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Starobinsky Inflation and beyond in Einstein—Cartan Gravity

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1. Introduction (Einstein—Cartan gravity; inflation theory)
2. Propagating degree of freedom in EC gravity as inflatons
3. Conclusions and future works

1. Introduction (Einstein—Cartan gravity; inflation theory)
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3. Conclusions and future works

Outline

- Torsion and metricity condition

- (textbook) metric formalism:
  torsion ✖ violation of metricity ✖

  Einstein—Cartan gravity:
  torsion ✔ violation of metricity ✖

Torsion and Metricity

- Metric tensor $g_{\mu\nu}(x)$
  invariant square of an infinitesimal line element:
  $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$

- Affine connection $\bar{\Gamma}^\lambda_{\nu\mu}(x)$
  infinitesimal parallel transport:
  $A_\mu(x+\Delta x)_{//} \equiv A_\mu(x) + \bar{\Gamma}^\lambda_{\nu\mu}(x)\Delta x^\nu A_\lambda(x)$
  covariant derivative:

$$\nabla_\nu A_\mu \equiv \lim_{\Delta x \to 0} \frac{1}{\Delta x^\nu}\{A_\mu(x+\Delta x) - A_\mu(x+\Delta x)_{//}\}$$
$$= \partial_\nu A_\mu(x) - \bar{\Gamma}^\lambda_{\nu\mu}(x)A_\lambda(x)$$

Torsion and Metricity

- Torsion $T^\lambda_{\ \mu\nu} \equiv \bar{\Gamma}^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\nu\mu}$

$$[\nabla_\mu, \nabla_\nu]V^\rho = (\partial_\mu \bar{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \bar{\Gamma}^\rho_{\mu\sigma} + \bar{\Gamma}^\rho_{\mu\lambda}\bar{\Gamma}^\lambda_{\nu\sigma} - \bar{\Gamma}^\rho_{\nu\lambda}\bar{\Gamma}^\lambda_{\mu\sigma})V^\sigma - (\bar{\Gamma}^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\nu\mu})\nabla_\lambda V^\rho$$

$$\equiv R^\rho_{\ \sigma\mu\nu}V^\sigma - T^\lambda_{\ \mu\nu}\nabla_\lambda V$$

- Curvature tensor $\bar{R}^\rho_{\ \sigma\mu\nu} \equiv \partial_\mu \bar{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \bar{\Gamma}^\rho_{\mu\sigma} + \bar{\Gamma}^\rho_{\mu\lambda}\bar{\Gamma}^\lambda_{\nu\sigma} - \bar{\Gamma}^\rho_{\nu\lambda}\bar{\Gamma}^\lambda_{\mu\sigma}$

  Ricci scalar $\bar{R} \equiv g^{\mu\nu}\bar{R}^\rho_{\ \mu\rho\nu}$

Torsion and Metricity

- Metricity condition $\nabla_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \bar{\Gamma}^{\mu}_{\lambda\rho} g^{\rho\nu} + \bar{\Gamma}^{\nu}_{\lambda\rho} g^{\mu\rho} = 0$

the length of a vector does not change with parallel transport

$$g^{\mu\nu}(x+\Delta x)A_\mu(x+\Delta x)_{//}A_\nu(x+\Delta x)_{//} = g^{\mu\nu}(x)A_\mu(x)A_\nu(x)$$

$$\downarrow$$

$$(g^{\mu\nu}(x) + \partial_\lambda g^{\mu\nu}(x)\Delta x^\lambda)(A_\mu(x) + \bar{\Gamma}^{\rho}_{\lambda\mu}A_\rho \Delta x^\lambda)(A_\nu(x) + \bar{\Gamma}^{\rho}_{\lambda\nu}A_\rho \Delta x^\lambda) = g^{\mu\nu}(x)A_\mu(x)A_\nu(x)$$

$$\downarrow$$

$$\nabla_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \bar{\Gamma}^{\mu}_{\lambda\rho} g^{\rho\nu} + \bar{\Gamma}^{\nu}_{\lambda\rho} g^{\mu\rho} = 0$$

- (textbook) metric formalism:
  torsion ✖ violation of metricity ✖

- Einstein—Cartan gravity:
  torsion ✔ violation of metricity ✖

- Palatini formalism:
  torsion ✖ violation of metricity ✔

- Metric Affine gravity:
  torsion ✔ violation of metricity ✔

(Textbook) metric formalism

- (textbook) metric formalism:
torsion ✖ violation of metricity ✖

- Torsion is zero: $T^{\lambda}_{\ \mu\nu} \equiv \bar{\Gamma}^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\lambda}_{\nu\mu} = 0$
Metricity condition is satisfied: $\nabla_{\lambda} g^{\mu\nu} = \partial_{\lambda} g^{\mu\nu} + \bar{\Gamma}^{\mu}_{\lambda\rho} g^{\rho\nu} + \bar{\Gamma}^{\nu}_{\lambda\rho} g^{\mu\rho} = 0$

↓

$$\bar{\Gamma}^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} \equiv \frac{1}{2} g^{\lambda\sigma} (\partial_{\alpha} g_{\beta\sigma} + \partial_{\beta} g_{\alpha\sigma} - \partial_{\sigma} g_{\alpha\beta})$$

The affine connection $\bar{\Gamma}^{\lambda}_{\alpha\beta}$ is equivalent to the Levi-Civita connection $\Gamma^{\lambda}_{\alpha\beta}$ in metric formalism

$\Gamma^{\lambda}_{\alpha\beta}$ is symmetric about the lower indices and determined by the metric tensor $g^{\mu\nu}$

Einstein—Cartan Gravity

- Einstein—Cartan gravity:
  torsion ✔ violation of metricity ✖

- Decomposing torsion $T^{\lambda}_{\ \mu\nu} \equiv \bar{\Gamma}^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\lambda}_{\nu\mu}$

- Using torsion to express the relation between $\bar{\Gamma}^{\lambda}_{\mu\nu}$ in EC gravity and $\Gamma^{\lambda}_{\alpha\beta} \equiv \frac{1}{2} g^{\lambda\sigma}(\partial_{\alpha} g_{\beta\sigma} + \partial_{\beta} g_{\alpha\sigma} - \partial_{\sigma} g_{\alpha\beta})$ in metric formalism

- Ricci scalar in EC gravity

Einstein—Cartan Gravity

- Decomposing torsion $T^{\lambda}_{\ \mu\nu} \equiv \bar{\Gamma}^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\lambda}_{\nu\mu}$

  vector $T_{\mu} \equiv T^{\alpha}_{\ \mu\alpha}$
  axial vector $S^{\beta} \equiv E^{\mu\nu\alpha\beta} T_{\mu\nu\alpha}$
  tensor $q_{\alpha\beta\gamma} \equiv T_{\alpha\beta\gamma} - \frac{1}{3}(g_{\alpha\gamma} T_{\beta} - g_{\alpha\beta} T_{\gamma}) + \frac{1}{6} E_{\alpha\beta\gamma\mu} S^{\mu}$

Einstein—Cartan Gravity

- Contorsion tensor $K^{\mu}{}_{\alpha\beta}$

$$K^{\mu}{}_{\alpha\beta} \equiv \bar{\Gamma}^{\mu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\alpha\beta}, \qquad \Gamma^{\lambda}_{\alpha\beta} \equiv \frac{1}{2} g^{\lambda\sigma}(\partial_{\alpha} g_{\beta\sigma} + \partial_{\beta} g_{\alpha\sigma} - \partial_{\sigma} g_{\alpha\beta})$$

- Metricity condition $\nabla_{\lambda} g^{\mu\nu} = \partial_{\lambda} g^{\mu\nu} + \bar{\Gamma}^{\mu}_{\lambda\rho} g^{\rho\nu} + \bar{\Gamma}^{\nu}_{\lambda\rho} g^{\mu\rho} = 0$

(torsion $T^{\lambda}_{\mu\nu} \equiv \bar{\Gamma}^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\lambda}_{\nu\mu}$)

↓

$$K^{\mu}{}_{\alpha\beta} = \frac{1}{2}(T^{\mu}{}_{\alpha\beta} + T_{\alpha}{}^{\mu}{}_{\beta} + T_{\beta}{}^{\mu}{}_{\alpha})$$

Einstein—Cartan Gravity

- Ricci Scalar

  $\bar{R}^{\rho}{}_{\sigma\mu\nu} \equiv \partial_\mu \bar{\Gamma}^{\rho}_{\nu\sigma} - \partial_\nu \bar{\Gamma}^{\rho}_{\mu\sigma} + \bar{\Gamma}^{\rho}_{\mu\lambda}\bar{\Gamma}^{\lambda}_{\nu\sigma} - \bar{\Gamma}^{\rho}_{\nu\lambda}\bar{\Gamma}^{\lambda}_{\mu\sigma}$

  $\bar{R} \equiv g^{\mu\nu}\bar{R}^{\rho}_{\mu\rho\nu}$

  in metric formalism: define R using $\Gamma^{\lambda}_{\alpha\beta} \equiv \frac{1}{2}g^{\lambda\sigma}(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta})$

- $\bar{R} = R + 2\nabla_\mu T^\mu - \frac{2}{3}T_\mu T^\mu + \frac{1}{24}S_\mu S^\mu + \frac{1}{2}q^{\mu\nu\rho}q_{\mu\nu\rho}$

Einstein—Cartan Gravity

- Einstein—Hilbert action in metric formalism
  $$S = \int \sqrt{-g} d^4x \frac{M_{Pl}^2}{2} R$$

- In EC gravity: $\bar{R} = R + 2\nabla_\mu T^\mu - \frac{2}{3} T_\mu T^\mu + \frac{1}{24} S_\mu S^\mu + \frac{1}{2} q^{\mu\nu\rho} q_{\mu\nu\rho}$

- The equation of motion from $S = \int \sqrt{-g} d^4x \frac{M_{Pl}^2}{2} \bar{R}$ is the same as that from metric formalism
  $2\nabla_\mu T^\mu$: boundary term
  T^μ, S^μ, $q^{\mu\alpha\beta}$ are not dynamical and constrained to zero

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Outline

Let's go back to (textbook) metric formalism for this section!

- Motivation of inflation theory
- Single-field slow-roll inflation
- Quick introduction to cosmological perturbations

Motivation of inflation theory

- Horizon Problem
  Flatness Problem

- Friedmann Equations
- Comoving Hubble radius
- Horizon problem and inflation

Friedmann Equations

- FRW metric
  $ds^2 = -dt^2 + a^2(t)(d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2))$
  3D flat, homogeneous and isotropic distance element

- Stress-energy tensor of perfect fluid
  $T^\mu_{\ \nu} = -((\varepsilon + p)u^\mu u_\nu + p\delta^\mu_{\ \nu})$
  in comoving frame $u^\mu = (1, 0, 0, 0)$

- Putting these in Einstein equation

  ↓

  first Friedmann equation $\ddot{a} = -\frac{4\pi}{3}(\varepsilon + 3p)a$

  second Friedmann equation $H^2 = \frac{8\pi}{3}\varepsilon$ with $H \equiv \frac{\dot{a}}{a}$

Comoving Hubble radius

- Energy conservation equation $T^{\alpha}_{\ 0;\alpha} = 0$

$$\dot{\varepsilon} = -3H(\varepsilon + p) \text{ with } H \equiv \frac{\dot{a}}{a}$$

$$\downarrow$$

$$\varepsilon \propto a^{-3(1+w)} \quad \text{with } w \equiv \frac{p}{\varepsilon}$$

