\dagger, (\pi \mapsto -\pi)$

✓ Parity transformation is $P = P_0(-1)^{N_\pi}$

WZW terms violate P_0 and $(-1)^{N_\pi}$, leaving P.

2.4. WZW terms

Consider 4-dim manifold M

Difficult to write down the action including both $\epsilon^{\alpha\beta\gamma\delta}$ and $tr(\cdots)$.

Equation of motion :

$$\frac{1}{2}{f_\pi}^2\partial_\mu\left(U^\dagger\partial^\mu U\right) = \frac{k}{48\pi^2}\epsilon^{\alpha\beta\gamma\delta}U^\dagger(\partial_\alpha U)U^\dagger(\partial_\beta U)U^\dagger(\partial_\gamma U)U^\dagger(\partial_\delta U)$$

Consider 5-dim manifold $N, (\partial N = M)$

Action :

$$S_{WZW} \propto \int_N d^5x\, \epsilon^{\alpha\beta\gamma\delta\varepsilon}U^\dagger(\partial_\alpha U)U^\dagger(\partial_\beta U)U^\dagger(\partial_\gamma U)U^\dagger(\partial_\delta U)U^\dagger(\partial_\varepsilon U)$$

Independent of extension N

2.4. WZW terms

Consider gauged WZW terms :

In SU(3) QCD with u,d,s quarks,

when $U(1)_{em} \subset SU(3)_V \subset SU(3)_L \times SU(3)_R$ is gauged,

WZW term is upgraded to gauge invariant one.

Gauged WZW term contains anomaly

$$\mathcal{L} = \frac{N_c e^2}{96\pi^2 f_\pi} \pi^0 \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$$

This term describes $\pi^0 \longrightarrow 2\gamma$

2.4. WZW terms

Consider spontaneous symmetry breaking $G \longrightarrow H$ on n-dim

It is known that

Anomaly polynomial : $\alpha \in H^{n+2}(BG;\mathbb{R})$

WZW terms : $\Gamma \in H^{n+1}(G/H;\mathbb{R})$

For $G/H \longrightarrow BH \longrightarrow BG$, $\alpha = \tau(\Gamma)$, where τ is transgression

Note :

Assuming NG field is the sole massless field.

For ungauged WZW terms.

Torsion part of anomaly is not considered.

Contents

3. Recent developments

- WZW terms are <u>not always well defined</u> on arbitrary manifolds.
- Sometimes requires a <u>spin structure</u>.
- Spin bordism seems to classify WZW terms, not homology.
- <u>Conjecture</u> : bordism classifies invertible QFT (including WZW)
- WZW terms of SU, SO, Sp gauge theories are studied.

3. Recent developments

For $Sp(N_c) \times SU(2N_f)$ QCD, quarks condensate as

$$\langle \bar{\psi}_i \psi_j \rangle = v^3 \Sigma_{ij} \in SU(2N_f)/Sp(N_f)$$

WZW terms are well-defined

$$exp\left(2\pi i N_c \int_{W_5} \tilde{c}_3\right)$$

Anomaly polynomial is

$$\tau(N_c \tilde{c}_3) = N_c c_3$$

where

$$\tilde{c}_3 = \sigma(c_3) \in Hom_{\mathbb{Z}}\left(\Omega_5^{spin}(SU(2N_f)/Sp(N_f)), \mathbb{Z}\right)$$

Without residual global anomaly in $Ext_{\mathbb{Z}}(\Omega_5^{spin}(BSp), \mathbb{Z})$

3. Recent developments

Future works :

- There are problems of torsion part (discrete WZW)

- Classification of WZW terms (invertible QFT) by bordism is conjecture.

  Proof or Counterexamples by other models

- Models with mixed anomalies are not investigated.

  Higher dimensional theories will suffer from mixed anomalies

Summary

- Homology and bordism can study the topology of manifolds.
- Spectral sequence can compute them.
- WZW terms can reproduce anomalies.
- The relation between anomalies and WZW terms is transgression.
- Classification by bordism is working well.

Reference

- Y. Lee, K. Ohmori, and Y. Tachikawa, *Revisiting Wess-Zumino-Witten terms*, SciPostPhys.10(2021)061,arXiv:2009.00033 [hep-th].
- E. Witten and K. Yonekura, *Anomaly Inflow and the* η*-Invariant*, The Shoucheng Zhang Memorial Workshop Stanford, CA, USA, May 2-4, 2019, 2019. arXiv:1909.08775 [hep-th].
- G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking*, NATO Sci. Ser. B 59 (1980) 135–157.
- E. Witten, *An SU(2) Anomaly*, Phys. Lett. B 117 (1982) 324–328.
- E. Witten, *Global Aspects of Current Algebra*, Nucl. Phys. B223 (1983) 422–432.
- E. Witten, *Current Algebra, Baryons, and Quark Confinement*, Nucl. Phys. B223 (1983) 433–444.
- J. Wess and B. Zumino, *Consequences of Anomalous Ward Identities*, Phys. Lett. B 37 (1971) 95–97.
- D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, *Generalized Global Symmetries*, JHEP 02 (2015) 172, arXiv:1412.5148 [hep-th].
- A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, *Fermionic Symmetry Protected Topological Phases and Cobordisms*, JHEP 12 (2015) 052, arXiv:1406.7329 [cond-mat.str-el].
- M. Nakahara, *Geometry, Topology and Physics*, CRC Press (2018)
- T. Kugo, *Quantum Theory of Gauge Field*, Baifukan (1989)
- D. Tong, *Gauge Theory*, lecture note on https://www.damtp.cam.ac.uk/user/tong/gaugetheory.html