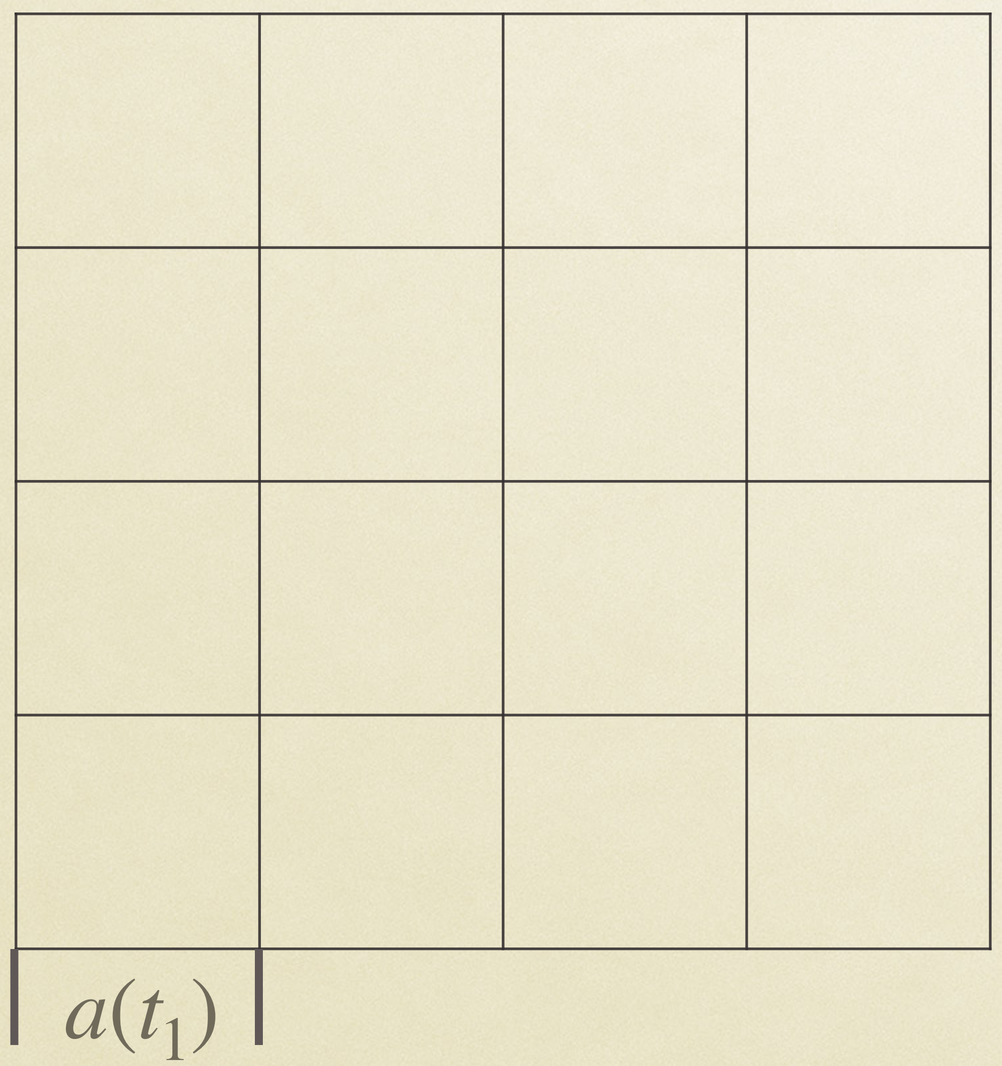
- Together with second Friedmann equation $H^2 = \frac{8\pi}{3}\varepsilon$

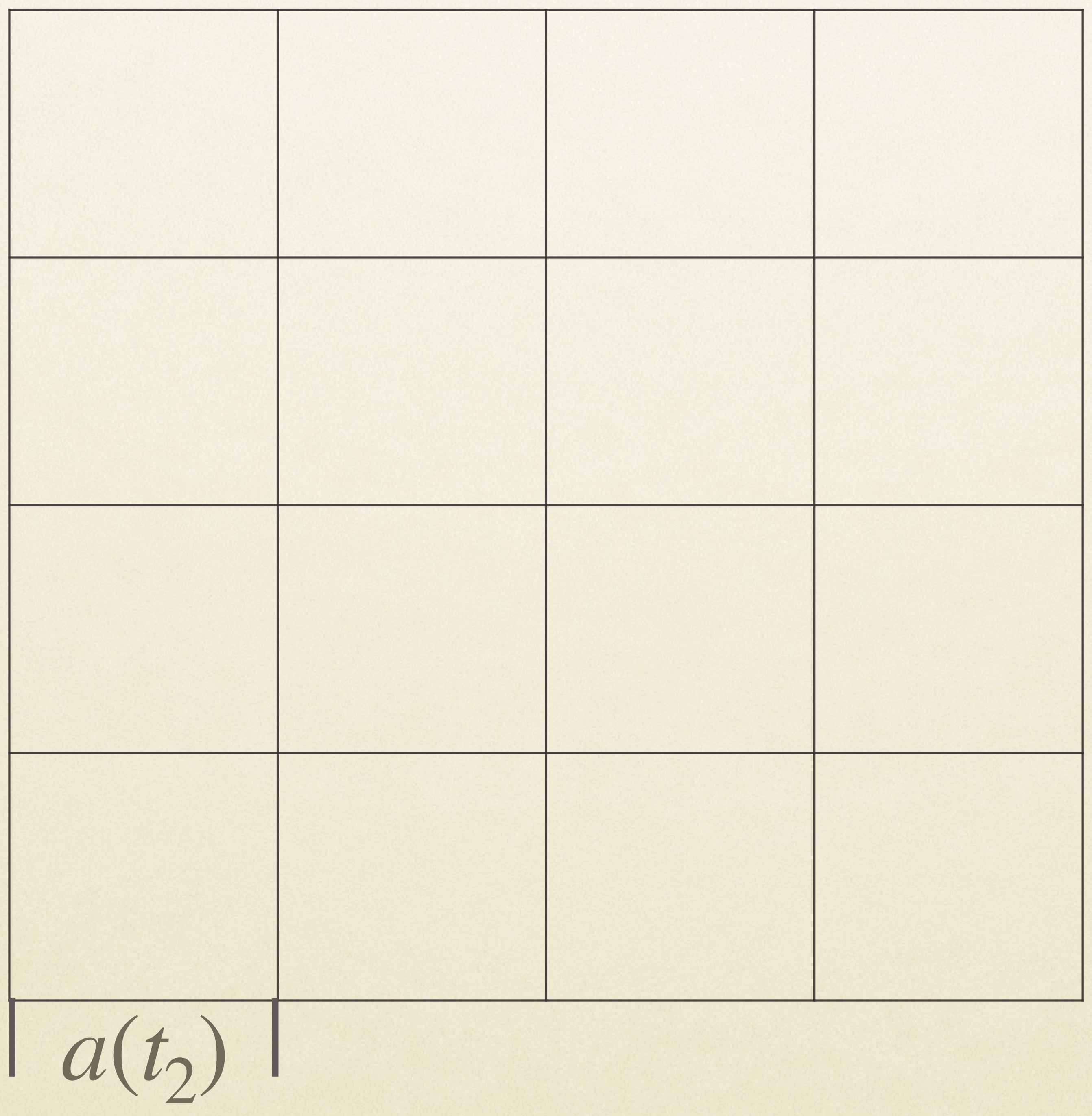
$$\downarrow$$

comoving Hubble radius $(aH)^{-1} \propto a^{\frac{1}{2}(1+3w)}$

What is comoving Hubble radius?

- Physical distance & comoving distance
  $ds^2 = -dt^2 + a^2(t)(d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2))$
  $dl = a(t)d\chi$





The length of the large square?

How many grids?

What is comoving Hubble radius?

- Characteristic length-scale
  Hubble length $d \sim H^{-1}$
  Within this scale, particles can "communicate with" each other

- comoving Hubble length $(aH)^{-1} \propto a^{\frac{1}{2}(1+3w)}$

Increasing comoving Hubble radius?

$(aH)^{-1} \propto a^{\frac{1}{2}(1+3w)}$

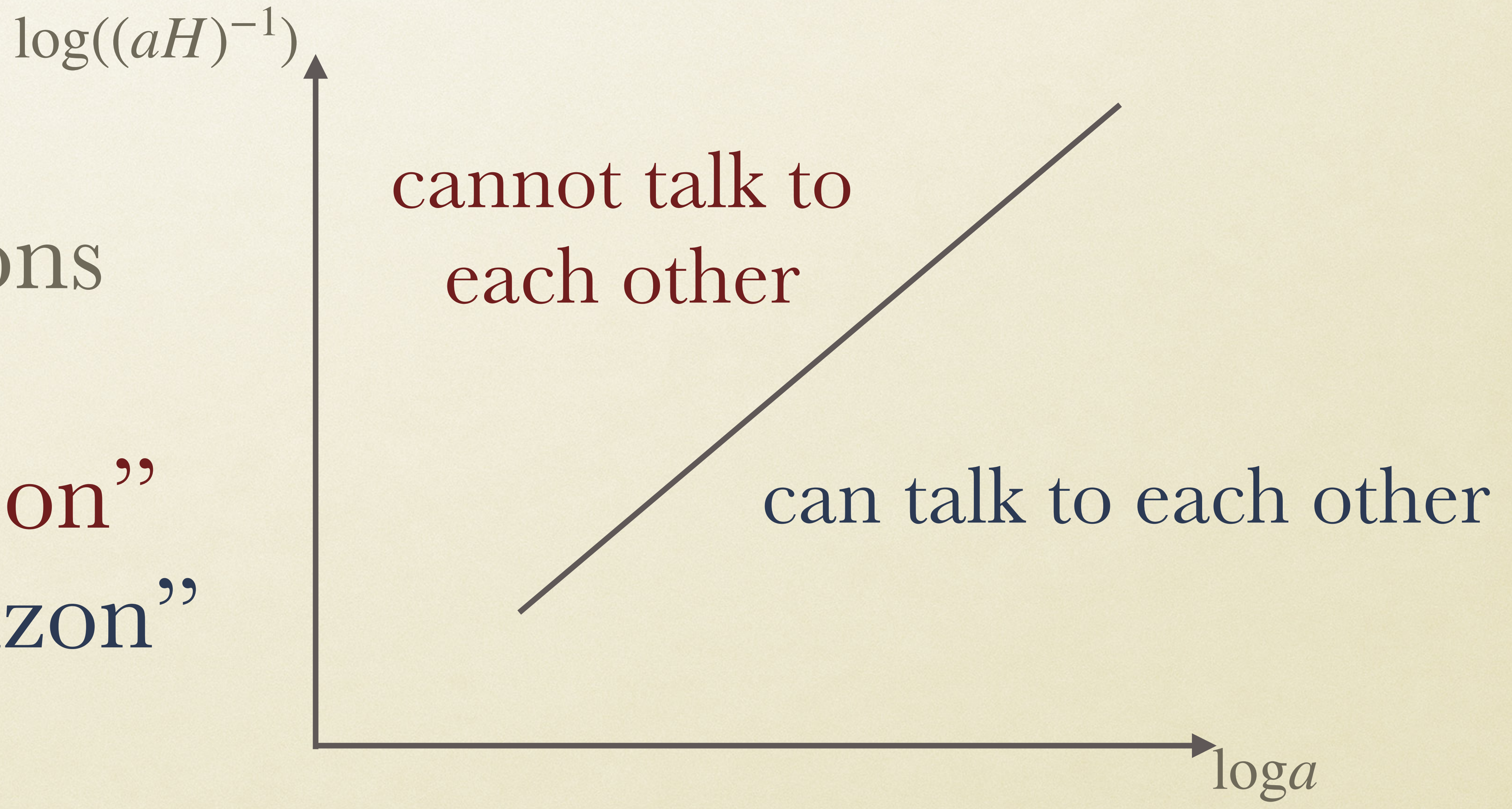
- Radiation-dominated $w = \dfrac{1}{3}$ & Matter-dominated $w = 0$

  increasing $(aH)^{-1}$

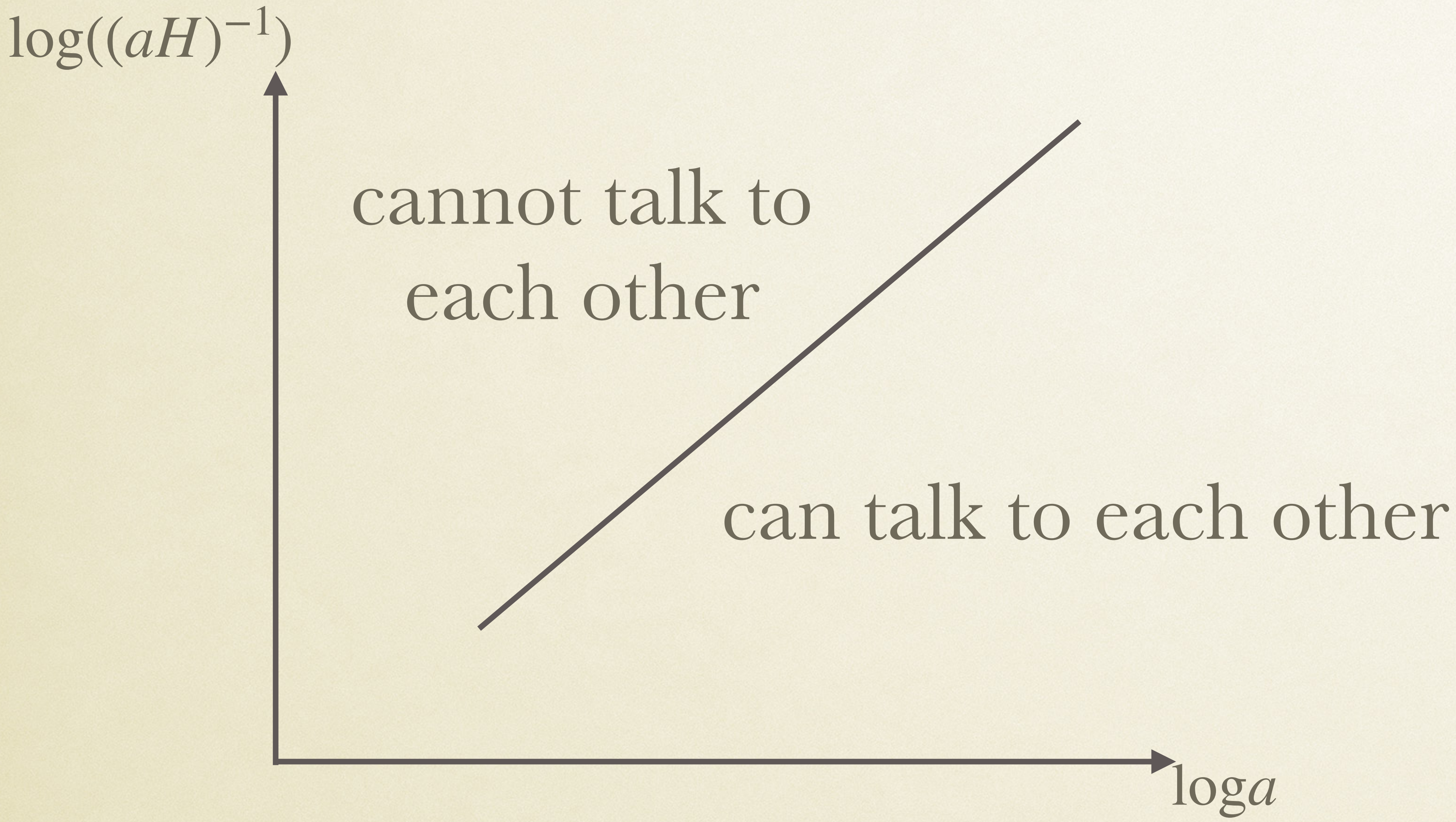
- comoving wave number k of fluctuations constant over time for each mode

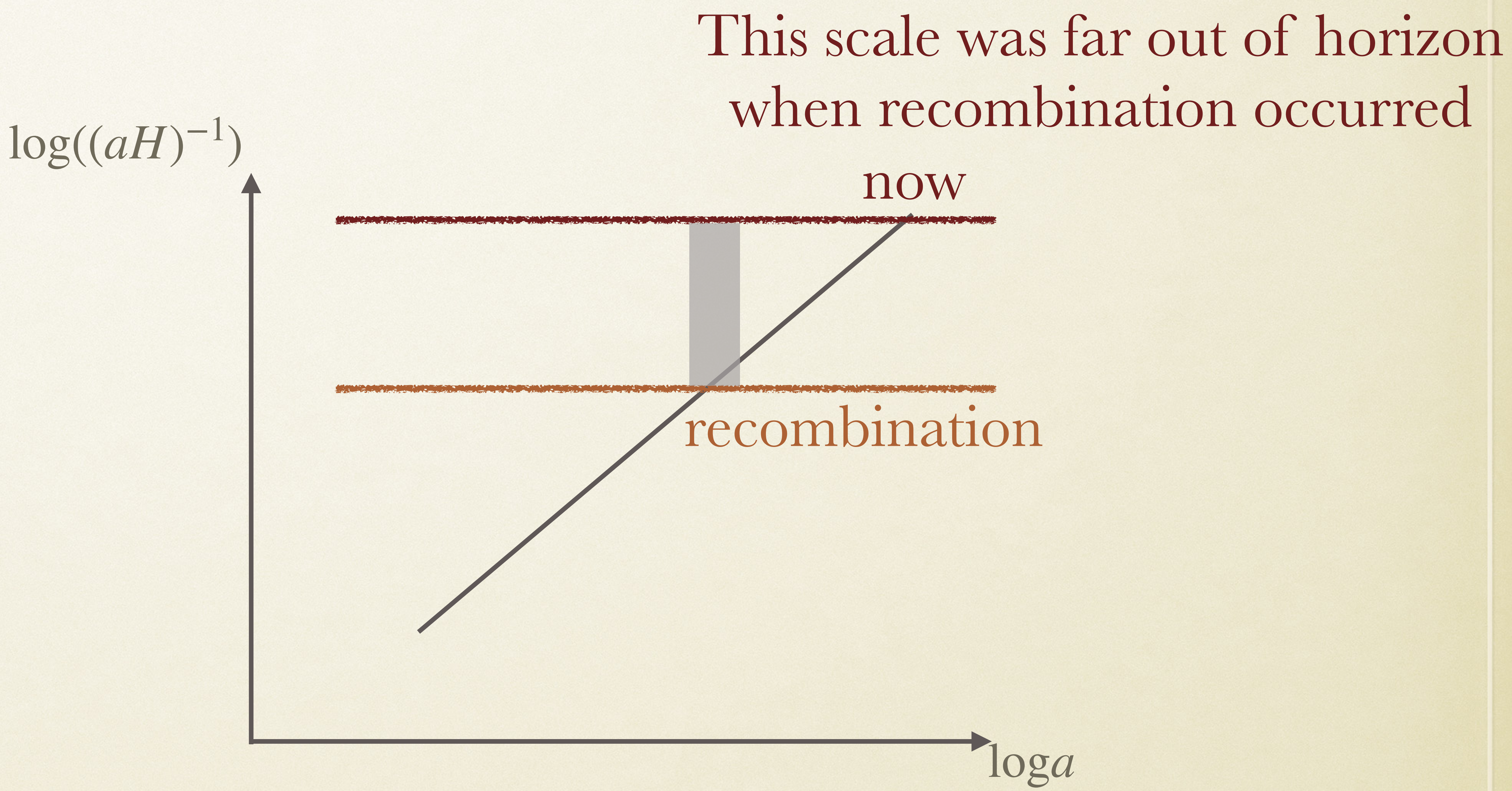
  $1/k \gg (aH)^{-1}$: mode far beyond "horizon"

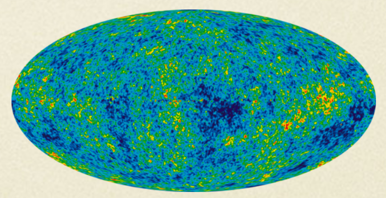
  $1/k \ll (aH)^{-1}$: mode deep within "horizon"



Horizon problem

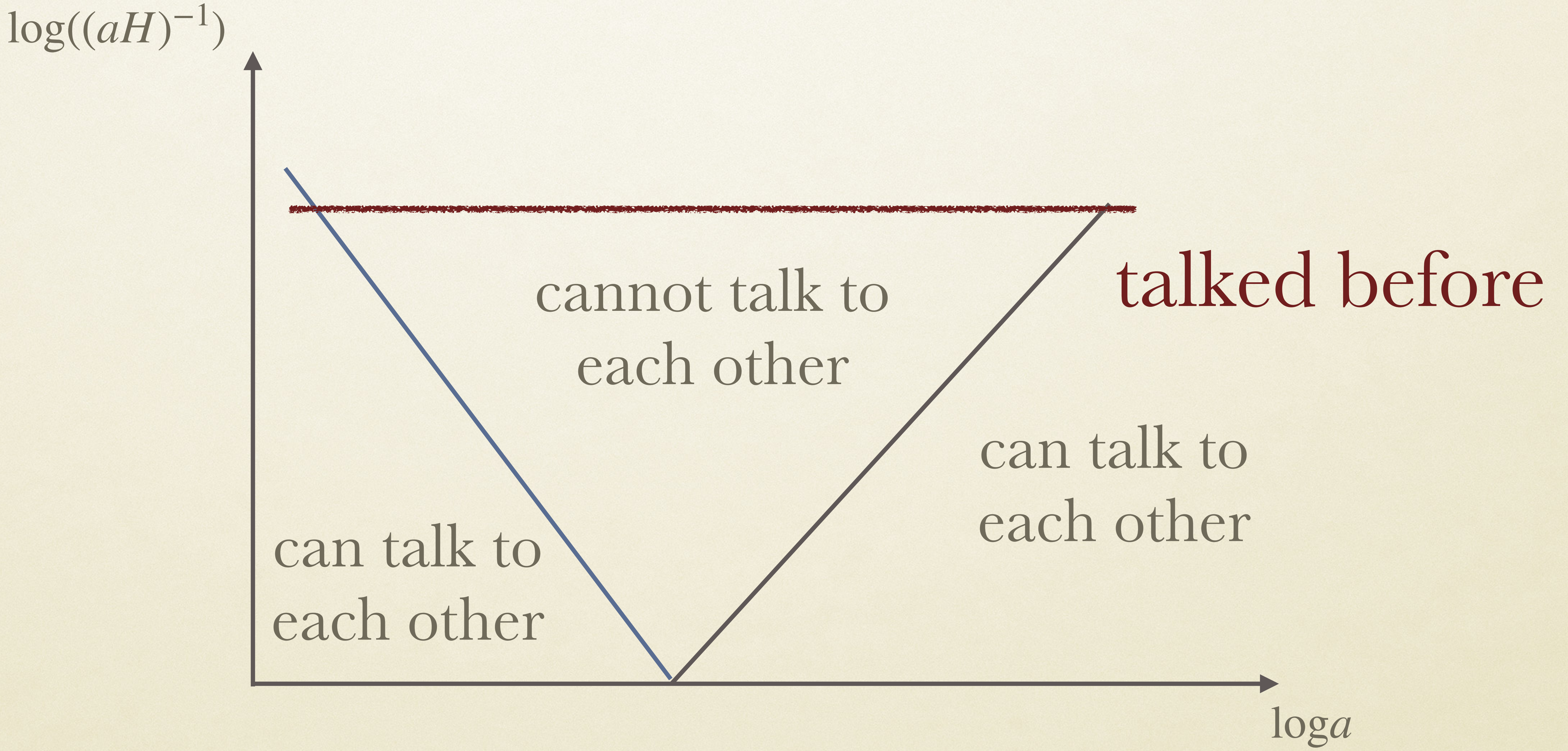




Horizon problem: how can 
cosmic microwave background (CMB) be so homogeneous and isotropic without ever talking to each other???

Inflation theory

- How about a period of decreasing $(aH)^{-1} \propto a^{\frac{1}{2}(1+3w)}$?
- $w < -\frac{1}{3}$



c.f. Flatness problem is also from an (forever) increasing comoving Hubble radius

How to realize inflation?

Single-Field Slow-roll inflation

- The simplest models of inflation
- inflaton $\phi(t)$
- $S = \int \sqrt{-g} d^4x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]$
- Stress-energy tensor

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \longrightarrow \begin{aligned} \varepsilon &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned}$$

$$w = \frac{p}{\varepsilon}$$

$\dot{\phi}^2 \ll V(\phi)$ to make $w \sim -1$

Single-Field Slow-roll inflation

- Dynamics of inflaton and FRW geometry
  inflaton's equation of motion $\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$
  the second Friedmann equation $H^2 = \frac{8\pi}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$

- Slow-roll parameters
  for $\dot{\phi}^2 \ll V(\phi)$, $\epsilon \equiv \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \ll 1$
  for this to last a sufficiently long time, $\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1$

Single-Field Slow-roll inflation

- Another set of slow-roll parameters — shape of the potential!

$$\epsilon_V \equiv \frac{M_{Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1$$

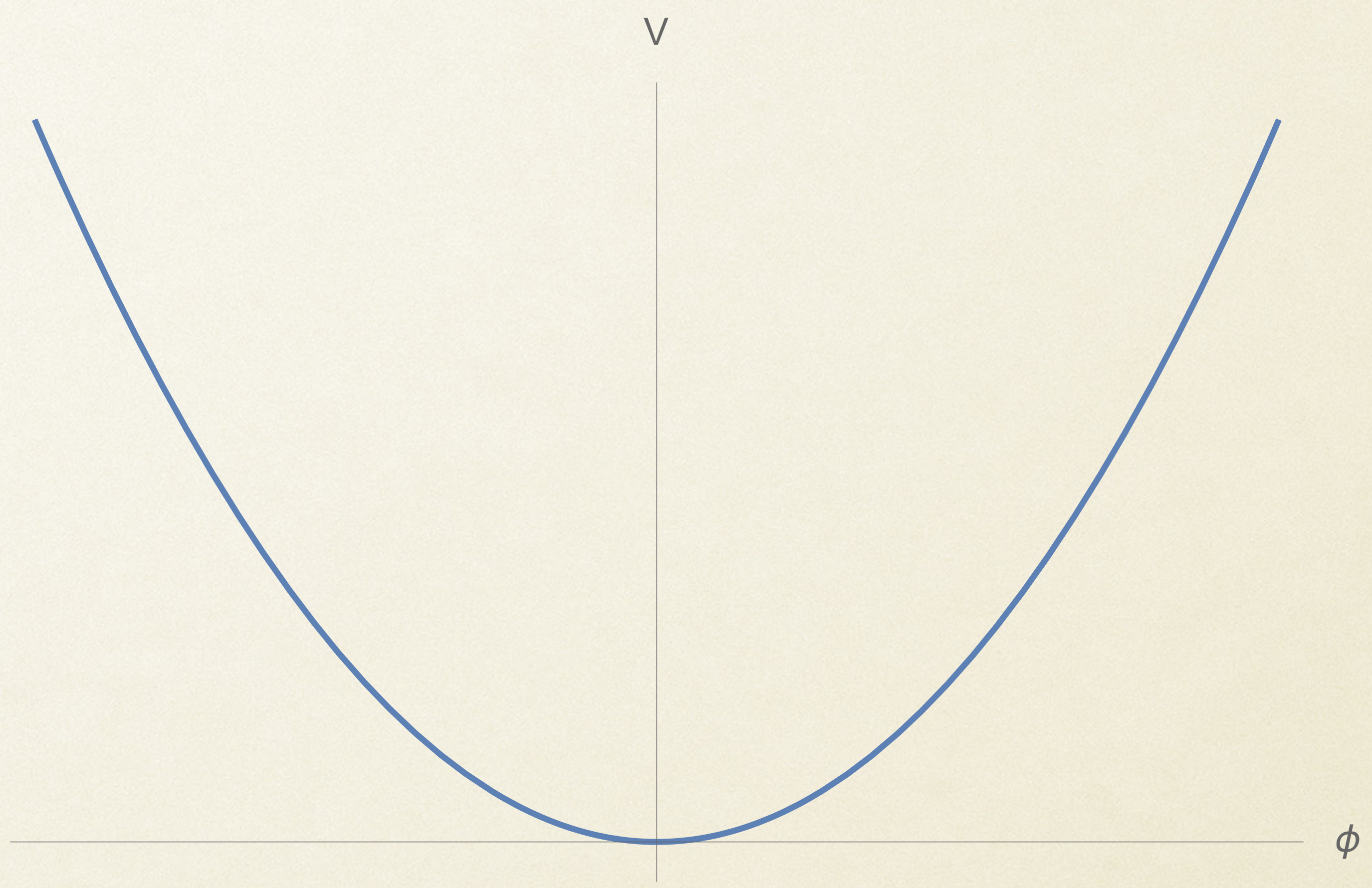
$$\eta_V \equiv M_{Pl}\frac{V_{,\phi\phi}}{V} \ll 1$$

- One can show $\epsilon \approx \epsilon_V$, $\eta \approx \eta_V - \epsilon_V$

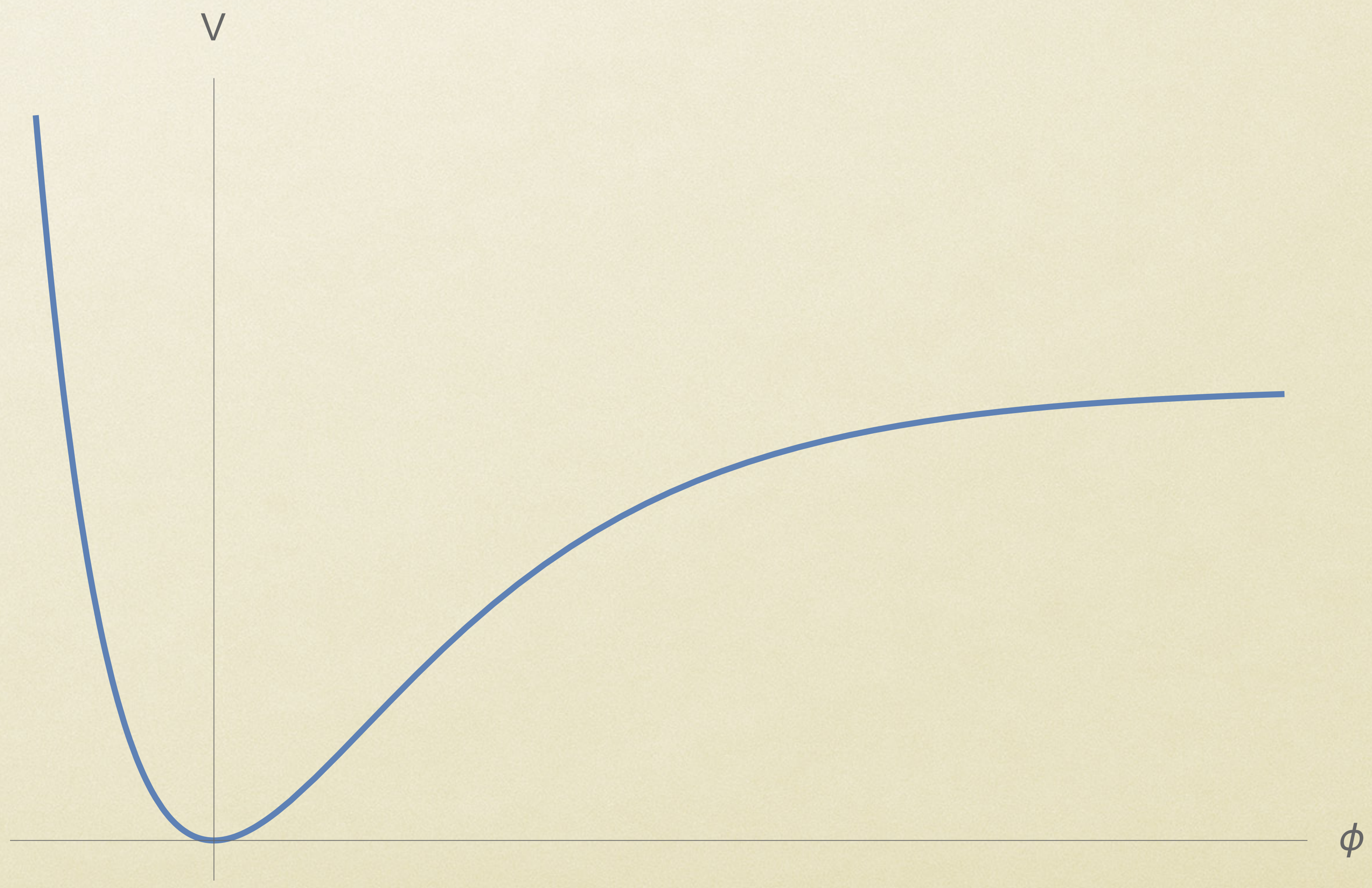
Single-Field Slow-roll inflation

Examples

- Quadratic chaotic inflation
  $V(\phi) \propto \phi^2$

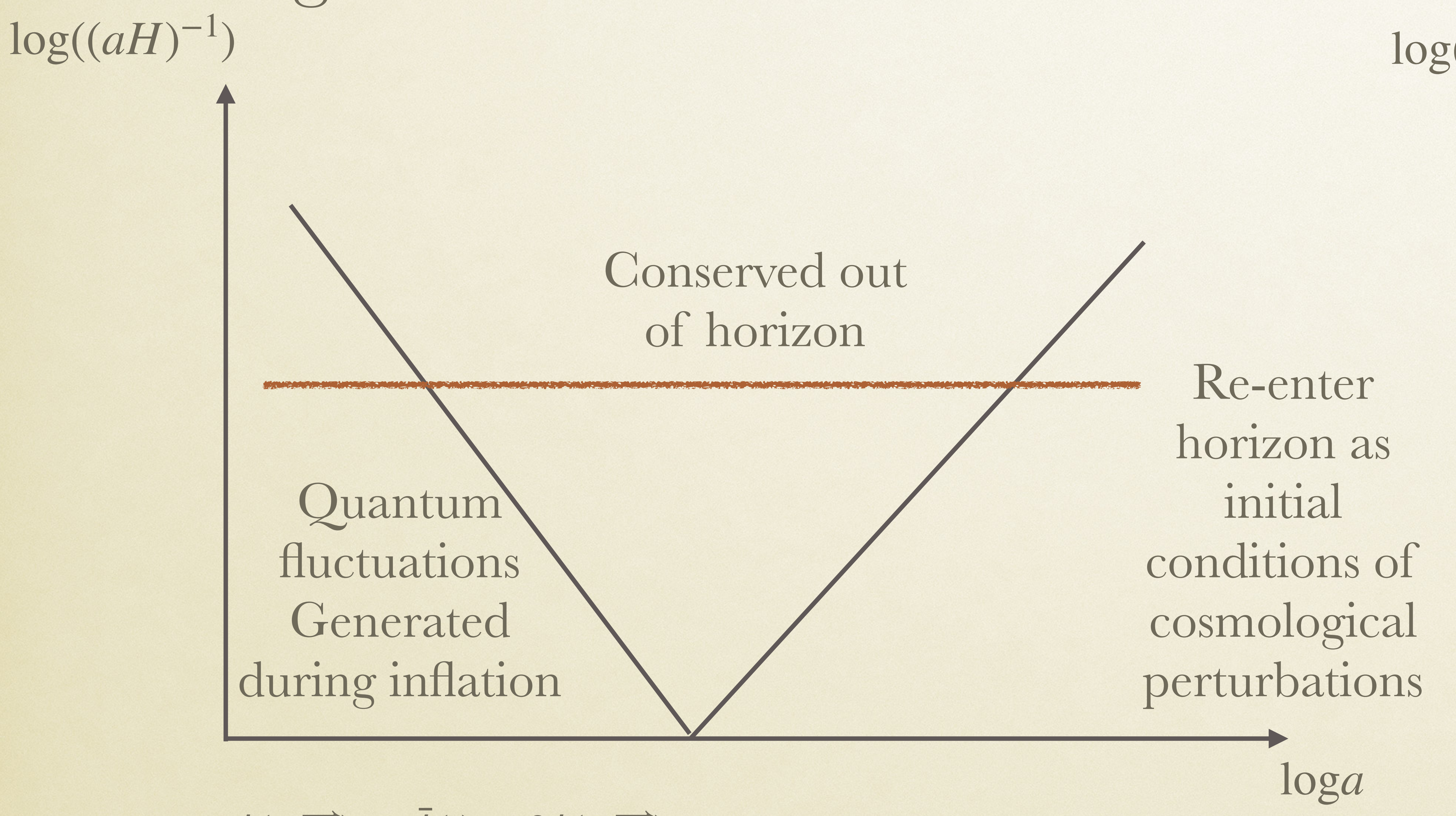


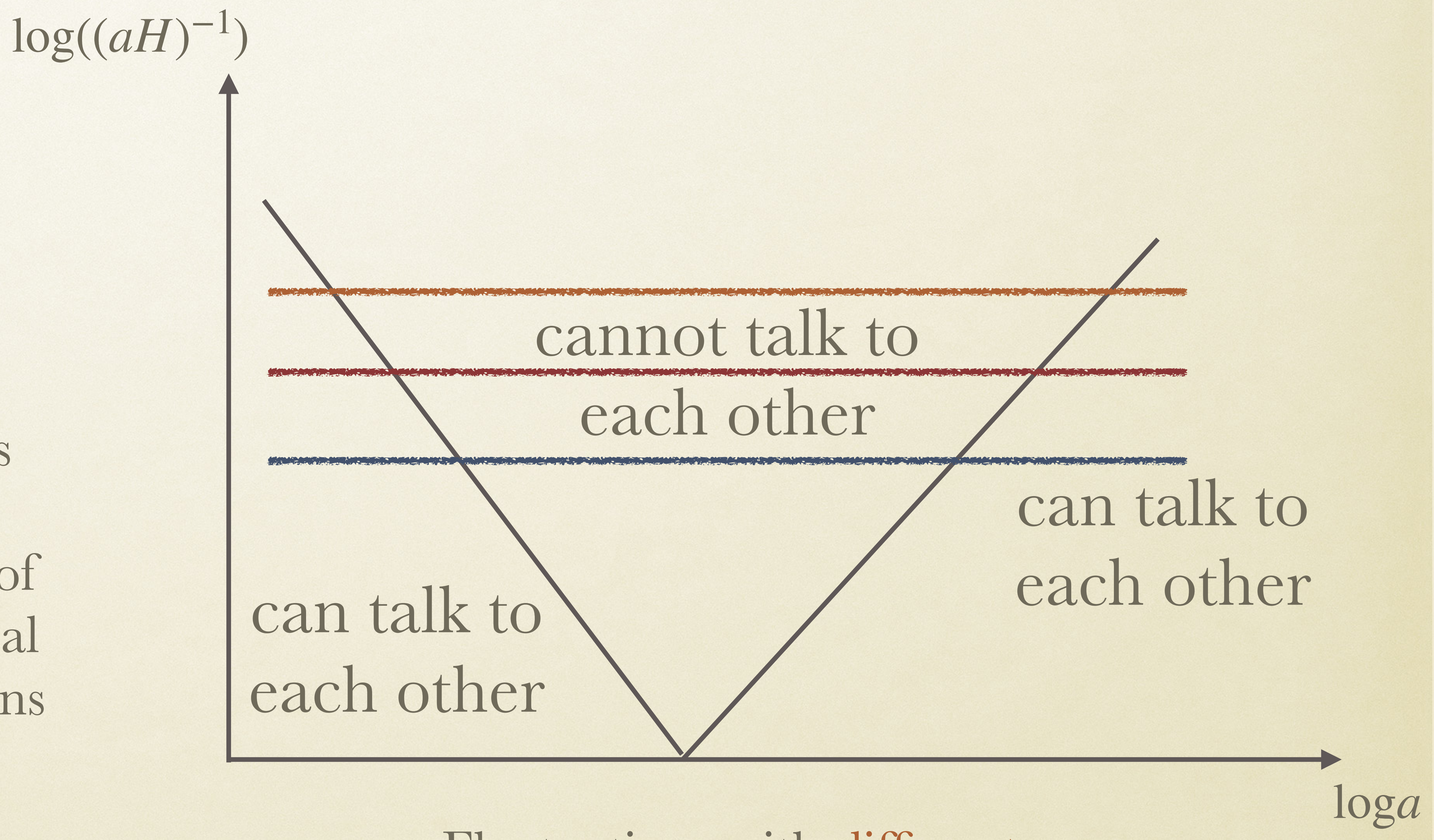
- Starobinsky inflation
  $$V(\phi) \propto \left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}}\right)^2$$



Inflation and cosmological perturbations

- The universe is not completely homogeneous and isotropic
  e.g. structure formations





$\phi(t, \vec{x}) = \bar{\phi}(t) + \delta\phi(t, \vec{x})$

$g_{\mu\nu}(t, \vec{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \vec{x})$

Fluctuations with different comoving wavenumber k exits/re-enter horizon at different timings

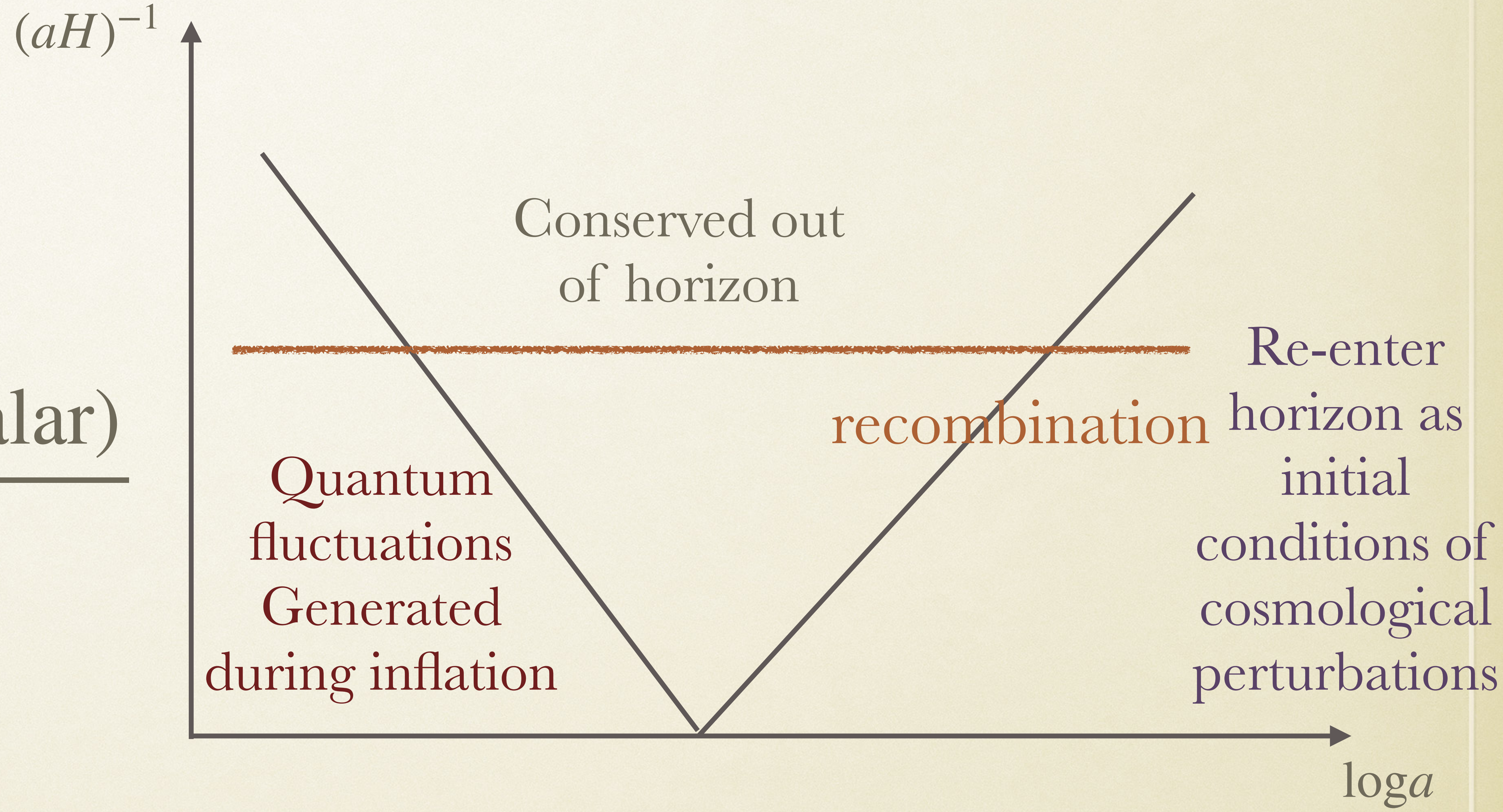
Inflation and cosmological perturbations

- $\delta g_{\mu\nu}(t, \vec{x})$ can be decomposed into scalar, vector, tensor parts

- During inflation:
  scalar perturbations: created by $\delta\phi$
  vector perturbations: not created by inflation
  tensor perturbations: gravitational waves

Observation from CMB

- Observational quantities (@CMB scale)
  scalar spectral index
  $$n_s - 1 = \frac{d\ln(\text{power spectrum about scalar})}{d\ln k}$$
  tensor-to-scalar ratio
  $$r = \frac{(\text{power spectrum about scalar})}{(\text{power spectrum about tensor})}$$



- These quantities can be expressed using slow-roll parameters in the case of single-field slow-roll inflation and compared with observation

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Outline

- Extra propagating degree of freedom (scalaron) in $f(R)$ theory of metric formalism

- No extra propagating degree of freedom in $f(\bar{R})$ theory of EC gravity

- Nieh—Yan term and Holst term

- Extra propagating degree of freedom as inflaton in EC gravity with Nieh—Yan term

- Extra propagating degree of freedom as inflaton in EC gravity with Nieh—Yan term and Holst term

Scaleron in $f(R)$ in metric formalism

- $S = \int \sqrt{-g_J} d^4x \frac{M_{Pl}^2}{2} f(R_J)$

- Introducing auxiliary field χ
  $S = \int \sqrt{-g_J} d^4x \frac{M_{Pl}^2}{2} [(R_J - \chi) f'(\chi) + f(\chi)]$

- Solving the constraint equation of χ will bring back the starting action
  $(R_J - \chi) f''(\chi) = 0$, assuming $f''(\chi) \neq 0$

Scaleron in $f(R)$ in metric formalism

- $S = \int \sqrt{-g_J} d^4x \frac{M_{Pl}^2}{2} [(R_J - \chi) f'(\chi) + f(\chi)]$

- Performing conformal transformation from Jordan frame to Einstein frame
  $g_{E\mu\nu} = \Omega^2 g_{J\mu\nu}$, $\Omega^2 \equiv f'(\chi)$, assuming $\Omega^2 > 0$ to maintain the sign before R
  $R_J = \Omega^2 R_E - 6\Omega^3 \Box_E \Omega^{-1}$

- $S = \int \sqrt{-g_E} d^4x \left[\frac{M_{Pl}^2}{2} R_E - \frac{3}{4} M_{Pl}^2 \partial_\mu \ln\Omega^2 \partial^\mu \ln\Omega^2 + V(\ln\Omega^2) \right]$
  Kinetic term!
  $\sigma \equiv \sqrt{\frac{3}{2}} M_{Pl} \ln\Omega^2$ is the scalaron field

- E.g. scaleron in $R + \alpha R^2$ is the inflaton in the Staronbinsky inflation

No scaleron in $f(\bar{R})$ in EC gravity

- $S = \int \sqrt{-g_J} d^4x \frac{M_{Pl}^2}{2} f(\bar{R}_J)$

- Introducing auxiliary field χ

$\Omega^2 \equiv f'(\chi)$

$$S = \int \sqrt{-g_J} d^4x \frac{M_{Pl}^2}{2} [(R_J + 2\nabla_\mu T^\mu - \frac{2}{3} T_\mu T^\mu + \frac{1}{24} S_\mu S^\mu + \frac{1}{2} q^{\mu\nu\rho} q_{\mu\nu\rho} - \chi) f'(\chi) + f(\chi)]$$

- S^μ and $q^{\mu\nu\rho}$ constrained themselves to zero

- Integral by part, and solve the constraint equation for T^μ, (and performing conformal transformation)

$$-\frac{1}{3} M_{Pl}^2 \Omega^2 \left(T^\mu + \frac{3}{2} \partial_\mu \ln\Omega^2 \right)^2 + \frac{3}{4} M_{Pl}^2 \Omega^2 \partial_\mu \ln\Omega^2 \partial^\mu \ln\Omega^2$$

A new term appears and cancels the "kinetic term" $-\frac{3}{4} M_{Pl}^2 \partial_\mu \ln\Omega^2 \partial^\mu \ln\Omega^2$

Introducing the Nieh—Yan term or/and the Holst term to realizing propagating degree of freedom in EC gravity and sustain inflation

Nieh—Yan term and Holst term

- Nieh—Yan term H. T. Nieh and M. L. Yan, J. Math. Phys. 23 (1982) 373

$$\int d^4x \partial_\mu \left(\sqrt{-g} E^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}\right) = - \int d^4x \partial_\mu \left(\sqrt{-g} S^\mu\right) = \int \sqrt{-g} d^4x \nabla_\mu S^\mu$$

- Holst term S. Holst, *Phys. Rev. D* **53** (1996) 5966

$$\int d^4x \sqrt{-g} E^{\mu\nu\rho\sigma} \bar{R}_{\mu\nu\rho\sigma} = \int \sqrt{-g} d^4x \left(\nabla_\mu S^\mu - \frac{2}{3} S_\mu T^\mu + \frac{1}{2} E^{\rho\sigma\mu\nu} q_{\lambda\rho\sigma} q^\lambda{}_{\mu\nu} \right)$$

- $q^{\mu\nu\rho}$ constrained itself to zero, so let's drop it from now on

- Considering linear combinations of the Nieh—Yan term and the Holst term is equivalent to considering linear combinations of Nieh—Yan term and $S_\mu T^\mu$

Nieh—Yan term and Holst term

Previous studies including Nieh—Yan term and Holst term

- Higgs inflation with non-minimal coupling to Higgs field

M. He, K. Kamada, and K. Mukaida, *JHEP* **01** (2024) 014
M. Långvik, J.-M. Ojanperä, S. Raatikainen, and S. Rasanen, *Phys. Rev. D* **103** no. 8, (2021) 083514
M. Shaposhnikov, A. Shkerin, I. Timiryasov, and S. Zell, *JCAP* **02** (2021) 008

- Including only Nieh—Yan term in the context of modified gravity

F. Bombacigno, S. Boudet, and G. Montani, *Nucl. Phys. B* **963** (2021) 115281
S. Boudet, PhD thesis, University of Trento, 2023

- Including only Holst term

e.g.
G. Pradisi and A. Salvio, Eur. Phys. J. C 82 no. 9, (2022) 840
A. Salvio, *Phys. Rev. D* **106** no. 10, (2022) 103510

With Nieh—Yan term

- Let's start with the simplest model
- $$S = \int \sqrt{-g} d^4x \left[\frac{M_{Pl}^2}{2} \bar{R} + \beta \left(\nabla_\mu S^\mu \right)^2 \right]$$

Similar procedures as shown before:

- introducing an auxiliary field to rewrite the last term
- solving constraint equations for torsion
- field redefinition

$$f\left(\nabla_\mu S^\mu \right) = \beta \left(\nabla_\mu S^\mu \right)^2$$

$$S = \int \sqrt{-g} d^4x \left(\frac{M_{Pl}^2}{2} R - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{M_{Pl}^2}{96\beta} \sigma^2 \right)$$

Chaotic quadratic inflation!

With Nieh—Yan term

- One can also consider
- $$S = \int \sqrt{-g_{\rm J}} d^4x \left[\frac{M_{Pl}^2}{2} \bar{R}_{\rm J} + \alpha_{\rm R} \left(\bar{R}_{\rm J} + \frac{\alpha}{2} \nabla_\mu S^\mu \right)^2 \right]$$

Similar procedures as shown before:

- introducing an auxiliary field
- solving constraint equations for torsion
- conformal transformation
- field redefinition

$$S_{\rm E} = \int \sqrt{-g_{\rm E}} d^4x \left[\frac{M_{Pl}^2}{2} R_{\rm E} - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{M_{Pl}^4}{16\alpha_{\rm R}} \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\sigma}{\alpha M_{Pl}}} \right)^2 \right]$$

alpha-attractor inflation!

Starobinsky inflation with $\alpha = 1$

With Nieh—Yan term

- "General" case
- $$S = \int \sqrt{-g} d^4x \left[\frac{M_{Pl}^2}{2} \bar{R} + \alpha_{\rm R} \bar{R}^2 + \alpha_{\rm RS} \bar{R} \, \nabla_\mu S^\mu + \alpha_{\rm S} \left(\nabla_\mu S^\mu \right)^2 \right]$$

$$\downarrow$$

$$\alpha_{\rm R} \left(\bar{R} + \frac{\alpha}{2} \nabla_\mu S^\mu \right)^2 + \beta \left(\nabla_\mu S^\mu \right)^2$$

- Introducing two auxiliary fields for each completed square to rewrite this action

With Nieh—Yan term

- With two auxiliary fields, one generally ends up with a $p(\phi, X)$-type theory, where X represents the kinetic term of ϕ
- When solving the constraint equation for the "non-dynamical" field (Ω^2), kinetic term of the dynamical field (Σ) enters the denominator

$$\int \sqrt{-g_{\mathrm{E}}} d^4x \left[\frac{M_{Pl}^2}{2} R_{\mathrm{E}} - \frac{1}{2\Omega^4} \left(\partial_\mu \Sigma\right)^2 - \frac{M_{Pl}^4}{16\alpha_{\mathrm{R}}} \frac{(\Omega^2 - 1)^2}{\Omega^4} - \frac{\alpha^2 M_{Pl}^4}{64\beta} \left(\sqrt{\frac{2}{3}} \frac{\Sigma}{M_{Pl}\alpha\Omega^2} + \frac{1}{\Omega^2} - 1 \right)^2 \right]$$

- With both Nieh—Yan term and Holst term, we consider actions with only one completed square

With Nieh—Yan term and Holst term

- Models with only one completed square in the action — one auxiliary field to analyze
- $$S = \int \sqrt{-g_{\mathrm{J}}} d^4x \left[\frac{M_{Pl}^2}{2} \left(\bar{R}_{\mathrm{J}} + \zeta S_\mu T^\mu \right) + \alpha_{\mathrm{R}} \left(\bar{R}_{\mathrm{J}} + \frac{\alpha}{2} \nabla_\mu S^\mu + \frac{\tilde{\alpha}}{2} S_\mu T^\mu \right)^2 \right]$$

↓ following the same procedures as shown before

$$S = \int \sqrt{-g_{\mathrm{E}}} d^4x \left[\frac{M_{Pl}^2}{2} R_{\mathrm{E}} - \frac{M_{Pl}^2}{2} \frac{3 \left\{ \alpha e^\sigma + 3 \left[\zeta + (e^\sigma - 1)\, \tilde{\alpha}/2 \right] \right\}^2}{2e^{2\sigma} + 18 \left[\zeta + (e^\sigma - 1)\, \tilde{\alpha}/2 \right]^2} \partial_\mu \sigma \partial^\mu \sigma - \frac{M_{Pl}^4}{16 \alpha_{\mathrm{R}}} (1 - e^{-\sigma})^2 \right]$$

- $f(\phi)X + V(\phi)$ can generally be turned to a canonical kinetic term and a potential by field redefinition
- Deformation of Starobinsky inflation/alpha-attractor inflation for some parameter ranges

With Nieh—Yan term and Holst term

- Consistency check

ζ

- $$S=\int\sqrt{-g_{\rm J}}d^4x\left[\frac{M_{Pl}^2}{2}\left(\bar{R}_{\rm J}+\frac{\tilde{\alpha}}{2}S_\mu T^\mu\right)+\alpha_{\rm R}\left(\bar{R}_{\rm J}+\frac{\alpha}{2}\nabla_\mu S^\mu+\frac{\tilde{\alpha}}{2}S_\mu T^\mu\right)^2\right]$$

following the same procedures as shown before or taking limit of the general action

alpha-attractor inflation!

- This action is equivalent to

$$S=\int\sqrt{-g_{\rm J}}d^4x\left[\frac{M_{Pl}^2}{2}\bar{R}_{\rm J}+\alpha_{\rm R}\left(\bar{R}_{\rm J}+\frac{\alpha}{2}\nabla_\mu S^\mu\right)^2\right]$$

by redefining $T'_\mu=T_\mu-\frac{3}{4}\zeta S_\mu$, $S'_\mu=\sqrt{1+9\zeta^2}S_\mu$

With Nieh—Yan term and Holst term

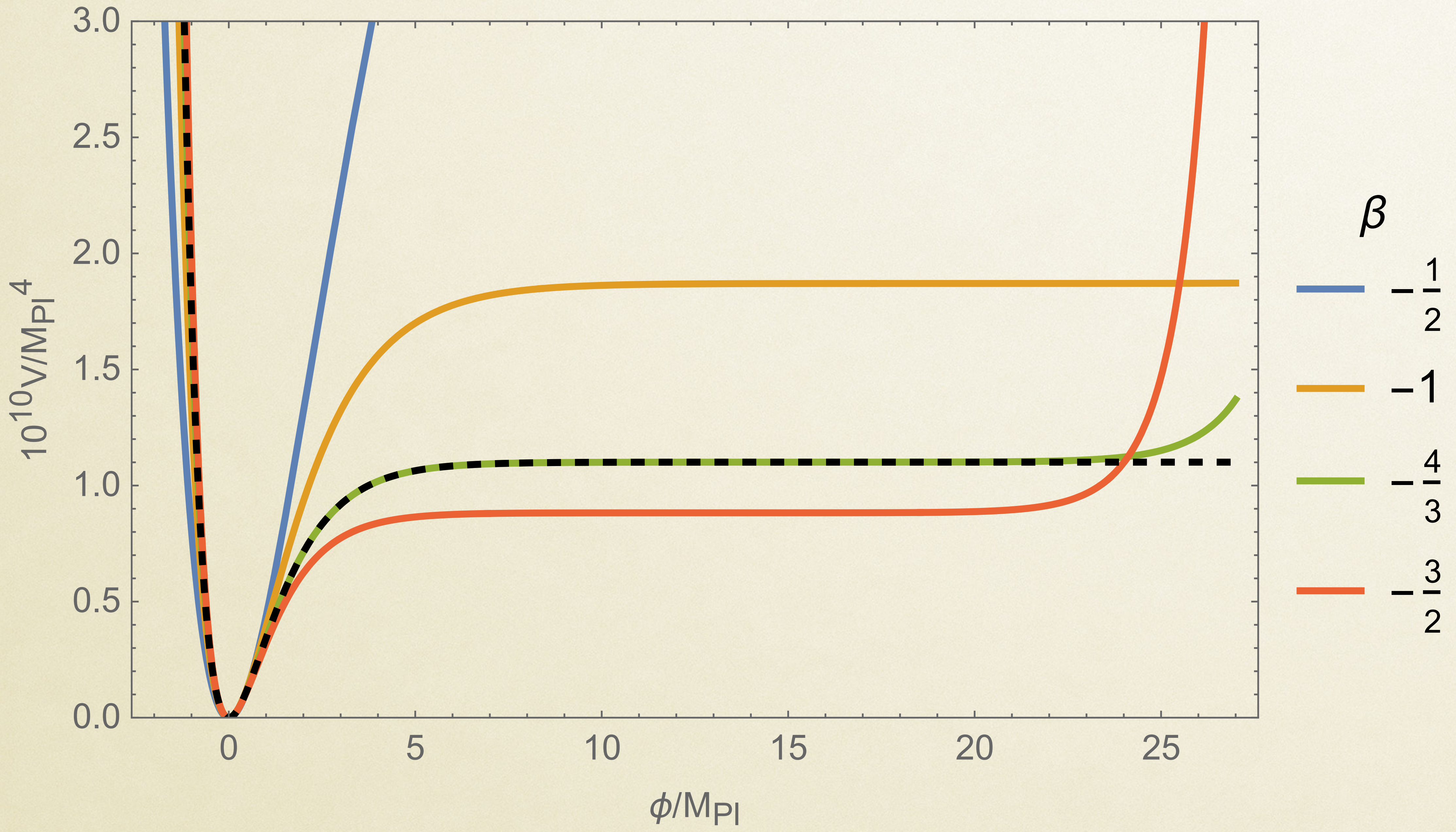
- Let's consider a simplified case
- $$S = \int \sqrt{-g_{\mathrm{J}}} d^4x \left[\frac{M_{Pl}^2}{2} \left(\bar{R}_{\mathrm{J}} + \zeta S_\mu T^\mu \right) + \alpha_{\mathrm{S}} \left(\nabla_\mu S^\mu + \frac{\beta}{2} S_\mu T^\mu \right)^2 \right]$$

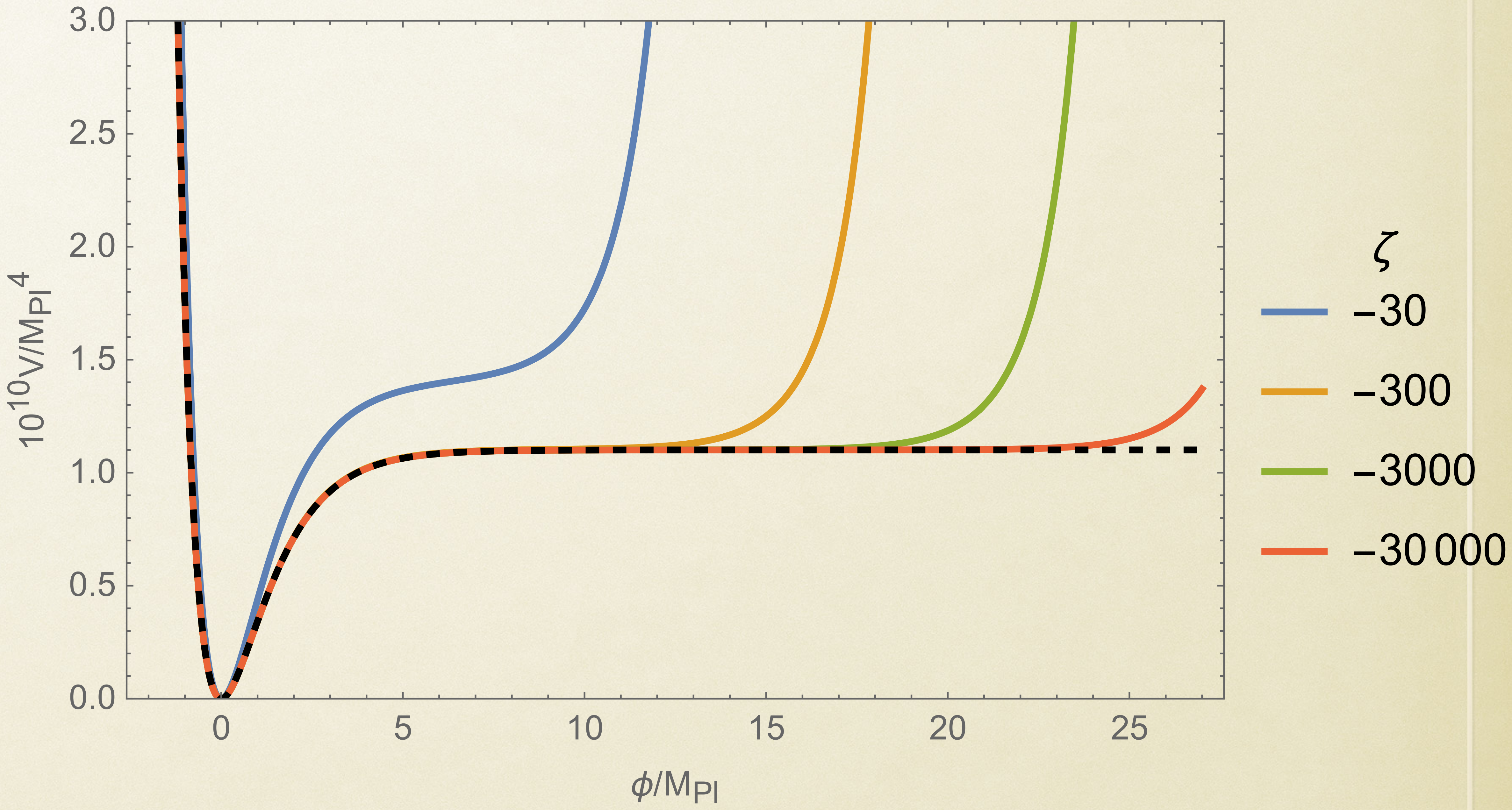
↓ following the same procedures as shown before
or
taking limit of the general action

$$S = \int \sqrt{-g_{\mathrm{E}}} d^4x \left[\frac{M_{Pl}^2}{2} R_{\mathrm{E}} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{M_{Pl}^4}{36 \alpha_{\mathrm{S}} \beta^2} \left(3\zeta + \sinh \sqrt{\frac{3}{8}} \frac{\beta \phi}{M_{Pl}} \right)^2 \right]$$

With Nieh—Yan term and Holst term

$$S = \int \sqrt{-g_{\mathrm{E}}} d^4x \left[\frac{M_{Pl}^2}{2} R_{\mathrm{E}} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{M_{Pl}^4}{36 \alpha_S \beta^2} \left(3\zeta + \sinh \sqrt{\frac{3}{8}} \frac{\beta \phi}{M_{Pl}} \right)^2 \right]$$





β plays the role of the factor in the exponential of alpha-attractor inflation $\left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi}{\alpha M_{Pl}}}\right)^2$

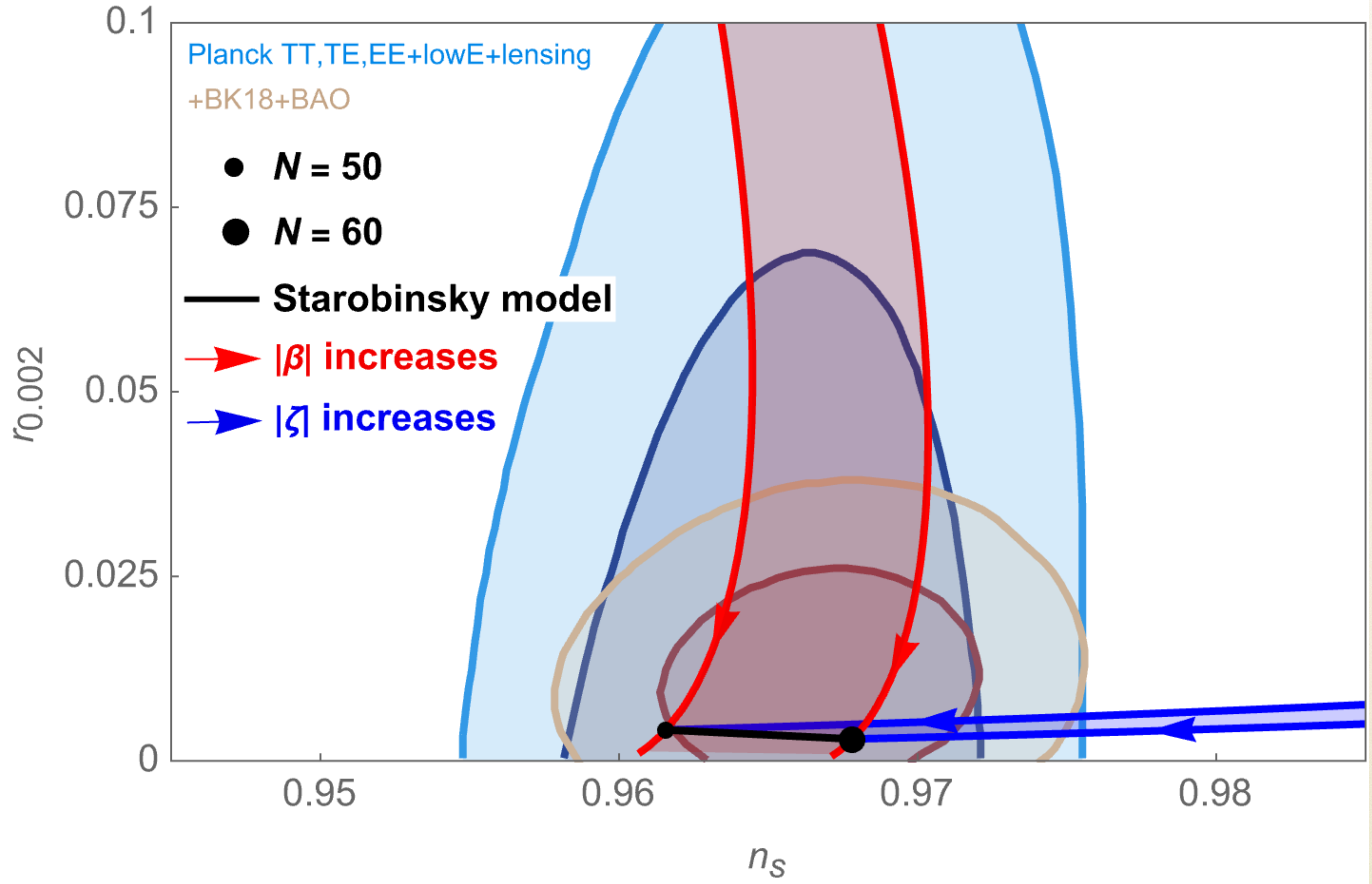
ζ controls the deviation from alpha-attractor

Starobinsky Inflation and beyond in Einstein-Cartan Gravity, Muzi Hong (RESCEU), arXiv: 2402.05358

With Nieh—Yan term and Holst term

- Observational constraint

$$S = \int \sqrt{-g_J} d^4x \left[\frac{M_{Pl}^2}{2} \left(\bar{R}_J + \zeta S_\mu T^\mu \right) + \alpha_S \left(\nabla_\mu S^\mu + \frac{\beta}{2} S_\mu T^\mu \right)^2 \right]$$



Red trajectory
(alpha-attractor inflation limit)
$\zeta = -3 \times 10^4$
β from $-5/2$ to $-1/25$

Blue trajectory
$\beta = -4/3$
ζ from -3×10^4 to -13

1. Introduction (Einstein—Cartan gravity; inflation theory)
2. Propagating degree of freedom in EC gravity as inflatons
3. Conclusions and future works

Conclusion and future works

- By adding Nieh—Yan term or/and Holst term into EC gravity, one can obtain Starobinsky inflation and its deformations

- Future work
  For fermions, torsion is naturally coupled to $j^{5\mu}$
  reheating, baryogengesis…

Backups

Motivation of EC gravity

- spinor fields in curved spacetime?
- Poincare gauge theory
  introducing veirbeins and spin connections

  extra degrees of freedom than metric formalism

Geometric meaning of torsion

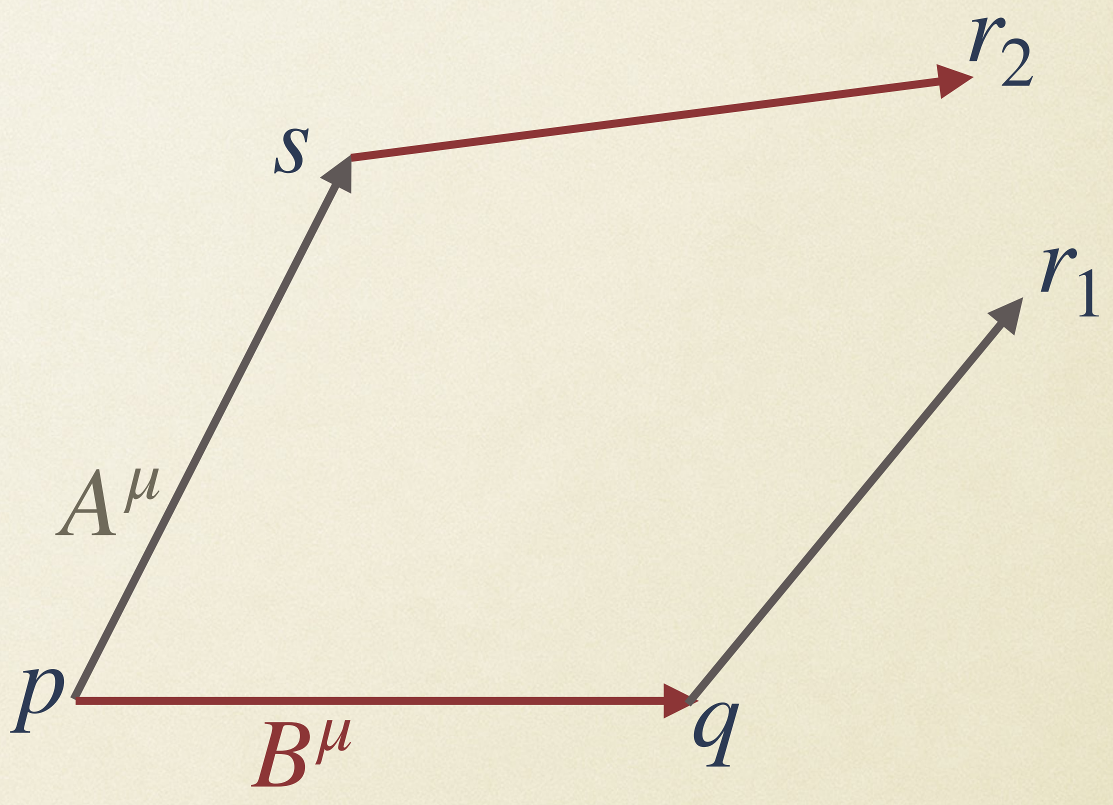
- prepare two infinitesimal vectors A^μ and B^μ

- parallel transport A^μ along B^μ
  vector $pr_1 = B^\mu + A^\mu - \bar{\Gamma}^\mu_{\ \nu\lambda} A^\lambda B^\nu$

- Parallel transport B^μ along A^μ
  $pr_2 = A^\mu + B^\mu - \bar{\Gamma}^\mu_{\ \nu\lambda} B^\lambda A^\nu$

- $r_1 r_2 = pr_2 - pr_1 = (\bar{\Gamma}^\mu_{\ \nu\lambda} - \bar{\Gamma}^\mu_{\ \lambda\nu}) A^\lambda B^\nu = T^\mu_{\ \nu\lambda} A^\lambda B^\nu$

- Torsion tensor measures the part that does not close



Horizon problem explained using particle horizon

- $ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2)$ with $dt = ad\eta$
- Light propagating along $\chi = \pm\eta + \text{const}$
- Particle horizon: maximum comoving distance light can propagate (in causal contact), (talked before)

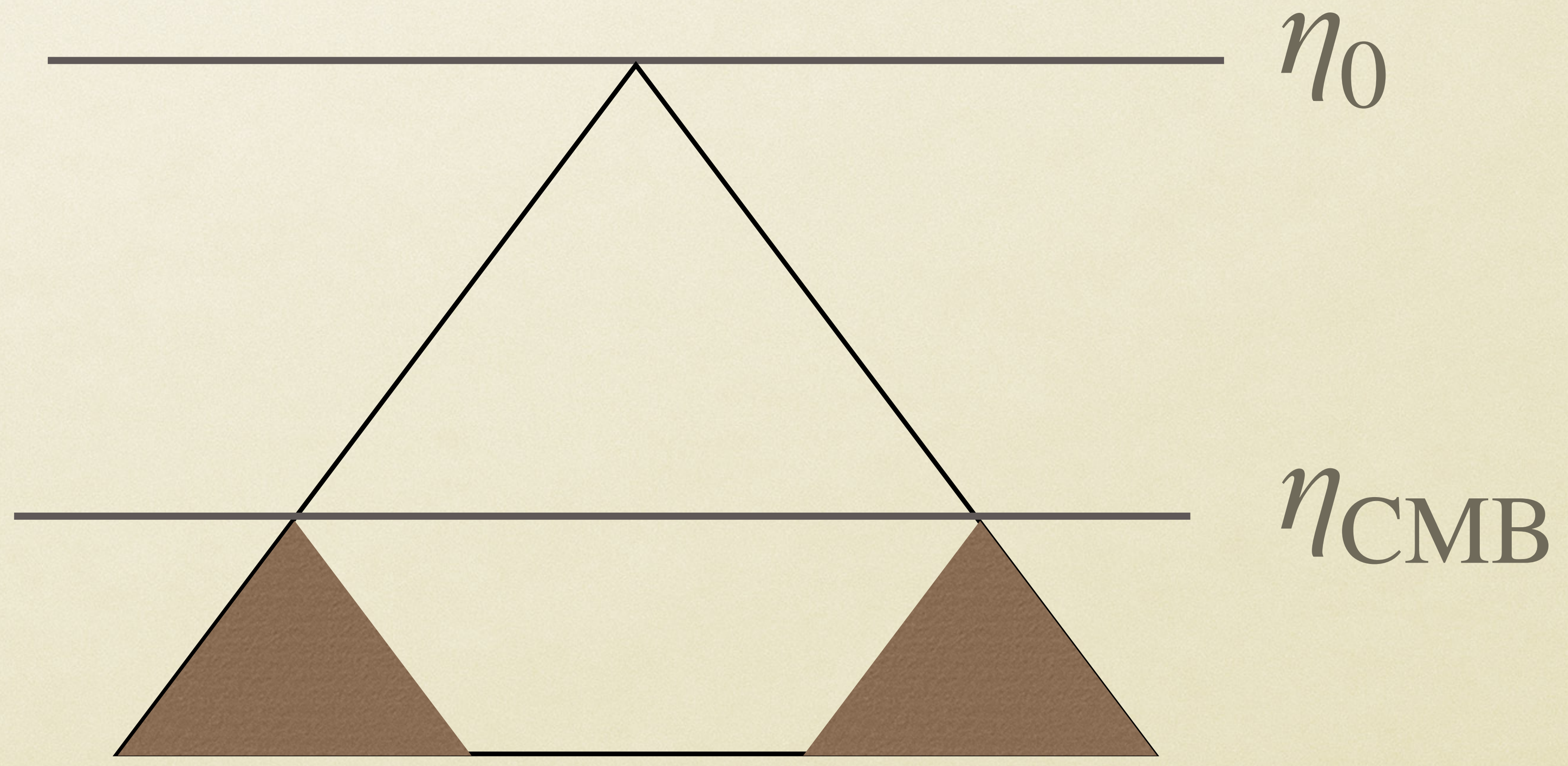
$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a} = \int_{a_i}^{a} d\ln a \, \frac{1}{aH}$$

Horizon problem explained using particle horizon

- Particle horizon: maximum comoving distance light can propagate

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a} = \int_{a_i}^{a} d\ln a \; \frac{1}{aH}$$

- Forever increasing $\frac{1}{aH}$: the closer to today the more contributions to causally connected region

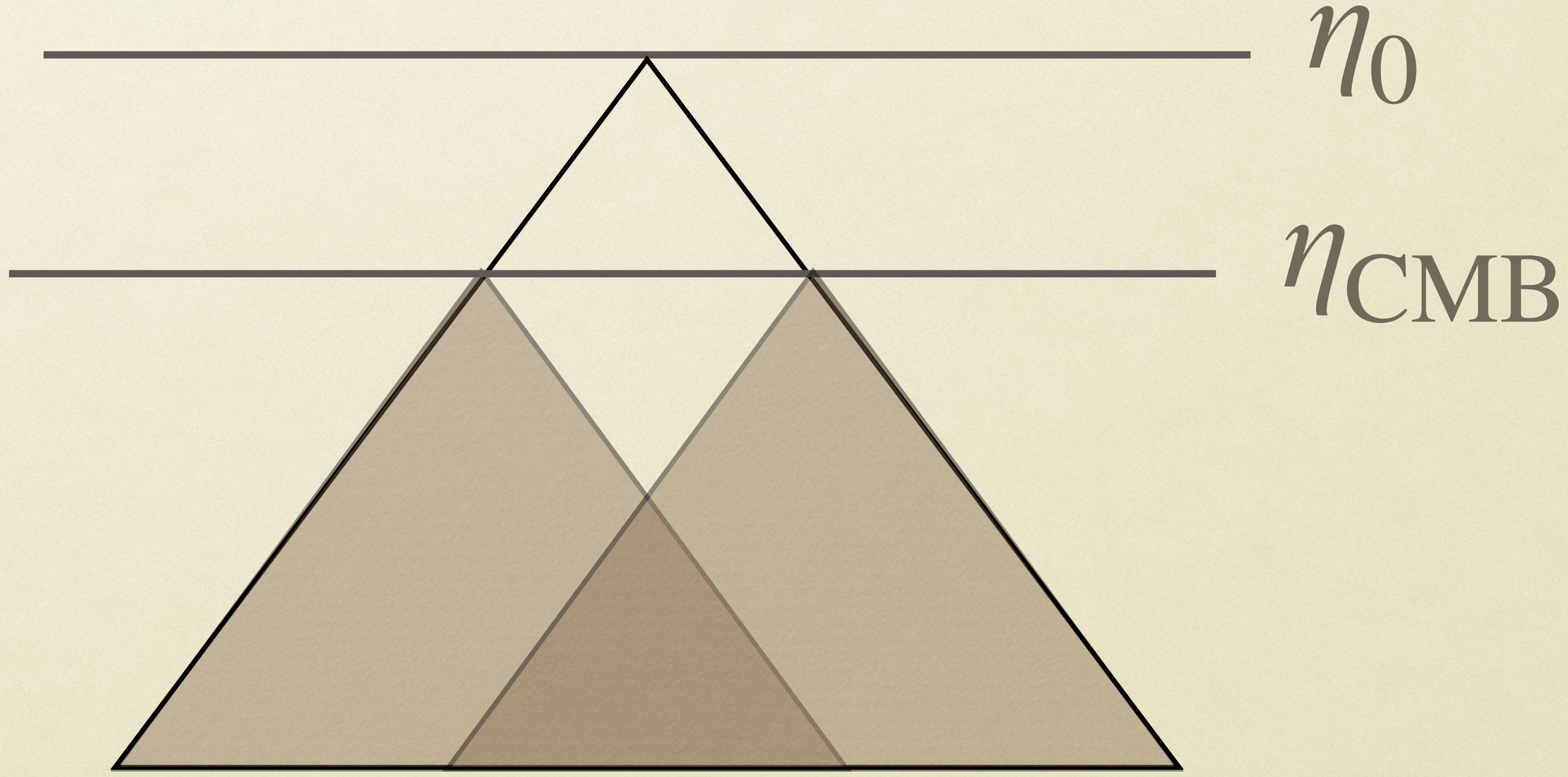


Horizon problem explained using particle horizon

- Particle horizon: maximum comoving distance light can propagate

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a} = \int_{a_i}^{a} d\ln a \, \frac{1}{aH}$$

- Inflation period (decreasing $\frac{1}{aH}$) existed: contributions to $\chi_p(\eta)$ mainly comes from the inflation period



Observational quantities

- $ds^2 = -(1+2\Phi)dt^2 + 2aB_i dx^i dt + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j$

- Comoving curvature perturbation
  $\mathscr{R} = \Psi - \frac{H}{\dot{\phi}}\delta\phi$
  gauge-invariant & conserved on super-horizon scales

- Mukhanov-Sasaki variable
  $v \equiv z\mathscr{R}$ with $z \equiv a^2\frac{\dot{\phi}^2}{H^2}$

Observational quantities

- $v_k'' + (k^2 - \frac{z''}{z})v_k = 0$
- One can quantize v_k as quantizing a harmonic oscillator with time-dependent frequency, and calculate the vacuum expectations
- $< \mathscr{R}_{\mathbf{k}} \mathscr{R}_{\mathbf{k}'} > = (2\pi^3)\delta(\mathbf{k} + \mathbf{k}')P_{\mathscr{R}}(k)$
- Similarly $< h_{\mathbf{k}} h_{\mathbf{k}'} > = (2\pi^3)\delta(\mathbf{k} + \mathbf{k}')P_h(k)$ for $h \equiv h^+, h^\times$

Observational quantities

- $$\Delta_s^2 \equiv \frac{k^3}{2\pi^2} P_{\mathscr{R}}(k) = \frac{H^2}{(2\pi)^2} \frac{H^2}{\dot{\phi}^2}$$
  $$\Delta_t^2 \equiv 2\frac{k^3}{2\pi^2} P_h(k) = \frac{2}{\pi^2} \frac{H^2}{M_{Pl}^2}$$

- $$n_s - 1 \equiv \frac{d\ln\Delta_s^2}{d\ln k} \approx 2\eta_V - 6\epsilon_V$$
  $$r \equiv \frac{\Delta_t^2}{\Delta_s^2} \approx 16\epsilon_V$$

How to decide α_{R} or α_{S}?

- $\Delta_s^2(k) \approx \frac{1}{24\pi^2} \frac{V}{M_{Pl}^4} \frac{1}{\epsilon_V} |_{k=aH}$
  $\sim 10^{-9}$ @CMB

- $\alpha_{\mathrm{R}} \sim 10^9$

Shifting the last $V(\phi)$

- $$S = \int \sqrt{-g_{\mathrm{E}}}d^4x \left[\frac{M_{Pl}^2}{2}R_{\mathrm{E}} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{M_{Pl}^4}{36\alpha_{\mathrm{S}}\beta^2}\left(3\zeta + \sinh\sqrt{\frac{3}{8}}\frac{\beta\phi}{M_{Pl}}\right)^2 \right]$$
- $$\phi_{\mathrm{min}} = \sqrt{\frac{8}{3}}\frac{M_{Pl}}{\beta}\ln(-3\zeta + \sqrt{1+9\zeta^2})$$
- $$V(\phi) = \frac{M_{Pl}^4}{144\alpha_{\mathrm{S}}\beta^2}\left[6\zeta + e^{\sqrt{\frac{3}{8}}\frac{\beta}{M_{Pl}}(\phi+\phi_{\mathrm{min}})} - e^{-\sqrt{\frac{3}{8}}\frac{\beta}{M_{Pl}}(\phi+\phi_{\mathrm{min}})}\right]^2$$
- If large $|\zeta|$, the last term can be dropped ($\zeta < 0$, $\beta < 0$ here), and one obtains the alpha-attractor inflation

Matter & Dark energy